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Research Article

On the Algebraic Characteristics of Fuzzy Sub e-Groups

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Fuzzy set is a modern tool for depicting uncertainty. This paper introduces the concept of fuzzy sub e-group as an extension of fuzzy subgroup. The concepts of identity and inverse are generalized in fuzzy sub e-groups. Every fuzzy subgroup is proven to be a fuzzy sub e-group, but the converse is not true. Various properties of fuzzy sub e-groups are established. Moreover, the concepts of proper fuzzy sub e-group and super fuzzy sub e-group are discussed. Further, the concepts of fuzzy e-coset and normal fuzzy sub e-group are presented. Finally, we describe the effect of e-group homomorphism on normal fuzzy sub e-groups.

1. Introduction

Many decades ago, researchers developed an algebraic structure and named it as a group. Various properties of groups are proposed later on. In group theory, every group contains a unique identity element and every element has a unique inverse. In 2018, Saeid et al. [1] generalized the notion of groups to a new algebraic structure as e-groups. They generalized the notion of the identity of a group. Instead of choosing a single element as an identity element, Saeid et al. [1] considered a subset of the main set as an identity set. So, in an e-group, the identity element needs not to be unique. They proved that every group is an e-group, but the converse is not true. They defined homomorphism on e-groups in a different manner. E-group is an important tool for classifying isotopes. It is also a physical background in the unified Gauge theory.

Uncertainty is a massive component in the life of a person. In 1965, in his pioneer paper, Zadeh [2] first defined fuzzy set to handle uncertainty in real-life problems. In 1971, utilizing the concept of fuzzy set, Rosenfeld [3] first defined fuzzy subgroup. In 1979, using the t-norm concept of the fuzzy subgroup was restructured by Anthony and Sherwood [4, 5]. In 1981, the idea of fuzzy level subgroup was introduced by Das [6]. In 1988, Choudhury et al. [7] proved various properties of fuzzy homomorphism. In 1990, Dixit et al. [8] discussed the union of fuzzy subgroups and fuzzy level subgroups. The concept of antifuzzy subgroups was proposed by Biswas [9]. In 1992, Ajmal and Prajapati [10] developed fuzzy cosets and fuzzy normal subgroups. Chakraborty and Khare [11] studied various properties of fuzzy homomorphism. Ajmal [12] also studied the homomorphism of fuzzy subgroups. Later, many researchers studied various properties of fuzzy subgroups [13–16]. In 2015, Tarnauceanu [17] classified fuzzy normal subgroups of finite groups. In 2016, Onasanya [18] reviewed some antifuzzy properties of fuzzy subgroups. In 2018, Shuaib and Shaheryar [19] introduced omicron fuzzy subgroups. In 2018, Addis [20] developed fuzzy homomorphism theorems on groups. In 2019, Bhunia and Ghorai [21] studied (α, β)-Pythagorean fuzzy subgroups. In 2021, Bhunia et al. [22, 23] developed Pythagorean fuzzy subgroups. Abuhijleh et al. [24] worked on complex fuzzy subgroups in 2021. Alolaiyan et al. [25] studied algebraic structure of (α, β)

-complex fuzzy subgroups. Alolaiyan et al. [26] developed bipolar fuzzy subrings in 2021. In 2021, Talafha et al. [27] studied fuzzy fundamental groups and fuzzy folding of fuzzy Minkowski space.

In a fuzzy subgroup, the identity element of the group has the highest membership value. Also, in a fuzzy subgroup, the membership value of an element and its inverse are equal. But in an e-group, there is no unique identity element and no direct concept of the inverse of elements. Till now, no fuzzification has been done for e-groups. So, it is a challenge for us to fuzzify e-groups. In this study, we construct the concept of fuzzy sub e-groups. Here, we show that every fuzzy subgroup is a fuzzy sub e-group, but the converse is not true. So, the idea of fuzzy sub e-group is a much more generalized concept. In the study of isotopes, we notice that isotopes decay through neutron emission. So, the membership degree of these unstable neutrons must lie in $[0, 1]$. Therefore, fuzzy sub e-group will be more efficient in studying these unstable isotopes rather than a crisp e-group.

This paper is arranged in the following order. In Section 2, we recall some important concepts. In Section 3, utilizing the concept of e-groups, we generalize fuzzy subgroups. We develop the concept of fuzzy sub e-group and show that any fuzzy subgroup is also a fuzzy sub e-group. We also prove many algebraic properties of fuzzy sub e-groups. Further, we define fuzzy e-coset and normal fuzzy sub e-group in Section 4. Moreover, in Section 5, we show that after e-group homomorphism, a fuzzy sub e-group remains a fuzzy sub e-group. Finally, the conclusion is given in Section 6.

2. Preliminaries

Here, we will go over some basic definitions and concepts, which will be useful in the following sections.

