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Research Article

Faber Polynomial Coefficient Bounds for m-Fold Symmetric Analytic and Bi-univalent Functions Involving q-Calculus

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In our present investigation, by applying q-calculus operator theory, we define some new subclasses of m-fold symmetric analytic and bi-univalent functions in the open unit disk $\mathscr{U} = \{z \in \mathbb{C} : |z| < 1\}$ and use the Faber polynomial expansion to find upper bounds of $|a_{mk+1}|$ and initial coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ as well as Fekete-Szego inequalities for the functions belonging to newly defined subclasses. Also, we highlight some new and known corollaries of our main results.

1. Introduction, Definitions, and Motivation

Let $\mathscr{A}$ denote the class of all analytic functions $f(z)$ in the open unit disk $\mathscr{U} = \{z : |z| < 1\}$ and have the series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

By $\mathscr{S}$, we mean the subclass of $\mathscr{A}$ consisting of univalent functions. The inverse f^{-1} of univalent function f can be defined as

$$\begin{aligned} &f^{-1}(f(z)) = z,\ z \in \mathscr{U}, \\ &f(f^{-1}(w)) = w, \quad |w| < r_0(f), r_0(f) \geq \frac{1}{4}, \end{aligned} \tag{2}$$

where

$$\begin{aligned} g_1(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 \\ - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + . \cdots \end{aligned} \tag{3}$$

According to the Koebe one-quarter theorem [1], an analytic function f is called bi-univalent in $\mathscr{U}$ if both f and f^{-1} are univalent in $\mathscr{U}$. Let Σ denote the class all bi-univalent functions in $\mathscr{U}$. For $f \in \Sigma$, Lewin [2] showed that $|a_2| < 1.51$ and Brannan and Cluni [3] proved that $|a_2| \leq \sqrt{2}$. Netanyahu [4] showed that $\max |a_2| = 4/3$. Brannan and Taha [5] introduced a certain subclass of bi-univalent functions for class Σ. In recent years, Srivastava et al. [6], Frasin and Aouf [7], Altinkaya and Yalcin [8, 9], and Hayami and Owa [10] studied the various subclasses of analytic and bi-univalent function. For a brief history, see [11].

In [12], Faber introduced Faber polynomials, and after that, Gong [13] studied Faber polynomials in geometric function theory. In their published works, some contributions have been made to finding the general coefficient bounds $|a_n|$ by applying Faber polynomial expansions. By using Faber polynomial expansions, very little work has been done for the coefficient bounds $|a_n|$ for $n \geq 4$ of Maclaurin's series. For more studies, see [14–17].

A domain $\mathscr{U}$ is said to be m-fold symmetric if

$$f\left(e^{i(2\pi/m)} z\right) = e^{i(2\pi/m)} f(z), \quad z \in \mathscr{U}, f \in \mathscr{A}, m \in \mathbb{N}. \tag{4}$$

The univalent function $h(z)$ maps the unit disk $\mathcal{U}$ into a region with m-fold symmetry and can be defined as

$$h(z) = \sqrt[m]{f(z^m)}, \quad f \in \mathcal{S}. \tag{5}$$

A function f is said to be m-fold symmetric [18] if it has the series expansion of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}. \tag{6}$$

The class of all m-fold symmetric univalent functions is denoted by $\mathcal{S}^m$, and for $m = 1$, then $\mathcal{S}^m = \mathcal{S}$.

In [19], Srivastava et al. proved the inverse f_m^{-1} series expansion for $f \in \Sigma_m$, which is given as follows:

$$\begin{aligned} g(w) = f_m^{-1}(w) = {} & w - a_{m+1} w^{m+1} + \left((m+1)a_{m+1}^2 - a_{2m+1}\right) w^{2m+1} \\ & - \left\{ \frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1} \right) \Big\} w^{3m+1} \\ & +. \cdots \end{aligned} \tag{7}$$

Here, we will denote m-fold symmetric bi-univalent functions by Σ_m. For $m = 1$, equation (7) coincides with equation (3) of the class Σ. The coefficient problem for $f \in \Sigma_m$ is one of the favorite subjects of geometric function theory in these days (see [20–23]).

