Hindawi
Journal of Function Spaces
Volume 2021, Article ID 5285962, 8 pages
https://doi.org/10.1155/2021/5285962

Research Article

Weighted Central BMO Spaces and Their Applications

Huan Zhao and Zongguang Liu

Department of Mathematics, China University of Mining and Technology, Beijing 100083, China

Correspondence should be addressed to Zongguang Liu; liuzg@cumtb.edu.cn

Received 3 June 2021; Accepted 16 July 2021; Published 4 August 2021

Academic Editor: Andrea Scapellato

In this paper, the central BMO spaces with Muckenhoupt A_p weight is introduced. As an application, we characterize these spaces by the boundedness of commutators of Hardy operator and its dual operator on weighted Lebesgue spaces. The boundedness of vector-valued commutators on weighted Herz spaces is also considered.

1. Introduction

For $1 < p < \infty$ and a nonnegative locally integrable function ω on $\mathbb{R}^n$, it is said that ω is in the Muckenhoupt A_p class if it satisfies the condition

$$[\omega]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{-1/(p-1)}dx\right)^{p-1} < \infty. \quad (1)$$

A weight function ω belongs to the class A_1 if

$$[\omega]_{A_1} := \sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)\left(\operatorname*{ess\,sup}_{x\in Q}\omega(x)^{-1}\right) < \infty. \quad (2)$$

A weight ω is called an A_∞ weight if

$$[\omega]_{A_\infty} := \sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)\exp\left(\frac{1}{|Q|}\int_Q \log \omega(x)^{-1}dx\right) < \infty. \quad (3)$$

It is well-known that $A_\infty = \bigcup_{1\le p<\infty} A_p$. Let $\omega \in A_\infty$ and $p \in (0,\infty)$; we denote $L^p(\omega)$ as the space of all measurable functions f such that

$$\|f\|_{L^p(\omega)} := \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx\right)^{1/p} < \infty. \quad (4)$$

The definition of A_p weight was introduced by Muckenhoupt [1]. Weighted inequalities arise naturally in Fourier analysis, but their use is best justified by the variety of applications in which they appear. For example, the theory of weights plays an important role in the study of boundary value problems for the Laplace equation on Lipschitz domains. Other applications of weighted inequalities include vector-valued inequalities, extrapolation of operators, and applications to certain classes of integral equations and nonlinear partial differential equations. There are a number of classical results which demonstrate that the Muckenhoupt A_p classes are the right collections of weights to do harmonic analysis on weighted spaces. The main results along these lines are the equivalence between the $\omega \in A_p$ condition and the $L^p(\omega)$ boundedness (or weak boundedness) of maximal operator and singular integral operators.

A well-known result of Muckenhoupt [1] showed that the Hardy-Littlewood maximal operator M, that is

$$Mf(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f(y)|dy, \quad (5)$$

is (weak) bounded on weighted Lebesgue spaces $L^p(\omega)$ if and only if $\omega \in A_p$ for $1 < p < \infty$ (for the case $n = 1$). Hunt et al. [2] proved that the A_p condition also characterizes the $L^p(\omega)$ boundedness of the Hilbert transform $\mathbf{H}$, where

$$\mathbf{H}f(x) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{f(y)}{x-y}dy. \quad (6)$$

Later, Coifman and Fefferman [3] extended the A_p theory to the case $n \geq 1$ and the general Calderón-Zygmund operators; they also proved that A_p weights satisfy the crucial reverse Hölder condition.

On the other hand, it is well-known that $\text{BMO}(\mathbb{R}^n)$ is just the dual space of Hardy space $H^1(\mathbb{R}^n)$. Like this, the dual space of Herz-type Hardy space is the so-called central BMO space which is defined by

$$\text{CBMO}^p(\mathbb{R}^n) = \left\{f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{\text{CBMO}^p(\mathbb{R}^n)} < \infty\right\}, \quad (7)$$

with

$$\|f\|_{\text{CBMO}^p(\mathbb{R}^n)} = \sup_{r>0}\left(\frac{1}{|B(0,r)|}\int_{B(0,r)}\left|f(x) - f_{B(0,r)}\right|^p dx\right)^{1/p}, \quad (8)$$

where

$$f_{B(0,r)} = \frac{1}{|B(0,r)|}\int_{B(0,r)} f(x)dx. \quad (9)$$

The space $\text{CBMO}^p(\mathbb{R}^n)$ can be regarded as a local version of $\text{BMO}(\mathbb{R}^n)$ at the origin, that is, $\text{BMO}(\mathbb{R}^n) \subsetneq \text{CBMO}^p(\mathbb{R}^n)$ for $1 \leq p < \infty$ (see [4]). However, they have quite different properties. For example, there is no analysis of the famous John-Nirenberg inequality of $\text{BMO}(\mathbb{R}^n)$ for $\text{CBMO}^p(\mathbb{R}^n)$. See also [5–9] and [10] for more details. In 2007, Fu et al. [11] characterized $\text{CBMO}^p(\mathbb{R}^n)$ space in terms of the boundedness of commutators of the Hardy operator.