Definition 1 (see [2]). A fuzzy set (FS) (D, κ) on a crisp set D is an object having the form $(D, \kappa) = \{(d, \kappa(d)) | d \in D\}$, where $\kappa \longrightarrow [0, 1]$ is the membership function.

Definition 2 (see [3]). Let (D, κ) be a FS on a group D. Then, (D, κ) is referred to be a fuzzy subgroup (FSG) of D if the following conditions hold:

(i) $\kappa(d_1 d_2) \geq \kappa(d_1) \wedge \kappa(d_2) \forall d_1, d_2 \in D$

(ii) $\kappa(d^{-1}) \geq \kappa(d) \forall d \in D$

Definition 3 (see [2]). Let (D, κ) be a FS on D. Then, for any $a \in [0, 1]$, the set $\kappa_a = \{d \mid d \in D, \kappa(d) \geq a\}$ is called a-cut of (D, κ).

Clearly, κ_a is a subset of D.

Proposition 4 (see [28]). *Let h be a mapping from D_1 into D_2. Let (D_1, κ_1) and (D_2, κ_2) be the two FSs on D_1 and D_2, respectively. Then, $(D_2, h(\kappa_1))$ and $(D_1, h^{-1}(\kappa_2))$ are FSs on D_2 and D_1, respectively, where for all $d_2 \in D_2$*

$$h(\kappa_1)(d_2) = \begin{cases} \vee\{\kappa_1(d_1) | d_1 \in D_1, h(d_1) = d_2\}, \\ \text{when } h^{-1}(d_2) \neq \varnothing, \\ 0, \text{elsewhere}, \end{cases} \quad (1)$$

and for all $d_1 \in D_1$, $(h^{-1}(\kappa_2))(d_1) = \kappa_2(h(d_1))$.

Definition 5 (see [1]). Let D be a nonempty crisp set and $L \subseteq D$. Then, $(D,\circ,L)$ is an e-group, where $\circ$ is the binary operation on D, which meets the following criteria:

(i) $d_1 \circ (d_2 \circ d_3) = (d_1 \circ d_2) \circ d_3 \forall d_1, d_2, d_3 \in D$

(ii) For every $d \in D$, $\exists$ an element $l \in L$ such that $d \circ l = l \circ d = d$

(iii) For every $d_1 \in D$, $\exists$ an element $d_2 \in D$ such that $d_1 \circ d_2$ and $d_2 \circ d_1 \in L$

Definition 6 (see [1]). Let $(D_1, \circ_1, L_1)$ and $(D_2, \circ_2, L_2)$ be the two e-groups. If a mapping $h : D_1 \longrightarrow D_2$ meets the following criteria, it is referred to as a homomorphism:

(i) $h(L_1) \subseteq L_2$

(ii) $h(d_1 \circ_1 d_2) = h(d_1) \circ_2 h(d_2) \forall d_1, d_2 \in D_1$

3. Fuzzy Sub e-Group and Its Properties

In this section, fuzzy sub e-group is briefly described as a generalization of fuzzy subgroup. The notions of identity and inverse are generalized in fuzzy sub e-group. We investigate its properties. We define super fuzzy sub e-group. We check whether union and intersections of fuzzy sub e-group are fuzzy sub e-groups.

Definition 7. A FS (D, κ) is referred to be a fuzzy sub e-group of an e-group $(D,\circ,L)$ if the following conditions hold:

(i) $\kappa(d_1 \circ d_2) \geq \kappa(d_1) \wedge \kappa(d_2) \forall d_1, d_2 \in D$

(ii) $\kappa(l) \geq \kappa(d) \forall l \in L$ and $d \in D/L$

where $\kappa : D \longrightarrow [0, 1]$ is the membership function.

Example 8. Let $D = \{d_1, d_2, d_3\}$ and $L = \{d_1, d_2\}$. Define a binary operation $\circ$ on D as below.

o	d_1	d_2	d_3
d_1	d_3	d_1	d_2
d_2	d_1	d_2	d_3
d_3	d_2	d_3	d_1

(2)

Then, $(D,\circ,L)$ is an e-group.

Here, we assign membership degrees to the elements of D by $\kappa(d_1) = \kappa(d_3) = 0.8$ and $\kappa(d_2) = 0.9$.

Now, $\kappa(d_1 \circ d_3) = \kappa(d_2) = 0.9 > 0.8 = \kappa(d_1) \wedge \kappa(d_3)$.

Similarly, we can check for other elements of D.

Therefore, $\kappa(d_i \circ d_j) \geq \kappa(d_i) \wedge \kappa(d_j)$ for all $d_i, d_j \in D$.

Also, $\kappa(d_1) = 0.8 = \kappa(d_3)$ and $\kappa(d_2) = 0.9 > 0.8 = \kappa(d_3)$.

Thus, (D, κ) forms a fuzzy sub e-group of the e-group $(D,\circ,L)$.