The quantum (or q-) calculus has great importance because of its applications in several fields of mathematics, physics, and some related areas. The importance of q -derivative operator (D_q) is pretty recognizable by its applications in the study of numerous subclasses of analytic functions. Initially, in 1908, Jackson [24] introduced a q -derivative operator and studied its applications. Further, in [25], Ismail et al. defined a class of q-starlike functions; after that, Srivastava [26] studied q-calculus in the context of univalent function theory; also, numerous mathematicians studied q-calculus in the context of univalent function theory. Further, the q-analogue of the Ruscheweyh differential operator was defined by Kanas and Raducanu [27] and Arif et al. [28] discussed some of its applications for multivalent functions while Zhang et al. in [29] studied q-starlike functions related with the generalized conic domain. Srivastava et al. published the articles (see [30, 31]) in which they studied the class of q-starlike functions. For some more recent investigations about q-calculus, we may refer to [32–34].

For a better understanding of the article, we recall some concept details and definitions of the q-difference calculus. Throughout the article, we presume that

$$0 < q < 1. \tag{8}$$

Definition 1. The q-factorial $[n]_q!$ is defined as

$$[n]_q! = \prod_{k=1}^{n} [k]_q \quad (n \in \mathbb{N}), \tag{9}$$

and the q-generalized Pochhammer symbol $[t]_{n,q}$, $t \in \mathbb{C}$, is defined as

$$[t]_{n,q} = [t]_q [t+1]_q [t+2]_q \cdots [t+n-1]_q \quad (n \in \mathbb{N}). \tag{10}$$

Remark 2. For $n = 0$, then $[n]_q! = 1$, and $[t]_{n,q} = 1$.

Definition 3. The q-number $[t]_q$ for $q \in (0, 1)$ is defined as

$$[t]_q = \begin{cases} \dfrac{1-q^t}{1-q} & (t \in \mathbb{C}), \\ \displaystyle\sum_{k=0}^{n-1} q^k & (t = n \in \mathbb{N}). \end{cases} \tag{11}$$

Definition 4 (see [24]). The q-derivative (or q-difference) operator D_q of a function f is defined, in a given subset of $\mathbb{C}$, by

$$(D_q f)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0, \end{cases} \tag{12}$$

provided that $f'(0)$ exists.

From Definition 4, we can observe that

$$\lim_{q \longrightarrow 1^-} (D_q f)(z) = \lim_{q \longrightarrow 1^-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z), \tag{13}$$

for a differentiable function f in a given subset of $\mathbb{C}$. It is also known from (1) and (12) that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \tag{14}$$

Here, in this paper, we use the q-difference operator to define new subclasses of m-fold symmetric analytic and bi-univalent functions and then apply the Faber polynomial expansion technique to determine the general coefficient bounds $|a_{mk+1}|$ and initial coefficient bounds $|a_{m+1}|$ and $|a_{2m+1}|$ as well as Fekete-Szego inequalities.

Definition 5. A function $f \in \Sigma_m$ is said to be in the class $\mathcal{R}_b(\varphi, m, q)$ if and only if

$$\begin{aligned} 1 + \frac{1}{b}\left(D_q f(z) - 1\right) \prec \varphi(z), \\ 1 + \frac{1}{b}\left(D_q g(w) - 1\right) \prec \varphi(w), \end{aligned} \tag{15}$$

where $\varphi \in \mathcal{P}$, $b \in \mathbb{C} \setminus \{0\}$, and$z, w \in \mathcal{U}$, and$g(w) = f_m^{-1}(w)$ is defined by (7).

Remark 6. For $q \longrightarrow 1^-$ and $m = 1$, then the class $\mathscr{R}_b(\varphi, m, q)$ reduces into the class $\mathscr{R}_b(\varphi)$ introduced by Hamidi and Jahangiri in [35].

Definition 7. A function $f \in \Sigma_m$ is said to be in the class $\mathcal{S}^*_{\Sigma_m}(\varphi, q)$ if and only if

$$\frac{zD_q f(z)}{f(z)} \prec \varphi(z), \qquad \frac{wD_q g(w)}{g(w)} \prec \varphi(w), \tag{16}$$

where $\varphi \in \mathscr{P}$, $b \in \mathbb{C} \setminus \{0\}$, and$z, w \in \mathscr{U}$, and$g(w) = f_m^{-1}(w)$ is defined by (7).

Remark 8. For $q \longrightarrow 1^-$, $m = 1$, and $\varphi(z) = (1 + Az)/(1 + Bz)$, then the class $S^*_{\Sigma_m}(\varphi, q)$ reduces into the class $\mathcal{S}(A, B)$, introduced by Hamidi and Jahangiri in [36].