In this paper, we will introduce the space of central BMO with Muckenhoupt A_p weight and characterize these spaces by the boundedness of commutator of the Hardy operator and its dual operator on weighted Lebesgue spaces. The boundedness of vector-valued commutators on weighted Herz spaces is also considered.

Throughout this paper, the letter C denotes constants which are independent of the main variables and may change from one occurrence to another. Denote $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$, and χ_k is the characteristic function for $k \in \mathbb{Z}$.

2. Weighted Central BMO Spaces

In this section, we will introduce the definition of weighted central BMO spaces and give some properties of $\text{CBMO}^p(\omega)$.

Let $1 \leq p < \infty$, and ω is a nonnegative locally integrable function. A function $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ is said to belong to the weighted central BMO spaces, if

$$\|f\|_{\text{CBMO}^p(\omega)} := \sup_{r>0}\frac{\left\|\left(f - f_{B(0,r)}\right)\chi_{B(0,r)}\right\|_{L^p(\omega)}}{\left\|\chi_{B(0,r)}\right\|_{L^p(\omega)}} < \infty. \quad (10)$$

When $\omega \equiv 1$ is a constant, $\text{CBMO}^p(\omega)$ is just $\text{CBMO}^p(\mathbb{R}^n)$.

We recall some properties of the weighted Lebesgue spaces. Let $\mathcal{Y}^n$ denote the set of all families of disjoint and open cubes in $\mathbb{R}^n$. In [12], Diening et al. obtained the following lemma in the general case on Musielak-Orilicz spaces. But we only describe the special case on the weighted Lebesgue spaces now.

Lemma 1. *If $\omega \in A_\infty$, then there exist $0 < \delta < 1$ and $C > 0$ which only depend on the A_∞-constant of ω such that*

$$\left\|\sum_{Q\in\mathcal{Q}} t_Q\left|\frac{f}{f_Q}\right|^\delta \chi_Q\right\|_{L^p(\omega)} \leq C\left\|\sum_{Q\in\mathcal{Q}} t_Q\chi_Q\right\|_{L^p(\omega)}, \quad (11)$$

for all $\mathcal{Q} \in \mathcal{Y}^n$; all $\{t_Q\}_{Q\in\mathcal{Q}}$, $t_Q \geq 0$; and all $f \in L^1_{loc}(\mathbb{R}^n)$ with $f_Q \neq 0$, $Q \in \mathcal{Q}$.

Lemma 2 (see [1]). *Let $\omega \in A_p$, $1 \leq p < \infty$; then, there exist constants $C_1, C_2, \delta > 0$ such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $E \subset B$,*

$$C_1\left(\frac{|E|}{|B|}\right)^p \leq \frac{\omega(E)}{\omega(B)} \leq C_2\left(\frac{|E|}{|B|}\right)^\delta. \quad (12)$$

In fact, the first inequality of Lemma 2 can be improved as follows.

Lemma 3. *Let $\omega \in A_p$, $1 < p < \infty$. Then, there exist p_0 with $1 < p_0 < p$ and $C > 0$ such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $E \subset B$,*

$$\frac{\|\chi_B\|_{L^p(\omega)}}{\|\chi_E\|_{L^p(\omega)}} \leq C\left(\frac{|B|}{|E|}\right)^{1/p_0}. \quad (13)$$

Proof. By the fact that $A_p = \cup_{q<p} A_q$ and $\omega \in A_p$ (see [13]), there exist $1 < p_0 < p$ such that $\omega \in A_{p/p_0}$. Applying Lemma 2, there exists a constant $C > 0$ such that for any ball B and any measurable set $E \subset B$,

$$\left(\frac{|E|}{|B|}\right)^{p/p_0} \leq C\frac{\omega(E)}{\omega(B)}. \quad (14)$$

That is,

$$\frac{\|\chi_B\|_{L^p(\omega)}}{\|\chi_E\|_{L^p(\omega)}} \le C\left(\frac{|B|}{|E|}\right)^{1/p_0}. \tag{15}$$

Therefore we have proved Lemma 3. □

Now, we show the relationship between $\mathrm{CBMO}^p(\omega)$ and central BMO spaces.