Theorem 9. *Let $(D,\circ,L)$ stand for an e-group and (D, κ) be a FS on D. Then, (D, κ) is referred to be a fuzzy sub e-group of $(D,\circ,L)$ if for all d_1 and $d_2 \in D$, $\kappa(d_1 \circ d_2') \geq \kappa(d_1) \wedge \kappa(d_2)$ for some $d_2' \in D$ such that $d_2 \circ d_2'$, $d_2' \circ d_2 \in L$.*

Proof. Let (D, κ) stand for a fuzzy sub e-group of $(D,\circ,L)$.

Then, for all d_1 and d_2 in D, $\kappa(d_1 \circ d_2) \geq \kappa(d_1) \wedge \kappa(d_2)$.

Let $d_1, d_2 \in D$. Then, $\exists$ some $d_2' \in D$ such that $d_2 \circ d_2'$ and $d_2' \circ d_2 \in L$.

Therefore, $\kappa(d_1 \circ d_2') \geq \kappa(d_1) \wedge \kappa(d_2') \geq \kappa(d_1) \wedge \kappa(d_2)$.

Conversely, assume that $\forall d_1, d_2 \in D$, $\kappa(d_1 \circ d_2') \geq \kappa(d_1) \wedge \kappa(d_2)$ for some $d_2' \in D$ such that $d_2 \circ d_2'$ and $d_2' \circ d_2 \in L$. Let $d_1 \in D$, $d_2 \in D/L$.

Here, $d_2 \circ d_2' \in E$, then $\kappa(d_2 \circ d_2') \geq \kappa(d_2) \wedge \kappa(d_2) = \kappa(d_2)$.

Therefore, $\kappa(l) \geq \kappa(d)$, where $l \in L$ and $d \in D/L$.

Let $d_2 = (d_2')'$ for all $d_2, d_2' \in D$ such that $d_2 \circ d_2'$ and $d_2' \circ d_2 \in L$.

Therefore, $(d_1 \circ d_2) = \kappa(d_1 \circ (d_2')') \geq \kappa(d_1) \wedge \kappa(d_2') \geq \kappa(d_1) \wedge \kappa(d_2) \forall d_1, d_2 \in D$.

Hence, the e-group $(D,\circ,L)$ has a fuzzy sub e-group (D, κ). □

Remark 10. The above theorem gives the necessary and sufficient condition for a FS of an e-group to be a fuzzy sub e-group.

Now, we will demonstrate that any FSG within a group D is also a fuzzy sub e-group of the e-group $(D,\circ,\{l\})$, where l is the group's identity element. But the converse needs not to be true.

Theorem 11. *Any fuzzy subgroup of a group is a fuzzy sub e-group.*

Proof. Let (D, κ) stand for a FSG of a group D.

Then, $\kappa(d_1 \circ d_2) \geq \kappa(d_1) \wedge \kappa(d_2)$ and $\kappa(d_1^{-1}) = \kappa(d_1)$ for all d_1 and $d_2 \in D$.

So, the first condition of fuzzy sub e-group is satisfied.

As D is a group, then there is a unique identity element l in D.

Since (D, κ) is a FSG of D, $\kappa(l) \geq \kappa(d) \forall d \in D$.

Now, we take $L = \{l\}$; then, $L \subseteq D$.

Therefore, $\kappa(l) \geq \kappa(d)$, where $l \in L$ and $d \in D/L$.

Hence, the FSG (D, κ) of D is a fuzzy sub e-group of the e-group $(D, \cdot, L)$. □

Example 12. Let $D = \{d_1, d_2, d_3, d_4\}$ and $L = \{d_1, d_2\}$.

Define $\circ$ on D as binary operation by the following:

$\circ$	d_1	d_2	d_3	d_3
d_1	d_1	d_1	d_1	d_1
d_2	d_1	d_2	d_3	d_4
d_3	d_1	d_3	d_1	d_1
d_4	d_1	d_4	d_3	d_4

(3)

Then, $(D,\circ,L)$ is an e-group.

Now, we assign a membership value to each of the elements of D by the following:

$$\begin{aligned} \kappa(d_1) &= 0.8, \\ \kappa(d_2) &= 0.9, \\ \kappa(d_3) &= 0.6, \\ \kappa(d_4) &= 0.7. \end{aligned} \tag{4}$$

Now, we can verify that (D, κ) is a fuzzy sub e-group of the e-group $(D,\circ,L)$.

But $|L| > 1$. So, $(D, \circ)$ is not a group. Hence, the FS (D, κ) is not a FSG.

Remark 13. A fuzzy sub e-group of an e-group is not necessarily a FSG.

Definition 14. A fuzzy sub e-group of an e-group which is not a FSG is said to be a proper fuzzy sub e-group.

The fuzzy sub e-group (D, κ) in Example 12. is a proper fuzzy sub e-group.

Now, we will check about union and intersection of fuzzy sub e-groups.