2. Main Results

Using the Faber polynomial expansion of functions $f \in \mathscr{A}$ of the form (1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as [15] given by

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots) w^n, \tag{17}$$

for an expansion of K_{n-1}^{-n} (see [37]). In particular, the first three terms of K_{n-1}^{-n} are

$$\begin{aligned} \frac{1}{2}K_1^{-2} &= -a_2, \\ \frac{1}{3}K_2^{-3} &= 2a_2^2 - a_3, \\ \frac{1}{4}K_3^{-4} &= -\left(5a_2^3 - 5a_2a_3 + a_4\right). \end{aligned} \tag{18}$$

In general, for any $p \in \mathbb{N}$ and $n \geq 2$, an expansion of K_{n-1}^p is as (see [15])

$$\begin{aligned} K_{n-1}^p = pa_n + \frac{p(p-1)}{2} E_{n-1}^2 + \frac{p!}{(p-3)!3!} E_{n-1}^3 + \cdots \\ + \frac{p!}{(p-n+1)!(n-1)!} E_{n-1}^{n-1}, \end{aligned} \tag{19}$$

where $E_{n-1}^p = E_{n-1}^p(a_2, a_3, \cdots)$, and by [37],

$$E_{n-1}^m(a_2, \cdots, a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1} \cdots (a_n)^{\mu_{n-1}}}{\mu_1!, \cdots, \mu_{n-1}!}, \quad \text{for } m \leq n, \tag{20}$$

while $a_1 = 1$, and the sum is taken over all nonnegative integers $\mu_1, \cdots, \mu_n$ satisfying

$$\begin{aligned} \mu_1 + \mu_2 + \cdots + \mu_n &= m, \\ \mu_1 + 2\mu_2 + \cdots + (n-1)\mu_{n-1} &= n - 1. \end{aligned} \tag{21}$$

Evidently, $E_{n-1}^{n-1}(a_2, \cdots, a_n) = a_2^{n-1}$ (see [14]), or equivalently,

$$E_n^m(a_1, a_2, \cdots, a_n) = \sum_{n=1}^{\infty} \frac{m!(a_1)^{\mu_1} \cdots (a_n)^{\mu_n}}{\mu_1!, \cdots, \mu_n!}, \quad \text{for } m \leq n, \tag{22}$$

while $a_1 = 1$, and the sum is taken over all nonnegative integers $\mu_1, \cdots, \mu_n$ satisfying

$$\begin{aligned} \mu_1 + \mu_2 + \cdots + \mu_n &= m, \\ \mu_1 + 2\mu_2 + \cdots + (n)\mu_n &= n. \end{aligned} \tag{23}$$

It is clear that $E_n^n(a_1, \cdots, a_n) = E_1^n$, and the first and last polynomials are $E_n^n = a_1^n$ and $E_n^1 = a_n$.

Similarly, using the Faber polynomial expansion of functions $f \in \mathscr{A}$ of the form (6), that is,

$$f(z) = z + \sum_{k=1}^{\infty} K_k^{1/m}(a_2, a_3, \cdots, a_{k+1}) z^{mk+1}. \tag{24}$$

The coefficients of its inverse map $g = f_m^{-1}$ may be expressed as

$$\begin{aligned} g(z) = f_m^{-1}(z) = w + \sum_{k=1}^{\infty} \frac{1}{(mk+1)} K_k^{-(mk+1)} \\ \cdot (a_{m+1}, a_{2m+1}, \cdots, a_{mk+1}) w^{mk+1}. \end{aligned} \tag{25}$$

Theorem 9. *For $b \in \mathbb{C} \setminus \{0\}$, let $f \in \mathscr{R}_b(\varphi, m, q)$be given by (6), and if$a_{mj+1} = 0$, $1 \leq j \leq k - 1$, then*

$$|a_{mk+1}| \leq \frac{2|b|}{1 + mk}, \quad \text{for } k \geq 2. \tag{26}$$

Proof. By definition, for the function $f \in \mathscr{R}_b(\varphi, m, q)$ of the form (6), we have

$$1 + \frac{1}{b}\left(D_q f(z) - 1\right) = 1 + \sum_{k=1}^{\infty} \frac{[1 + mk]_q}{b} a_{mk+1} z^{mk}, \tag{27}$$

and for its inverse map $g = f_m^{-1}$, we have

$$1 + \frac{1}{b}\left(D_q g(w) - 1\right) = 1 + \sum_{k=1}^{\infty} \frac{[1 + mk]_q}{b} A_{mk+1} w^{mk}, \tag{28}$$

where

$$A_{mk+1} = \frac{1}{mk+1} K_k^{-(mk+1)}(a_{m+1}, a_{2m+1}, \cdots, a_{mk+1}), \quad k \geq 1. \tag{29}$$