Proposition 4. *If $\omega \in A_p$ and $1 < p < q < \infty$, then $CBMO^q(\omega) \subset CBMO^p(\omega) \subsetneq CBMO(\mathbb{R}^n)$.*

Proof. Let $f \in \mathrm{CBMO}^q(\omega)$. For any $B := B(0,r)$, by Hölder's inequality, we have

$$\begin{aligned}\left(\frac{1}{\omega(B)}\int_B |f(x)-f_B|^p\omega(x)dx\right)^{1/p} &\le \left(\frac{1}{\omega(B)}\int_B |f(x)-f_B|^q\omega(x)dx\right)^{1/q}\\ &\le C\|f\|_{\mathrm{CBMO}^q(\omega)}.\end{aligned} \tag{16}$$

On the other hand, Let $f \in \mathrm{CBMO}^p(\omega)$, For any $B := B(0, r)$, by Hölder's inequality and the condition $\omega \in A_p$, we have

$$\begin{aligned}\frac{1}{|B|}\int_B |f(x)-f_B|dx &\le \left(\frac{1}{|B|}\int_B |f(x)-f_B|^p\omega(x)dx\right)^{1/p}\left(\frac{1}{|B|}\int_B \omega(x)^{-p'/p}dx\right)^{1/p'}\\ &\le C\|f\|_{\mathrm{CBMO}^p(\omega)}\left(\frac{\omega(B)}{|B|}\right)^{1/p}\left(\frac{1}{|B|}\int_B \omega(x)dx\right)^{-1/p}\\ &\le C\|f\|_{\mathrm{CBMO}^p(\omega)}.\end{aligned} \tag{17}$$

Therefore, we only need to prove that there exists a function f such that $f \in \mathrm{CBMO}(\mathbb{R}^n) \setminus \mathrm{CBMO}^p(\omega)$. Without loss of generality, we may assume that $n = 1$.

Let $A_k = \{x \in \mathbb{R} : 2^k < |x| \le 2^k + 1\}, k \in \mathbb{Z}_+$. Taking $f(x) = \sum_{k=0}^{\infty} 2^k \chi_{A_k}(x)\ \mathrm{sgn}\ (x)$, then for any $B := B(0, r)$,

$$f_B = \frac{1}{|B|}\int_B f(x)dx = 0. \tag{18}$$

When $r \le 1$, we have $f(x) \equiv 0$ and

$$\sup_{0<r\le 1}\frac{1}{|B|}\int_B |f(x)-f_B|dx = 0. \tag{19}$$

When $r > 1$, there exists $k_0 \in \mathbb{Z}_+$ such that $2^{k_0} < r \le 2^{k_0+1}$; then,

$$\sup_{r>1}\frac{1}{|B|}\int_B |f(x)-f_B|dx \le \sup_{r>1} C2^{-k_0}\sum_{k=0}^{k_0+1}\int_{A_k} 2^k dx \le C. \tag{20}$$

From (19) and (20), it follows that $f \in \mathrm{CBMO}(\mathbb{R}^n)$.

When $r > 4$, there exists $k_0 \in \mathbb{Z}_+$ with $k_0 \ge 2$ such that $2^{k_0} < r \le 2^{k_0+1}$; then,

$$|(f(x)-f_B)\chi_B(x)| \ge \sum_{k=0}^{k_0-1} 2^k\chi_{A_k}(x) \ge 2^{k_0-1}\chi_{A_{k_0-1}}(x) \ge Cr\chi_{A_{k_0-1}}(x), \tag{21}$$

which implies that

$$\sup_{r>0}\frac{\|(f-f_B)\chi_B\|_{L^p(\omega)}}{\|\chi_B\|_{L^p(\omega)}} \ge \sup_{r>4}\frac{\|(f-f_B)\chi_B\|_{L^p(\omega)}}{\|\chi_B\|_{L^p(\omega)}} \ge \sup_{r>4} Cr\frac{\left\|\chi_{A_{k_0-1}}\right\|_{L^p(\omega)}}{\|\chi_B\|_{L^p(\omega)}}. \tag{22}$$

Since $A_{k_0-1} \subset B$, by Lemma 3, there exists p_0 with $1 < p_0 < p$; we have

$$\sup_{r>0}\frac{\|(f-f_B)\chi_B\|_{L^p(\omega)}}{\|\chi_B\|_{L^p(\omega)}} \ge \sup_{r>4} Cr\left(\frac{|B|}{|A_{k_0-1}|}\right)^{-1/p_0} = \sup_{r>4} Cr^{1-1/p_0} = \infty. \tag{23}$$

Therefore, $f \notin \mathrm{CBMO}^p(\omega)$. □

Proposition 5. *If $\omega \in A_p$ and $1 < p < \infty$, then there exists a constant $q > p$ such that $CBMO^q(\mathbb{R}^n) \subset CBMO^p(\omega)$.*