Theorem 15. *Intersection of fuzzy sub e-groups of an e-group is also a fuzzy sub e-group of that e-group.*

Proof. Let (D, κ_1) and (D, κ_2) be the two fuzzy sub e-groups of an e-group $(D,\circ,L)$.

Then, $\forall d_1, d_2 \in D$, $\kappa_1(d_1 \circ d_2) \geq \kappa_1(d_1) \wedge \kappa_1(d_2)$ and $\kappa_1(l) \geq \kappa_1(d)$, where $l \in L$ and $d \in D/L$.

Also, $\forall d_1, d_2 \in D$, $\kappa_2(d_1 \circ d_2) \geq \kappa_2(d_1) \wedge \kappa_2(d_2)$ and $\kappa_2(l) \geq \kappa_2(d)$, where $l \in L$ and $d \in D/L$.

Let (D, κ) be the intersection of (D, κ_1) and (D, κ_2), where $\kappa = \kappa_1 \cap \kappa_2$ is given by $\kappa(d) = \kappa_1(d) \wedge \kappa_2(d) \forall d \in D$.

Now for all $d_1, d_2 \in D$,

$$\begin{aligned}\kappa(d_1 \circ d_2) &= \kappa_1(d_1 \circ d_2) \wedge \kappa_2(d_1 \circ d_2) \\ &\geq (\kappa_1(d_1) \wedge \kappa_1(d_2)) \wedge (\kappa_2(d_1) \wedge \kappa_2(d_2)) \\ &= (\kappa_1(d_1) \wedge \kappa_2(d_1)) \wedge (\kappa_1(d_2) \wedge \kappa_2(d_2)) \\ &= \kappa(d_1) \wedge \kappa(d_2).\end{aligned} \tag{5}$$

Therefore, $\kappa(d_1 \circ d_2) \geq \kappa(d_1) \wedge \kappa(d_2)$ for all $d_1, d_2 \in D$.
Again for $l \in L$ and $d \in D/L$, we have the following:

$$\kappa(l) = \kappa_1(l) \wedge \kappa_2(l) \geq \kappa_1(d) \wedge \kappa_2(d) = \kappa(d). \tag{6}$$

Therefore, (D, κ) is a fuzzy sub e-group of the e-group $(D,\circ,L)$.

Hence, the intersection of two fuzzy sub e-groups of an e-group is also a fuzzy sub e-group of that e-group. □

Corollary 16. *Intersection of any fuzzy sub e-groups of an e-group $(D,\circ,L)$ is also a fuzzy sub e-group of that e-group $(D,\circ,L)$.*

Remark 17. Union of two fuzzy sub e-groups of an e-group may not be a fuzzy sub e-group of that e-group.

Example 18. Let us take the e-group $(D,\circ,L)$, where $D = \mathbb{Z}$, $L = 2\mathbb{Z}$, and $\circ$ is the addition of integers.

Let (D, κ_1) and (D, κ_2) be the two fuzzy sub e-groups of the e-group $(D,\circ,L)$, where κ_1 and κ_2 are presented by the following:

$$\kappa_1(d) = \begin{cases} 0.6, & \text{when } d \in 2\mathbb{Z}, \\ 0.3, & \text{when } d \in 5\mathbb{Z}/2\mathbb{Z}, \\ 0, & \text{elsewhere}, \end{cases}$$
$$\kappa_2(d) = \begin{cases} 0.8, & \text{when } d \in 2\mathbb{Z}, \\ 0.2, & \text{when } d \in 3\mathbb{Z}/2\mathbb{Z}, \\ 0, & \text{elsewhere}. \end{cases} \tag{7}$$

Let (D, κ) be the union of (D, κ_1) and (D, κ_2), where $\kappa = \kappa_1 \cup \kappa_2$ is given by $\kappa(d) = \kappa_1(d) \vee \kappa_2(d) \forall d \in D$.

Therefore,

$$\kappa(d) = \begin{cases} 0.8, & \text{when } d \in 2\mathbb{Z}, \\ 0.3, & \text{when } d \in 5\mathbb{Z}/2\mathbb{Z}, \\ 0.2, & \text{when } d \in 3\mathbb{Z}/(2\mathbb{Z} \cap 5\mathbb{Z}), \\ 0, & \text{elsewhere}. \end{cases} \tag{8}$$

Now, $\kappa(5 + (-4)) = \kappa(1) = 0$, but $\kappa(5) \wedge \kappa(-4) = \min\{0.3, 0.8\} = 0.3$.

So, $\kappa(5 + (-4)) \ngeq \kappa(5) \wedge \kappa(-4)$.

Hence, (D, κ) is not a fuzzy sub e-group of the e-group $(D,\circ,L)$.

Definition 19. Let (D, κ_1) and (D, κ_2) be the two fuzzy sub e-groups of an e-group $(D,\circ,L)$ such that $\kappa_2(l) \geq \kappa_1(l)$ for all $l \in L$ and $\kappa_2(d) \leq \kappa_1(d)$ for all $d \in D/L$, then (D, κ_2) is referred to be a super fuzzy sub e-group of (D, κ_1).