On the other hand, since $f \in \mathcal{R}_b(\varphi, m, q)$ and $g = f_m^{-1} \in \mathcal{R}_b(\varphi, m, q)$ by definition, we have

$$\begin{aligned} p(z) &= c_1 z^m + c_2 z^{2m} + \cdots = \sum_{k=1}^{\infty} c_k z^{mk}, \\ q(w) &= d_1 w^m + d_2 w^{2m} + \cdots = \sum_{k=1}^{\infty} d_k w^{mk}, \end{aligned} \tag{30}$$

where

$$\varphi(p(z)) = 1 + \sum_{k=1}^{\infty} \sum_{l=1}^{k} \varphi_l K_k^l (c_1, c_2, \cdots, c_k) z^{mk}, \tag{31}$$

$$\varphi(q(w)) = 1 + \sum_{k=1}^{\infty} \sum_{l=1}^{k} \varphi_l K_k^l (d_1, d_2, \cdots, d_k) w^{mk}. \tag{32}$$

Comparing the coefficients of (27) and (31), we have

$$\frac{1}{b}[1+mk]_q a_{mk+1} = \sum_{l=1}^{k-1} \varphi_l K_k^l (c_1, c_2, \cdots, c_k). \tag{33}$$

Similarly, comparing coefficients of (28) and (32), we have

$$\frac{1}{b}[1+mk]_q A_{mk+1} = \sum_{l=1}^{k-1} \varphi_l K_k^l (d_1, d_2, \cdots, d_k). \tag{34}$$

Note that for $a_{mj+1} = 0$, $1 \le j \le k-1$, we have

$$A_{mk+1} = -a_{mk+1}, \tag{35}$$

and so

$$\frac{1}{b}[1+mk]_q a_{mk+1} = \varphi_1 c_k, \tag{36}$$

$$-\frac{1}{b}[1+mk]_q a_{mk+1} = \varphi_1 d_k. \tag{37}$$

Now taking the absolute of (36) and (37) and using the fact that $|\varphi_1| \le 2$, $|c_k| \le 1$, and $|d_k| \le 1$, we have

$$\begin{aligned} |a_{mk+1}| &\le \frac{|b|}{[1+mk]_q} |\varphi_1 c_k| = \frac{|b|}{[1+mk]_q} |\varphi_1 d_k|, \\ |a_{mk+1}| &\le \frac{2|b|}{[1+mk]_q}, \end{aligned} \tag{38}$$

which completes the proof of Theorem 9. □

For $m = 1$ and $k = n - 1$, in Theorem 9, we obtain the following corollary.

Corollary 10. *For $b \in \mathbb{C} \setminus \{0\}$, let $f \in \mathcal{R}_b(\varphi, q)$, and if $a_{j+1} = 0$, $1 \le j \le n$, then*

$$|a_n| \le \frac{2|b|}{[n]_q}, \quad \text{for } n \ge 3. \tag{39}$$

For $q \longrightarrow 1^-$, $m = 1$, and $k = n - 1$, in Theorem 9, we obtain the following known corollary.

Corollary 11 (see [35]). *For $b \in \mathbb{C} \setminus \{0\}$, let $f \in \mathcal{R}_b(\varphi)$, and if $a_{j+1} = 0$, $1 \le j \le n$, then*

$$|a_n| \le \frac{2|b|}{n}, \quad \text{for } n \ge 3. \tag{40}$$

Theorem 12. *For $b \in \mathbb{C} \setminus \{0\}$, let $f \in \mathcal{B}_b(\varphi, m, q)$ be given by (6), and then*