Proof. We can take a cube Q_B so that $B \subset Q_B \subset \sqrt{n}B$. By Lemma 1, there exists a constant $0 < \delta < 1$ independent of B such that for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\left\||f|^\delta\chi_{Q_B}\right\|_{L^p(\omega)} \le C\left(|f|_{Q_B}\right)^\delta\left\|\chi_{Q_B}\right\|_{L^p(\omega)}. \tag{24}$$

Let $q = 1/\delta$ and $f = (b - b_B)^q\chi_B$. We conclude that

$$\left(|f|_{Q_B}\right)^\delta = \left(\frac{1}{|Q_B|}\int_B |b(x)-b_B|^q dx\right)^{1/q} \le C\|b\|_{\mathrm{CBMO}^q(\mathbb{R}^n)}. \tag{25}$$

By Lemma 3, there exists a constant $1 < p_0 < p$ such that

$$\begin{aligned}\left\|\chi_{Q_B}\right\|_{L^p(\omega)} &\le \frac{\left\|\chi_{\sqrt{n}B}\right\|_{L^p(\omega)}}{\|\chi_B\|_{L^p(\omega)}}\|\chi_B\|_{L^p(\omega)} \le C\left(\frac{|\sqrt{n}B|}{|B|}\right)^{1/p_0}\|\chi_B\|_{L^p(\omega)}\\ &\le C\|\chi_B\|_{L^p(\omega)}.\end{aligned} \tag{26}$$

This gives us

$$\|b\|_{\mathrm{CBMO}^p(\omega)} \le C\|b\|_{\mathrm{CBMO}^q(\mathbb{R}^n)}. \tag{27}$$

Hence, the proof of Proposition 5 is completed. □

Proposition 6. *If $\omega \in A_p$ and $1 < p < \infty$, then $f \in CBMO^p(\omega)$ if and only if there exist a collection of numbers $\{c_{B(0,r)}\}_{r>0}$*

(i.e., for each ball $B(0,r)$, there exists $c_{B(0,r)} \in \mathbb{R}$) such that

$$\sup_{r>0} \left\| \chi_{B(0,r)} \right\|_{L^p(\omega)}^{-1} \left\| \left(f - c_{B(0,r)} \right) \chi_{B(0,r)} \right\|_{L^p(\omega)} < \infty. \tag{28}$$

Proof. We set $c_{B(0,r)} = f_{B(0,r)}$ for all balls $B(0,r)$; the necessity of the condition in Proposition 6 holds. Let us check the sufficiency of Proposition 6.

A similar argument as Proposition 4, we have, for any $B := B(0,r)$,

$$\frac{1}{|B|} \int_B |f(x) - c_B| dx \le C \|\chi_B\|_{L^p(\omega)}^{-1} \|(f - c_B)\chi_B\|_{L^p(\omega)}. \tag{29}$$

Thus,

$$\begin{aligned} \frac{\|(f - f_B)\chi_B\|_{L^p(\omega)}}{\|\chi_B\|_{L^p(\omega)}} &\le \frac{\|(f - c_B)\chi_B\|_{L^p(\omega)}}{\|\chi_B\|_{L^p(\omega)}} + \frac{\|(c_B - f_B)\chi_B\|_{L^p(\omega)}}{\|\chi_B\|_{L^p(\omega)}} \\ &\le C + |c_B - f_B| \le C. \end{aligned} \tag{30}$$

Therefore, $f \in \mathrm{CBMO}^p(\omega)$; the proof of Proposition 6 is completed. □

Proposition 7. *If $\omega \in A_p$ and $1 < p < \infty$, then $f \in CBMO^p(\omega)$ if and only if*

$$\|f\|_{CBMO_*^p(\omega)} := \sup_{r>0} \inf_{c\in\mathbb{C}} \left\| \chi_{B(0,r)} \right\|_{L^p(\omega)}^{-1} \left\| (f - c)\chi_{B(0,r)} \right\|_{L^p(\omega)} < \infty. \tag{31}$$

Proof. The proof of Proposition 7 is similar as that of Proposition 6; we omit the details. □

3. Characterization of $\mathrm{CBMO}^p(\omega)$ Spaces via Commutators

We first review the definitions of the n-dimensional Hardy operator and its dual operator. For a locally integrable function f in $\mathbb{R}^n$, the n-dimensional Hardy operator H is defined by

$$Hf(x) = \frac{1}{|x|^n} \int_{|y|\le|x|} f(y) dy, \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{32}$$

The dual Hardy operator H^* is defined by

$$H^* f(x) = \int_{|y|>|x|} \frac{f(y)}{|y|^n} dy, \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{33}$$

Let b be a locally integrable function on $\mathbb{R}^n$. The commutators of H and H^* are defined by

$$[b, H](f) = b(Hf) - H(bf) \tag{34}$$

and

$$[b, H^*](f) = b(H^* f) - H^*(bf). \tag{35}$$

The study of the Hardy operator has a very long history, and a number of papers involved its generalizations, variants, and applications. For the earlier development of this kind of integrals and many important applications, we refer the interested reader to the masterpiece [14]. We are interested in the characterization of commutator of the Hardy operator.