Example 20. In Example 12., we take another FS (D, κ_1) on $(D,\circ,L)$, where

$$\begin{aligned}\kappa_1(d_1) &= 0.85, \\ \kappa_1(d_2) &= 0.93, \\ \kappa_1(d_3) &= 0.57, \\ \kappa_1(d_4) &= 0.68.\end{aligned} \tag{9}$$

Then, we can simply verify that $(D,\circ,L)$ has a fuzzy sub e-group (D, κ_1).

Now, we can see that for all $l \in L$, $\kappa_1(l) \geq \kappa(l)$ and for all $d \in D/L$, $\kappa_1(d) \leq \kappa(d)$.

Hence, (D, κ_1) is a super fuzzy sub e-group of (D, κ).

Theorem 21. *Let (D, κ) stand for a fuzzy sub e-group of an e-group $(D,\circ,L)$. Then, the set $K = \{d \mid d \in D, \kappa(d) = p\}$ forms a sub e-group $(K,\circ,L)$ of the e-group $(D,\circ,L)$, where $p = \wedge\{\kappa(l) \mid l \in L\}$.*

Proof. Given $K = \{d \mid d \in D, \kappa(d) = p\}$, where $p = \wedge\{\kappa(l) | l \in L\}$.

To show that the e-group $(D,\circ,L)$ has a sub e-group $(K,\circ,L)$, we have to show that $(K,\circ,L)$ itself forms an e-group.

Since $(D,\circ,L)$ is an e-group, the associative law holds.

Clearly, K is a subset of D. Then, associative law also holds in K. Instead of showing the other two conditions of e-group, we will show that for all k_1 and $k_2 \in K$, $\exists$ a $k_2' \in K$ such that $k_1 \circ k_2'$ and $k_2' \circ k_1 \in L$.

Let k_1, k_2, and $k_2' \in K$. Then, $\kappa(k_1) = \kappa(k_2) = \kappa(k_2') = p$.

Since (D, κ) is a fuzzy sub e-group of $(D,\circ,L)$, by Theorem 9, we have the following:

$$\kappa\left(k_1 \circ k_2'\right) \geq \kappa(k_1) \wedge \kappa(k_2) = p. \tag{10}$$

Similarly, we can show that $\kappa(k_2' \circ k_1) \geq p$.

Since $p = \wedge\{\kappa(l) | l \in L\}$, $k_1 \circ k_2'$ and $k_2' \circ k_1 \in L$.

Hence, $(K,\circ,L)$ forms a sub e-group of $(D,\circ,L)$. □

4. Normal Fuzzy Sub e-Group and Level Fuzzy Sub e-Group

This section will describe fuzzy e-cosets and normal fuzzy sub e-groups. We will also introduce the concept of level fuzzy sub e-groups.

Definition 22. Let (D, κ) stand for a fuzzy sub e-group of an e-group $(D,\circ,L)$. Then, $\forall s, d \in D$, the left fuzzy e-coset $s\kappa = \kappa(l)_{\{s\}} \circ \kappa$ is defined by $s\kappa(d) = \kappa(s' \circ d)$ and the right fuzzy e-coset $\kappa s = \kappa \circ \kappa(l)_{\{s\}}$ is defined by $\kappa s(d) = \kappa(d \circ s')$, where l is any element of L and $s' \in D$ such that $s \circ s'$ and $s' \circ s \in L$.

If a left fuzzy e-coset is also a right fuzzy e-coset, then we will simply call it is a fuzzy e-coset.

Definition 23. Let (d, κ) stand for a fuzzy sub e-group of an e-group $(D,\circ,L)$. Then, (D, κ) forms a normal fuzzy sub e-group of the e-group $(D,\circ,L)$ if every left fuzzy e-coset of (D, κ) is a right fuzzy e-coset of (D, κ) in $(D,\circ,L)$.

Equivalently, $s\kappa = \kappa s$ for all $s \in D$.

Example 24. Let us take the e-group $(\mathbb{Z},+,2\mathbb{Z})$. Now, $(\mathbb{Z}, \kappa)$ forms a fuzzy sub e-group on $\mathbb{Z}$, where κ is presented by the following:

$$\kappa(z) = \begin{cases} 0.9, & \text{when } z \in 2\mathbb{Z}, \\ 0.6, & \text{elsewhere.} \end{cases} \tag{11}$$

Let us take $s = 3 \in \mathbb{Z}$.

Then, $\forall d \in \mathbb{Z}$, the left fuzzy e-coset (3κ) is presented by $(3\kappa)(d) = \kappa(3' + d) = \kappa(-3 + d)$ and the right fuzzy e-coset $(\kappa 3)$ is presented by $(\kappa 3)(d) = \kappa(d + 3') = \kappa(d - 3)$.