$$\begin{aligned} |a_{m+1}| &\le \begin{cases} \dfrac{2|b|}{[m+1]_q}, & \text{if } |b| < \dfrac{8}{(m+1)[2m+1]_q}, \\ \sqrt{\dfrac{8|b|}{(m+1)[2m+1]_q}}, & \text{if } |b| \ge \dfrac{8}{(m+1)[2m+1]_q}, \end{cases} \\ |a_{2m+1}| &\le \begin{cases} \dfrac{2|b|}{[2m+1]_q} + \dfrac{2(m+1)|b|^2}{\left([m+1]_q\right)^2}, & \text{if } |b| < \dfrac{2}{[2m+1]_q}, \\ \dfrac{4|b|}{[2m+1]_q}, & \text{if } |b| \ge \dfrac{2}{[2m+1]_q}, \end{cases} \\ &\left|a_{2m+1} - (m+1)a_{m+1}^2\right| \le \frac{4|b|}{[2m+1]_q}, \\ &\left|a_{2m+1} - \frac{(m+1)}{2} a_{m+1}^2\right| \le \frac{2|b|}{[2m+1]_q}. \end{aligned} \tag{41}$$

Proof. Replacing k by 1 and 2 in (33) and (34), respectively, we have

$$\frac{1}{b}[m+1]_q a_{m+1} = \varphi_1 c_1, \tag{42}$$

$$\frac{1}{b}[2m+1]_q a_{2m+1} = \varphi_1 c_2 + \varphi_2 c_1^2, \tag{43}$$

$$-\frac{1}{b}[m+1]_q a_{m+1} = \varphi_1 d_1, \tag{44}$$

$$\frac{1}{b}[2m+1]_q \left\{(m+1)a_{m+1}^2 - a_{2m+1}\right\} = \varphi_1 d_2 + \varphi_2 d_1^2. \tag{45}$$

From (42) and (44), we have

$$|a_{m+1}| \le \frac{|b|}{[m+1]_q} |\varphi_1 c_1| = \frac{|b|}{[m+1]_q} |\varphi_1 d_1| \le \frac{2|b|}{[m+1]_q}. \tag{46}$$

Adding (43) and (45), we have

$$a_{m+1}^2 = \frac{b\{\varphi_1(c_2+d_2)+\varphi_2(c_1^2+d_1^2)\}}{(m+1)[2m+1]_q}. \tag{47}$$

Taking the absolute value (47), we have

$$|a_{m+1}| \le \sqrt{\frac{8|b|}{(m+1)[2m+1]_q}}. \tag{48}$$

Now, the bounds given for $|a_{m+1}|$ can be justified since

$$|b| < \sqrt{\frac{8|b|}{(m+1)[2m+1]_q}}, \quad \text{for } |b| < \frac{8}{(m+1)[2m+1]_q}. \tag{49}$$

From (43), we have

$$|a_{2m+1}| = \frac{|b||\varphi_1 c_2+\varphi_2 c_1^2|}{[2m+1]_q} \le \frac{4|b|}{[2m+1]_q}. \tag{50}$$

Next, we subtract (45) from (43), and we have

$$\begin{aligned}\frac{2[2m+1]_q}{b}\left\{a_{2m+1}-\frac{(m+1)}{2}a_{m+1}^2\right\}\\ = \varphi_1(c_2-d_2)+\varphi_2(c_1^2-d_1^2)=\varphi_1(c_2-d_2),\end{aligned} \tag{51}$$

or

$$a_{2m+1} = \frac{(m+1)}{2}a_{m+1}^2 + \frac{\varphi_1 b(c_2-d_2)}{2[2m+1]_q}. \tag{52}$$

After some simple calculation and by taking the absolute, we have

$$|a_{2m+1}| \le \frac{|\varphi_1||b||c_2-d_2|}{2(2m+1)} + \frac{(m+1)}{2}|a_{m+1}^2|. \tag{53}$$

Using the assertion (46), we have

$$|a_{2m+1}| \le \frac{2|b|}{[2m+1]_q} + \frac{2(m+1)|b|^2}{\left([m+1]_q\right)^2}. \tag{54}$$

From (50) and (54), we note that

$$\frac{2|b|}{[2m+1]_q} + \frac{2(m+1)|b|^2}{\left([m+1]_q\right)^2} \le \frac{4|b|}{[2m+1]_q}, \quad \text{if } |b| < \frac{2}{[2m+1]_q}. \tag{55}$$

Now, we rewrite (45) as

$$\frac{1}{b}[2m+1]_q\{(m+1)a_{m+1}^2 - a_{2m+1}\} = \varphi_1 d_2 + \varphi_2 d_1^2. \tag{56}$$

Taking the absolute value, we have

$$\left|a_{2m+1}-(m+1)a_{m+1}^2\right| \le \frac{4|b|}{[2m+1]_q}. \tag{57}$$

Finally, from (51), we have

$$\frac{2[2m+1]_q}{b}\left\{a_{2m+1}-\frac{(m+1)}{2}a_{m+1}^2\right\} = \varphi_1(c_2-d_2). \tag{58}$$

Taking the absolute value, we have

$$\left|a_{2m+1}-\frac{(m+1)}{2}a_{m+1}^2\right| \le \frac{2|b|}{[2m+1]_q}. \tag{59}$$

□

For $m=1$ and $k=n-1$, in Theorem 12, we obtain the following corollary.