Now, we give a remarkable result about the commutator of the Hardy operator; that is, Fu et al. [11] showed the following.

Theorem 8. *Let $1 < p < \infty$, $1/p + 1/p' = 1$, and $b \in CBMO^{\max(p,p')}(\mathbb{R}^n)$. Then, both $[b, H]$ and $[b, H^*]$ are bounded operators on $L^p(\mathbb{R}^n)$. Conversely,*

(i) if $[b, H]$ is a bounded operator on $L^p(\mathbb{R}^n)$, then $b \in CBMO^{p'}(\mathbb{R}^n)$

(ii) if $[b, H^*]$ is a bounded operator on $L^p(\mathbb{R}^n)$, then $b \in CBMO^p(\mathbb{R}^n)$

The following consequence improves Theorem 8.

Theorem 9. *If $\omega \in A_p$, $1 < p < \infty$ and $\mu = \omega^{1-p'}$. Then, the following statements are equivalent:*

(i) $b \in CBMO^p(\omega) \cap CBMO^{p'}(\mu)$

(ii) $[b, H]$ and $[b, H^*]$ are bounded from $L^p(\omega)$ to $L^p(\omega)$

Proof. (i) ⇒ (ii). We focus on the proof of the boundedness of $[b, H]$, since the arguments of $[b, H^*]$ are similar with necessary modifications.

For $f \in L^p(\omega)$, we have

$$\begin{aligned} \|[b, H](f)\|_{L^p(\omega)} &= \sum_{k=-\infty}^{\infty} \|\chi_k [b, H](f)\|_{L^p(\omega)} \\ &= \sum_{k=-\infty}^{\infty} \left\| \frac{\chi_k(\cdot)}{|\cdot|^n} \int_{B(0,|\cdot|)} (b(\cdot) - b(y)) f(y) dy \right\|_{L^p(\omega)} \\ &\le \sum_{k=-\infty}^{\infty} \left\| \chi_k(\cdot) \sum_{j=-\infty}^{k} \frac{1}{|\cdot|^n} \int_{C_j} |b(\cdot) - b(y)| |f(y)| dy \right\|_{L^p(\omega)}. \end{aligned} \tag{36}$$

It is easy to see that

$$\int_{C_j} |b(x) - b(y)| |f(y)| dy \le \int_{C_j} |b(x) - b_{B_k}| |f(y)| dy + \int_{C_j} |b_{B_k} - b(y)| |f(y)| dy. \tag{37}$$

By Hölder's inequality, we get

$$\begin{aligned}&\int_{C_j}|b(x)-b_{B_k}||f(y)|dy\\&\le C|b(x)-b_{B_k}|\left(\left(\int_{C_j}|f(y)|^p\omega(y)dy\right)^{1/p}\left(\int_{C_j}\omega(y)^{-p'/p}dy\right)^{1/p'}\right.\\&\le C|b(x)-b_{B_k}|\left\|f\chi_j\right\|_{L^p(\omega)}\left\|\chi_j\right\|_{L^{p'}(\mu)}.\end{aligned}\tag{38}$$

In [11], Fu et al. showed that for $b\in \mathrm{CBMO}(\mathbb{R}^n)$ and j, $k\in\mathbb{Z}$,

$$\left|b(t)-b_{B_k}\right|\le\left|b(t)-b_{B_j}\right|+C|j-k|\|b\|_{\mathrm{CBMO}(\mathbb{R}^n)}.\tag{39}$$

By Proposition 4, for $b\in\mathrm{CBMO}^p(\omega)\subset\mathrm{CBMO}(\mathbb{R}^n)$ and $j,k\in\mathbb{Z}$, we have

$$\left|b(t)-b_{B_k}\right|\le\left|b(t)-b_{B_j}\right|+C|j-k|\|b\|_{\mathrm{CBMO}^p(\omega)}.\tag{40}$$