Since addition is commutative on $\mathbb{Z}$, then $(3\kappa) = (\kappa 3)$. Similarly, we can check for other elements of $\mathbb{Z}$. Hence, $(\mathbb{Z}, \kappa)$ forms a normal fuzzy sub e-group of $(\mathbb{Z},+,2\mathbb{Z})$.

In the next theorem, we represent the necessary and sufficient condition for a fuzzy sub e-group to be a normal fuzzy sub e-group.

Theorem 25. *Let (D, κ) stand for a fuzzy sub e-group of an e-group $(D,\circ,L)$. Then, (D, κ) forms a normal fuzzy sub e-group of the e-group $(D,\circ,L)$ iff $\kappa(d_1 \circ d_2) = \kappa(d_2 \circ d_1)\forall d_1, d_2 \in D$.*

Proof. Let (D, κ) stand for a normal fuzzy sub e-group of the e-group $(D,\circ,L)$.

Then, every left fuzzy e-coset of (D, κ) is also a right fuzzy e-coset of (D, κ) in $(D,\circ,L)$.

Therefore, $(t\kappa) = (\kappa t)$ for all $t \in D$. That is $(t\kappa)(d_2) = (\kappa t)(d_2)\forall d_2, t \in D$.

This suggests that $\kappa(t' \circ d_2) = \kappa(d_2 \circ t')\forall t, t', d_2 \in D$ such that $t \circ t', t' \circ t \in L$.

Now, we put $t' = d_1 \in D$. Therefore, $\kappa(d_1 \circ d_2) = \kappa(d_2 \circ d_1)\forall d_1, d_2 \in D$.

Conversely, let $\kappa(d_1 \circ d_2) = \kappa(d_2 \circ d_1)\forall d_1, d_2 \in D$.

Let $s \in D$. Since $(D,\circ,L)$ is an e-group, then $\exists$ a $s' \in D$ such that $s \circ s', s' \circ s \in L$.

Suppose $s' = d_1$. Then, $\kappa(s' \circ d_2) = \kappa(d_2 \circ s')\forall d_2, s' \in D$.

Therefore, $(s\kappa)(d_2) = (\kappa s)(d_2)\forall s, d_2 \in D$. Thus, $(s\kappa) = (\kappa s)\forall s \in D$.

Hence, (D, κ) forms a normal fuzzy sub e-group of $(D,\circ,L)$. □

Theorem 26. *Let (D, κ) stand for a fuzzy sub e-group of an e-group $(D,\circ,L)$. Then, the a-cut κ_a of (D, κ) forms a sub e-group $(\kappa_a,\circ,L)$ of the e-group $(D,\circ,L)$, where $a \leq \wedge\{\kappa(l) \mid l \in L\}$.*

Proof. We have $\kappa_a = \{d \mid d \in D, \kappa(d) \geq a\}$, where $a \in [0, 1]$ and $a \leq \wedge\{\kappa(l) \mid l \in L\}$.

Clearly, κ_a is nonempty as $L \subseteq \kappa_a$.

To show that $(\kappa_a,\circ,L)$ is an e-subgroup of $(D,\circ,L)$, we need to prove that for $d_1, d_2, d_2' \in \kappa_a$, $d_1 \circ d_2' \in \kappa_a$, where $d_2 \circ d_2', d_2' \circ d_2 \in L$.

Let $d_1, d_2, d_2' \in \kappa_a$ and $d_2 \circ d_2', d_2' \circ d_2 \in L$. Then, $\kappa(d_1) \geq a$ and $\kappa(d_2) \geq a$.

Since (D, κ) forms a fuzzy sub e-group of the e-group $(D,\circ,L)$,

$$\kappa\left(d_1 \circ d_2'\right) \geq \kappa(d_1) \wedge \kappa\left(d_2'\right) \geq \kappa(d_1) \wedge \kappa(d_2) \geq a \wedge a = a. \tag{12}$$

Therefore, $d_1 \circ d_2' \in \kappa_a$, where $d_2 \circ d_2', d_2' \circ d_2 \in L$. Hence, $(\kappa_a,\circ,L)$ forms a sub e-group of the e-group $(D,\circ,L)$. □

Definition 27. The sub e-group $(\kappa_a,\circ,L)$ of the e-group $(D,\circ,L)$ is referred to as a level fuzzy sub e-group of (D, κ).

Example 28. We consider the fuzzy sub e-group (D, κ) of the e-group $(D,\circ,L)$ in Example 12..

Choose $a = 0.65$. Then, $\kappa_a = \{d_1, d_2, d_4\}$ and $L = \{d_1, d_2\}$.

Clearly, $\kappa_a \subseteq D$.

We can easily check that $(\kappa_a,\circ,L)$ is a sub e-group of the e-group $(D,\circ,L)$.

Therefore, $(\kappa_a,\circ,L)$ is a level fuzzy sub e-group of (D, κ).