Corollary 13. *For $b \in \mathbb{C}\setminus\{0\}$, let $f \in \mathscr{B}_b(\varphi, q)$ be given by (1), and then*

$$\begin{gathered}|a_2| \le \begin{cases}\dfrac{2|b|}{[2]_q}, & \text{if } |b| < \dfrac{4}{[3]_q},\\ \sqrt{\dfrac{4|b|}{[3]_q}}, & \text{if } |b| \ge \dfrac{4}{[3]_q},\end{cases}\\ |a_3| \le \begin{cases}\dfrac{2|b|}{[3]_q} + \dfrac{4|b|^2}{\left([2]_q\right)^2}, & \text{if } |b| < \dfrac{2}{[3]_q},\\ \dfrac{4|b|}{[3]_q}, & \text{if } |b| \ge \dfrac{2}{[3]_q},\end{cases}\\ \left|a_3 - 2a_{m+1}^2\right| \le \frac{4|b|}{[3]_q},\\ \left|a_{2m+1} - a_{m+1}^2\right| \le \frac{2|b|}{[3]_q}.\end{gathered} \tag{60}$$

For $q \longrightarrow 1^-$, $m=1$, and $k=n-1$, in Theorem 12, we obtain the following corollary.

Corollary 14 (see [35]). *For $b \in \mathbb{C} \setminus \{0\}$, let $f \in \mathscr{B}_b(\varphi)$ be given by (1), and then*

$$
\begin{aligned}
|a_2| &\le \begin{cases} |b|, & \text{if } |b| < \dfrac{4}{3}, \\ \sqrt{\dfrac{4|b|}{3}}, & \text{if } |b| \ge \dfrac{4}{3}, \end{cases} \\
|a_3| &\le \begin{cases} \dfrac{2|b|}{3} + |b|^2, & \text{if } |b| < \dfrac{2}{3}, \\ \dfrac{4|b|}{3}, & \text{if } |b| \ge \dfrac{2}{3}, \end{cases} \\
\left|a_3 - 2a_2^2\right| &\le \frac{4|b|}{3}, \\
\left|a_3 - a_2^2\right| &\le \frac{2|b|}{3}.
\end{aligned} \tag{61}
$$

Theorem 15. *Let $f \in \mathcal{S}^*_{\Sigma_m}(\varphi, q)$be given by (6), and if$a_{mj+1} = 0$, $1 \le j \le k-1$, then*

$$
|a_{mk+1}| \le \frac{2}{[mk]_q}, \quad \text{for } k \ge 2. \tag{62}
$$

Proof. By definition, for the function $f \in \mathcal{S}^*_{\Sigma_m}(\varphi, q)$ of the form (6), we have

$$
\frac{zD_q f(z)}{f(z)} = 1 - \sum_{k=1}^{\infty} F_k(a_{m+1}, a_{2m+1}, \cdots, a_{mk+1}) z^{mk}, \tag{63}
$$

where the first few coefficients of $F_k(a_{m+1}, a_{2m+1}, \cdots, a_{mk+1})$ are

$$
\begin{aligned}
F_1 &= -a_{m+1}, \\
F_2 &= a_{m+1}^2 - (m+1)a_{2m+1}, \\
F_3 &= \left\{-a_{m+1}^3 + (2m+1)a_{m+1}a_{2m+1} - (2m+1)a_{3m+1}\right\}.
\end{aligned} \tag{64}
$$

In general,

$$
\begin{aligned}
&F_k(a_{m+1}, a_{2m+1}, \cdots, a_{mk+1}) \\
&\quad = \sum_{i_1+2i_2+\cdots k i_{nk}=k} \left\{A(i_1, i_2, i_2, \cdots, i_k)(a_{m+1})^{i_1}(a_{2m+1})^{i_2}\cdots(a_{mk+1})^{i_k}\right\},
\end{aligned} \tag{65}
$$

where

$$
A(i_1, i_2, i_2, \cdots, i_k) = (-1)^{(k)+2i_1+\cdots(k+1)i_k} \frac{(i_1 + i_2 + i_2 \cdots + i_k - 1)!k}{(i_1!)(i_2!)\cdots(i_k!)}. \tag{66}
$$