This gives us

$$\begin{aligned}&\int_{C_j}|b(y)-b_{B_k}||f(y)|dy\\&\le\int_{C_j}\left|b(y)-b_{B_j}\right||f(y)|dy+\left|b_{B_k}-b_{B_j}\right|\int_{C_j}|f(y)|dy\\&\le\int_{C_j}\left|b(y)-b_{B_j}\right||f(y)|\omega(y)^{\frac1p}\omega(y)^{-\frac1p}dy\\&\quad+C(k-j)\|b\|_{\mathrm{CBMO}^p(\omega)}\int_{C_j}|f(y)|\omega(y)^{\frac1p}\omega(y)^{-\frac1p}dy\\&\le C\left\|\left(b-b_{B_j}\right)\chi_j\right\|_{L^{p'}(\mu)}\left\|f\chi_j\right\|_{L^p(\omega)}\\&\quad+C(k-j)\|b\|_{\mathrm{CBMO}^p(\omega)}\left\|f\chi_j\right\|_{L^p(\omega)}\left\|\chi_j\right\|_{L^{p'}(\mu)}\\&\le C\|b\|_{\mathrm{CBMO}^{p'}(\mu)}\left\|f\chi_j\right\|_{L^p(\omega)}\left\|\chi_{B_j}\right\|_{L^{p'}(\mu)}\\&\quad+C(k-j)\|b\|_{\mathrm{CBMO}^p(\omega)}\left\|f\chi_j\right\|_{L^p(\omega)}\left\|\chi_j\right\|_{L^{p'}(\mu)}\\&\le C(k-j)\left\|f\chi_j\right\|_{L^p(\omega)}\left\|\chi_{B_j}\right\|_{L^{p'}(\mu)}.\end{aligned}\tag{41}$$

Combing (38) and (41), we get

$$\begin{aligned}&\left\|\chi_k(\cdot)\frac{1}{|\cdot|^n}\int_{C_j}(b(\cdot)-b(y))\,|\,f(y)\,|\,dy\right\|_{L^p(\omega)}\\&\le C2^{-kn}\left\|\left(b-b_{B_k}\right)\chi_k\right\|_{L^p(\omega)}\left\|f\chi_j\right\|_{L^p(\omega)}\left\|\chi_j\right\|_{L^{p'}(\mu)}\\&\quad+C2^{-kn}(k-j)\|\chi_k\|_{L^p(\omega)}\left\|f\chi_j\right\|_{L^p(\omega)}\left\|\chi_{B_j}\right\|_{L^{p'}(\mu)}\\&\le C2^{-kn}(k-j)\left\|\chi_{B_k}\right\|_{L^p(\omega)}\left\|f\chi_j\right\|_{L^p(\omega)}\left\|\chi_{B_j}\right\|_{L^{p'}(\mu)}.\end{aligned}\tag{42}$$

From the condition $\omega\in A_p$ and Lemma 2, it follows that for $k\ge j$, there exists a constant $\delta\in(0,1)$ such that

$$2^{-kn}\left\|\chi_{B_k}\right\|_{L^p(\omega)}\left\|\chi_{B_j}\right\|_{L^{p'}(\mu)}\le C\frac{\left\|\chi_{B_j}\right\|_{L^{p'}(\mu)}}{\left\|\chi_{B_k}\right\|_{L^{p'}(\mu)}}\le C\left(\frac{|B_j|}{|B_k|}\right)^{\delta/p'}\le C2^{(j-k)n\delta/p'}.\tag{43}$$

Therefore, generalized Minkowski's inequality implies

$$\begin{aligned}\|[b,H](f)\|_{L^p(\omega)}&\le C\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k}(k-j)2^{(j-k)n\delta/p'}\left\|f\chi_j\right\|_{L^p(\omega)}\\&\le C\sum_{j=-\infty}^{\infty}\sum_{k=j}^{\infty}(k-j)2^{(j-k)n\delta/p'}\left\|f\chi_j\right\|_{L^p(\omega)}\\&\le C\|f\|_{L^p(\omega)}.\end{aligned}\tag{44}$$

(ii) ⇒ (i). The condition $b\in\mathrm{CBMO}^p(\omega)\cap\mathrm{CBMO}^{p'}(\mu)$ turns out to be necessary for the conclusion that both $[b,H]$ and $[b,H^*]$ are bounded on $L^p(\omega)$.

For any ball $B:=B(0,r)$ and $x\in B$, we have

$$\begin{aligned}|b(x)-b_B|&=\left|\frac{1}{|B|}\int_B(b(x)-b(y))dy\right|\\&\le C\left|\frac{1}{|x|^n}\int_{|y|\le|x|}(b(x)-b(y))\chi_B(y)dy\right|\\&\quad+C\left|\int_{|x|\le|y|}\frac{(b(x)-b(y))\chi_B(y)|y|^n|B|^{-1}}{|y|^n}dy\right|\\&\le C|[b,H](\chi_B)(x)|+C|[b,H^*](f_0)(x)|,\end{aligned}\tag{45}$$

where $f_0=|x|^n|B|^{-1}\chi_B(x)$.