5. Homomorphism of Fuzzy Sub e-Groups

We shall demonstrate some important theorems on fuzzy sub e-group homomorphism in this section.

Theorem 29. *Let $(D_1, \circ_1, L_1)$ and $(D_2, \circ_2, L_2)$ be the two e-groups. Let h be a bijective homomorphism from $(D_1, \circ_1, L_1)$ to $(D_2, \circ_2, L_2)$ and (D_2, κ) be a fuzzy sub e-group of $(D_2, \circ_2, L_2)$. Then, $(D_1, h^{-1}(\kappa))$ forms a fuzzy sub e-group of $(D_1, \circ_1, L_1)$.*

Proof. Let d_1 and l_1 be the two elements of D_1. Now,

$$\begin{aligned} \left(h^{-1}(\kappa)\right)(d_1 \circ_1 l_1) &= \kappa(h(d_1 \circ_1 l_1)) = \kappa(h(d_1) \circ_2 h(l_1)) \\ &\geq \kappa(h(d_1)) \wedge \kappa(h(l_1)) \\ &= \left(h^{-1}(\kappa)\right)(d_1) \wedge \left(h^{-1}(\kappa)\right)(l_1). \end{aligned} \tag{13}$$

Therefore, $(h^{-1}(\kappa))(d_1 \circ_1 l_1) \geq (h^{-1}(\kappa))(d_1) \wedge (h^{-1}(\kappa))(l_1)$ for all d_1 and $l_1 \in D_1$.

Let $l \in L_1$ and $d \in D_1/L_1$.

Since h is a homomorphism, $h(l) \in L_2$ as $h(L_1) \subseteq L_2$. Now,

$$\left(h^{-1}(\kappa)\right)(l) = \kappa(h(l)) \geq \kappa(h(d)) = \left(h^{-1}(\kappa)\right)(d).. \tag{14}$$

□

Therefore, $(h^{-1}(\kappa))(l) \geq (h^{-1}(\kappa))(d)$ for all $l \in L_1$ and $d \in D_1/L_1$.

Hence, $(D_1, h^{-1}(\kappa))$ forms a fuzzy sub e-group of $(D_1, \circ_1, L_1)$.

Theorem 30. *Let $(D_1, \circ_1, L_1)$ and $(D_2, \circ_2, L_2)$ be the two e-groups. Let h be a homomorphism from $(D_1, \circ_1, L_1)$ to $(D_2, \circ_2, L_2)$ and (D_1, κ) be a fuzzy sub e-group of $(D_1, \circ_1, L_1)$. Then, $(D_2, h(\kappa))$ forms a fuzzy sub e-group of $(D_2, \circ_2, L_2)$.*

Proof. Let d_2 and l_2 be the two elements of D_2.

If either $d_2 \notin h(D_1)$ or $l_2 \notin h(D_1)$ then,

$$(h(\kappa))(d_2) \wedge (h(\kappa))(l_2) = 0 \leq (h(\kappa))(d_2 \circ_2 l_2). \tag{15}$$

Suppose $d_2 = h(d_1)$ and $l_2 = h(l_1)$ for some $d_1, l_1 \in D_1$. Now,

$$\begin{aligned}(h(\kappa))(d_2 \circ_2 l_2) &= \vee\{\kappa(p) | h(p) = d_2 \circ_2 l_2\} \\ &\geq \vee\{\kappa(d_1 \circ_1 l_1) | d_1, l_1 \in D_1, h(d_1) = d_2, h(l_1) = l_2\} \\ &\geq \vee\{\kappa(d_1) \wedge \kappa(l_1) | d_1, l_1 \in D_1, h(d_1) = d_2, h(l_1) = l_2\} \\ &= (\vee\{\kappa(d_1) | d_1 \in D_1, h(d_1) = d_2\}) \wedge \\ &\quad \cdot (\vee\{\kappa(l_1) | l_1 \in D_1, h(l_1) = l_2\}) \\ &= (h(\kappa))(d_2) \wedge (h(\kappa))(l_2).\end{aligned} \tag{16}$$

Therefore $(h(\kappa))(d_2 \circ_2 l_2) \geq (h(\kappa))(d_2) \wedge (h(\kappa))(l_2)$ for all d_2 and $l_2 \in D_2$.

Let $l_2 \in L_2$ and $d_2 \in D_2/L_2$.

Since h is a homomorphism, $h(L_1) \subseteq L_2$. Now,

$$\begin{aligned}(h(\kappa))(l_2) &= \vee\{\kappa(l_1) | l_1 \in D_1, h(l_1) = l_2\} \\ &\geq \vee\{\kappa(d_1) | d_1 \in D_1, h(d_1) = d_2\} = (h(\kappa))(d_2).\end{aligned} \tag{17}$$

Therefore, $(h(\kappa))(l_2) \geq (h(\kappa))(d_2)$ for all $l_2 \in L_2$ and $d_2 \in D_2/L_2$.