For the inverse map $g = f_m^{-1} \in \mathcal{S}^*_{\Sigma_m}(\varphi, q)$, we obtain

$$
\frac{zD_q g(w)}{g(w)} = 1 - \sum_{k=1}^{\infty} F_k(b_{m+1}, b_{2m+1}, \cdots, b_{mk+1}) w^{mk}, \tag{67}
$$

where

$$
A_{mk+1} = \frac{1}{mk+1} K_k^{-(mk+1)}(a_{m+1}, a_{2m+1}, \cdots, a_{mk+1}), \quad k \ge 1. \tag{68}
$$

On the other hand, since $f \in \mathcal{S}^*_{\Sigma_m}(\varphi, q)$ and $g = f_m^{-1} \in \mathcal{S}^*_{\Sigma_m}(\varphi, q)$ by definition, we have

$$
\begin{aligned}
p(z) &= c_1 z^m + c_2 z^{2m} + \cdots = \sum_{k=1}^{\infty} c_k z^{mk}, \\
q(w) &= d_1 w^m + d_2 w^{2m} + \cdots = \sum_{k=1}^{\infty} d_k w^{mk},
\end{aligned} \tag{69}
$$

where

$$
\varphi(p(z)) = 1 + \sum_{k=1}^{\infty} \sum_{l=1}^{k} \varphi_l K_k^l(c_1, c_2, \cdots, c_k) z^{mk}, \tag{70}
$$

$$
\varphi(q(w)) = 1 + \sum_{k=1}^{\infty} \sum_{l=1}^{k} \varphi_l K_k^l(d_1, d_2, \cdots, d_k) w^{mk}. \tag{71}
$$

Comparing the coefficients of (63) and (70), we have

$$
-[mk]_q a_{mk+1} = \sum_{l=1}^{k-1} \varphi_l K_k^l(c_1, c_2, \cdots, c_k). \tag{72}
$$

Similarly, comparing the coefficients of (67) and (71), we have

$$
-[mk]_q b_{mk+1} = \sum_{l=1}^{k-1} \varphi_l K_k^l(d_1, d_2, \cdots, d_k). \tag{73}
$$

Note that for $a_{mj+1} = 0$, $1 \le j \le k-1$, we have

$$
A_{mk+1} = -a_{mk+1}, \tag{74}
$$

and so

$$
-[mk]_q a_{mk+1} = \varphi_1 c_k, \tag{75}
$$

$$
[mk]_q a_{mk+1} = \varphi_1 d_k. \tag{76}
$$

Taking the absolute values of (75) and (76) and using the fact that $|\varphi_1| \le 2$, $|c_k| \le 1$, and $|d_k| \le 1$, we have

$$\begin{aligned}|a_{mk+1}| &\le \frac{1}{[mk]_q}|\varphi_1 c_k| = \frac{1}{[mk]_q}|\varphi_1 d_k|,\\ |a_{mk+1}| &\le \frac{2}{[mk]_q}.\end{aligned} \tag{77}$$

Hence, Theorem 15 is complete. □

For $q \longrightarrow 1^-$, $m=1$, and $k=n-1$, in Theorem 15, we obtain the following corollary.

Corollary 16. *$f \in \mathcal{S}^*(\varphi)$, and if$a_{j+1}=0$, $1 \le j \le n$, then*

$$|a_n| \le \frac{2}{n-1}, \quad \text{for } n \ge 3. \tag{78}$$

Theorem 17. *Let $f \in \mathcal{S}^*_{\Sigma_m}(\varphi, q)$be given by (6), and then*

$$\begin{aligned}|a_{m+1}| &\le \frac{2}{[m]_q},\\ |a_{2m+1}| &\le \frac{4(m+1)}{m[2m]_q} + \frac{2}{[2m]_q},\\ \left|a_{2m+1} - \frac{[m]_q(2m+1)}{[2m]_q}a_{m+1}^2\right| &\le \frac{4}{[2m]_q},\\ \left|a_{2m+1} - \frac{[m]_q(m+1)}{[2m]_q}a_{m+1}^2\right| &\le \frac{2}{[2m]_q}.\end{aligned} \tag{79}$$