From $[b,H]$ and $[b,H^*]$ that are bounded on $L^p(\omega)$, it follows that

$$\begin{aligned}\|(b-b_B)\chi_B\|_{L^p(\omega)}&\le C\|[b,H](\chi_B)\|_{L^p(\omega)}+C\|[b,H^*](f_0)\|_{L^p(\omega)}\\&\le C\|\chi_B\|_{L^p(\omega)}+C\|f_0\|_{L^p(\omega)}\le C\|\chi_B\|_{L^p(\omega)}.\end{aligned}\tag{46}$$

Therefore, we obtain that b belongs to $\mathrm{CBMO}^p(\omega)$.

Note that $(L^p(\omega))'=L^{p'}(\omega^{1-p'})$ (see [15]). We know that $[b,H]$ and $[b,H^*]$ are bounded on $L^{p'}(\mu)$. Therefore, we obtain that $b\in\mathrm{CBMO}^{p'}(\mu)$.

This completes the proof of Theorem 9. □

4. Vector-Valued Inequality

In this section, we give the definition of weighted Herz spaces ([16]). Let $\alpha\in\mathbb{R}$, $0<p,q<\infty$, and ω be weight functions on $\mathbb{R}^n$. The homogeneous weighted Herz space $\dot{K}_p^{\alpha,q}(\omega)$ is defined by

$$\dot{K}_p^{\alpha,q}(\omega) := \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_p^{\alpha,q}(\omega)} < \infty \right\}, \tag{47}$$

where

$$\|f\|_{\dot{K}_p^{\alpha,q}(\omega)} := \left\| \left\{ 2^{\alpha k} \|f\chi_k\|_{L^p(\omega)} \right\}_{k=-\infty}^{\infty} \right\|_{\ell^q}. \tag{48}$$

We prove the boundedness of the vector-valued commutator of the Hardy operator on weighted Herz spaces.

Theorem 10. *Let $\omega \in A_p$, $1 < r, p < \infty$, $0 < q < \infty$, and $b \in CBMO^p(\omega) \cap CBMO^{p'}(\mu)$.*

(i) If $\alpha < n/p'$, then there exists a constant C such that

$$\left\| \left(\sum_{j=1}^{\infty} \left| [b, H]\left(f_j\right) \right|^r \right)^{1/r} \right\|_{\dot{K}_p^{\alpha,q}(\omega)} \le C \left\| \left(\sum_{j=1}^{\infty} \left| f_j \right|^r \right)^{1/r} \right\|_{\dot{K}_p^{\alpha,q}(\omega)}, \tag{49}$$

for all sequences of functions $\{f_j\}_{j=1}^{\infty}$ satisfying $\| \|\{f_j\}_j\|_{\ell^r} \|_{\dot{K}_p^{\alpha,q}(\omega)} < \infty$

(ii) If $\alpha > -n/p$, then there exists a constant C such that

$$\left\| \left(\sum_{j=1}^{\infty} \left| [b, H^*]\left(f_j\right) \right|^r \right)^{1/r} \right\|_{\dot{K}_p^{\alpha,q}(\omega)} \le C \left\| \left(\sum_{j=1}^{\infty} \left| f_j \right|^r \right)^{\frac{1}{r}} \right\|_{\dot{K}_p^{\alpha,q}(\omega)}, \tag{50}$$

for all sequences of functions $\{f_j\}_{j=1}^{\infty}$ satisfying $\| \|\{f_j\}_j\|_{\ell^r} \|_{\dot{K}_p^{\alpha,q}(\omega)} < \infty$

In order to prove Theorem 10, we additionally introduce the next lemma well-known as the generalized Minkowski inequality.

Lemma 11. *If $1 < r < \infty$, then there exists a constant $C > 0$ such that for all sequences of functions $\{f_j\}_{j=1}^{\infty}$ satisfying $\| \|\{f_j\}_j\|_{\ell^r} \|_{\dot{K}_p^{\alpha,q}(\omega)} < \infty$,*

$$\left\{ \sum_{j=1}^{\infty} \left(\int_{\mathbb{R}^n} \left| f_j(y) \right| dy \right)^r \right\}^{1/r} \le C \int_{\mathbb{R}^n} \left\{ \sum_{j=1}^{\infty} \left| f_j(y) \right|^r \right\}^{1/r} dy. \tag{51}$$