Hence, $(D_2, h(\kappa))$ forms a fuzzy sub e-group of $(D_2, \circ_2, L_2)$. □

Theorem 31. *Let $(D_1, \circ_1, L_1)$ and $(D_2, \circ_2, L_2)$ be the two e-groups. Let h be a bijective homomorphism from $(D_1, \circ_1, L_1)$ to $(D_2, \circ_2, L_2)$ and (D_2, κ) be a normal fuzzy sub e-group of $(D_2, \circ_2, L_2)$. Then, $(D_1, h^{-1}(\kappa))$ forms a normal fuzzy sub e-group of $(D_1, \circ_1, L_1)$.*

Proof. From Theorem 29, we can say that $(D_1, h^{-1}(\kappa))$ forms a fuzzy sub e-group of $(D_1, \circ_1, L_1)$.

Since (D_2, κ) forms a normal fuzzy sub e-group of $(D_2, \circ_2, L_2)$, $\kappa(d_2 \circ_2 l_2) = \kappa(l_2 \circ_2 d_2)$ for all $d_2, l_2 \in D_2$.

Let d_1 and l_1 be the two elements of D_1. Then,

$$\begin{aligned}\left(h^{-1}(\kappa)\right)(d_1 \circ_1 l_1) &= \kappa(h(d_1 \circ_1 l_1)) = \kappa(h(d_1) \circ_2 h(l_1)) \\ &= \kappa(h(l_1) \circ_2 h(d_1)) = \kappa(h(l_1 \circ_1 d_1)) \\ &= \left(h^{-1}(\kappa)\right)(l_1 \circ_1 d_1).\end{aligned} \tag{18}$$

Therefore, $(h^{-1}(\kappa))(d_1 \circ_1 l_1) = (h^{-1}(\kappa))(l_1 \circ_1 d_1)$ for all d_1 and $l_1 \in D_1$.

Hence, $(D_1, h^{-1}(\kappa))$ forms a normal fuzzy sub e-group of $(D_1, \circ_1, L_1)$. □

Theorem 32. *Let $(D_1, \circ_1, L_1)$ and $(D_2, \circ_2, L_2)$ be the two e-groups. Let h be a bijective homomorphism from $(D_1, \circ_1, L_1)$ to $(D_2, \circ_2, L_2)$ and (D_1, κ) be a normal fuzzy sub e-group of $(D_1, \circ_1, L_1)$. Then, $(D_2, h(\kappa))$ forms a normal fuzzy sub e-group of $(D_2, \circ_2, L_2)$.*

Proof. From Theorem 30, we can say that $(D_2, h(\kappa))$ is a fuzzy sub e-group of $(D_2, \circ_2, L_2)$.

Since (D_1, κ) forms a normal fuzzy sub e-group of $(D_1, \circ_1, L_1)$, $\kappa(d_1 \circ_1 l_1) = \kappa(l_1 \circ_1 d_1)$ for all $d_1, l_1 \in D_1$.

Let d_2 and l_2 be the two elements of D_2.

Suppose that there are unique d_1 and $l_1 \in D_1$, such that $d_2 = h(d_1)$ and $l_2 = h(l_1)$. Now,

$$\begin{aligned}(h(\kappa))(d_2 \circ_2 l_2) &= \vee\{\kappa(p) | h(p) = d_2 \circ_2 l_2\} \\ &= \vee\{\kappa(d_1 \circ_1 l_1) | d_1, l_1 \in D_1, h(d_1) = d_2, h(l_1) = l_2\} \vee \\ &\quad \cdot \{\kappa(l_1 \circ_1 d_1) | d_1, l_1 \in D_1, h(d_1) = d_2, h(l_1) = l_2\} \\ &= \vee\{\kappa(p) | h(p) = l_2 \circ_2 d_2\} = (h(\kappa))(l_2 \circ_2 d_2).\end{aligned} \tag{19}$$

Therefore, $(h(\kappa))(d_2 \circ_2 l_2) = (h(\kappa))(l_2 \circ_2 d_2) \forall d_2, l_2 \in D_2$.

Hence, $(D_2, h(\kappa))$ forms a normal fuzzy sub e-group of $(D_2, \circ_2, L_2)$. □

6. Conclusion

In this paper, we presented a brief demonstration of fuzzy sub e-groups and its properties. A condition is given for a FS of an e-group to be a fuzzy sub e-group. We have demonstrated that any fuzzy sub e-group forms a fuzzy subgroup. However, the reverse is not always true. Therefore, fuzzy sub e-group is the generalization of fuzzy subgroup. We have presented the difference between FSG and fuzzy sub e-group. We have discussed about the idea of normal fuzzy sub e-groups and level fuzzy e-subgroups. Finally, we have explained the effect of e-group homomorphism on fuzzy sub e-groups. In future, we will work on important theorems like Lagrange's theorem and Sylow theorem in fuzzy sub e-groups.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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