Proof. Replacing k by 1 and 2 in (72) and (73), respectively, we have

$$[m]_q a_{m+1} = \varphi_1 c_1, \tag{80}$$

$$[2m]_q a_{2m+1} - [m]_q a_{m+1}^2 = \varphi_1 c_2 + \varphi_2 c_1^2, \tag{81}$$

$$-[m]_q a_{m+1} = \varphi_1 d_1, \tag{82}$$

$$[m]_q(2m+1)a_{m+1}^2 - [2m]_q a_{2m+1} = \varphi_1 d_2 + \varphi_2 d_1^2. \tag{83}$$

From (80) and (82), we have

$$|a_{m+1}| \le \frac{1}{[m]_q}|\varphi_1 c_1| = \frac{1}{[m]_q}|\varphi_1 d_1| \le \frac{2}{[m]_q}. \tag{84}$$

Adding (81) and (83), we have

$$a_{m+1}^2 = \frac{\varphi_1(c_2+d_2) + \varphi_2\left(c_1^2+d_1^2\right)}{2m[m]_q}. \tag{85}$$

Taking the absolute value (85), we have

$$|a_{m+1}| \le \frac{2}{\sqrt{m[m]_q}}. \tag{86}$$

Next, we subtract (83) from (81), and we have

$$\begin{aligned}&\left\{2[2m]_q a_{2m+1} - 2[m]_q(m+1)a_{m+1}^2\right\}\\ &\quad = \varphi_1(c_2-d_2) + \varphi_2\left(c_1^2-d_1^2\right),\end{aligned} \tag{87}$$

or

$$a_{2m+1} = \frac{[m]_q(m+1)}{[2m]_q}a_{m+1}^2 + \frac{\varphi_1(c_2-d_2)}{2[2m]_q}. \tag{88}$$

After some simple calculation of (88) and by taking the absolute, we have

$$|a_{2m+1}| \le \frac{|\varphi_1||c_2-d_2|}{2[2m]_q} + \frac{[m]_q(m+1)}{[2m]_q}\left|a_{m+1}^2\right|. \tag{89}$$

Using the assertion (86), we have

$$|a_{2m+1}| \le \frac{4(m+1)}{m[2m]_q} + \frac{2}{[2m]_q}. \tag{90}$$

For the third part, we rewrite (83) as

$$\left|[m]_q(2m+1)a_{m+1}^2 - [2m]_q a_{2m+1}\right| = \left|\varphi_1 d_2 + \varphi_2 d_1^2\right|. \tag{91}$$

Taking the absolute value, we have

$$\left|a_{2m+1} - \frac{[m]_q(2m+1)}{[2m]_q}a_{m+1}^2\right| \le \frac{4}{[2m]_q}. \tag{92}$$

Finally, from (87), we have

$$2[2m]_q\left|a_{2m+1} - \frac{[m]_q(m+1)}{[2m]_q}a_{m+1}^2\right| = |\varphi_1(c_2-d_2)|. \tag{93}$$

Taking the absolute value, we have

$$\left|a_{2m+1} - \frac{[m]_q(m+1)}{[2m]_q}a_{m+1}^2\right| \le \frac{2}{[2m]_q}. \tag{94}$$

□

For $q \longrightarrow 1^-$, $m=1$, and $k=n-1$, in Theorem 17, we get the following corollary.

Corollary 18. *Let $f \in \mathcal{S}^*(\varphi)$ be given by (1), and then*

$$\begin{aligned} |a_2| &\le 2, \\ |a_3| &\le 5, \\ \left|a_3 - \frac{3}{2}a_2^2\right| &\le 2, \\ \left|a_3 - a_2^2\right| &\le 1. \end{aligned} \tag{95}$$

3. Conclusion

In this paper, we have applied q-calculus operator theory to define some new subclasses of m-fold symmetric analytic and bi-univalent functions in open unit disk $\mathcal{U}$ and used the Faber polynomial expansion to find upper bounds $|a_{mk+1}|$ and initial coefficient bounds $|a_{m+1}|$ and $|a_{2m+1}|$ as well as Fekete-Szego inequalities for the functions belonging to newly defined subclasses of m-fold symmetric analytic and bi-univalent function. Also, we highlighted some new and known consequences of our main results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors jointly worked on the results, and they read and approved the final manuscript.

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