Proof of Theorem 10. We focus on the proof of the boundedness of $[b, H]$, since the arguments of $[b, H^*]$ are similar with necessary modifications. For every $\{f_j\}_{j=1}^{\infty}$ with $\| \|\{f_j\}_j\|_{\ell^r} \|_{\dot{K}_p^{\alpha,q}(\omega)} < \infty$, we obtain

$$\begin{aligned} &\left\| \left\| \left\{ [b, H]\left(f_j\right) \right\}_j \right\|_{\ell^r} \right\|_{\dot{K}_p^{\alpha,q}(\omega)} \\ &= \left\{ \sum_{k=-\infty}^{\infty} 2^{\alpha q k} \left\| \chi_k \left\| \left\{ [b, H]\left(\sum_{l=-\infty}^{\infty} f_j \chi_l \right) \right\}_j \right\|_{\ell^r} \right\|_{L^p(\omega)}^q \right\}^{1/q} \\ &\le \left\{ \sum_{k=-\infty}^{\infty} 2^{\alpha q k} \left\| \chi_k \sum_{l=-\infty}^{k} \left\| \left\{ [b, H]\left(f_j \chi_l \right) \right\}_j \right\|_{\ell^r} \right\|_{L^p(\omega)}^q \right\}^{1/q}. \end{aligned} \tag{52}$$

For convenience, below, we denote $F := \|\{f_j\}_j\|_{\ell^r}$. For $x \in C_k$, generalized Hölder's inequality and generalized Minkowski's inequality (51) imply

$$\begin{aligned} \left\| \left\{ [b, H]\left(f_j \chi_l\right)(x) \right\}_j \right\|_{\ell^r} &\le C \left\| \left\{ \frac{1}{|x|^n} \int_{C_l} |b(x) - b(y)| \left| f_j(y) \right| dy \right\}_j \right\|_{\ell^r} \\ &\le C 2^{-kn} \left\| \left\{ \int_{C_l} |b(x) - b(y)| \left| f_j(y) \right| dy \right\}_j \right\|_{\ell^r} \\ &\le C 2^{-kn} \int_{C_l} |b(x) - b(y)| F(y) dy \\ &\le C 2^{-kn} \left\{ \left| b(x) - b_{B_l} \right| \int_{C_l} F(y) dy + \int_{C_l} \left| b_{B_l} \right. \right. \\ &\quad \left. \left. - b(y) \right| F(y) dy \right\} \\ &\le C 2^{-kn} \|F\chi_l\|_{L^p(\omega)} \left\{ \left| b(x) - b_{B_l} \right| \|\chi_l\|_{L^{p'}(\mu)} \right. \\ &\quad \left. + \left\| \left| b - b_{B_l} \right| \chi_l \right\|_{L^{p'}(\mu)} \right\}, \end{aligned} \tag{53}$$

from the fact that

$$\left| b(x) - b_{B_l} \right| \le \left| b(t) - b_{B_k} \right| + C|l - k| \|b\|_{\mathrm{CBMO}^p(\omega)}, \tag{54}$$

which gives us

$$\begin{aligned} &\left\| \chi_k \left\| \left\{ [b, H]\left(f_j \chi_l\right) \right\}_j \right\|_{\ell^r} \right\|_{L^p(\omega)} \\ &\le C 2^{-kn} \|F\chi_l\|_{L^p(\omega)} \times \left\{ (k - l) \|b\|_{CBMO^p(\omega)} \|\chi_k\|_{L^p(\omega)} \|\chi_l\|_{L^{p'}(\mu)} \right. \\ &\quad \left. + \|b\|_{\mathrm{CBMO}^{p'}(\mu)} \left\| \chi_{B_l} \right\|_{L^{p'}(\mu)} \|\chi_k\|_{L^p(\omega)} \right\} \\ &\le C(k - l) \|F\chi_l\|_{L^p(\omega)} 2^{-kn} \left\| \chi_{B_l} \right\|_{L^{p'}(\mu)} \|\chi_k\|_{L^p(\omega)} \\ &\le C(k - l) \|F\chi_l\|_{L^p(\omega)} 2^{-kn} \|\chi_k\|_{L^p(\omega)} \left(|B_l| \left\| \chi_{B_l} \right\|_{L^p(\omega)}^{-1} \right). \end{aligned} \tag{55}$$

By Lemma 3 and $\alpha/n < 1/p'$, there exists a constant $1 < p_0 < p$ such that $\alpha/n < 1/p'_0$ and

$$\frac{\|\chi_k\|_{L^p(\omega)}}{\left\|\chi_{B_l}\right\|_{L^p(\omega)}} \le \frac{\left\|\chi_{B_k}\right\|_{L^p(\omega)}}{\left\|\chi_{B_l}\right\|_{L^p(\omega)}} \le C2^{n(k-l)/p_0}. \tag{56}$$

Then,

$$\left\|\chi_k\left\|\left\{[b,H]\left(f_j\chi_l\right)\right\}_j\right\|_{\ell^r}\right\|_{L^p(\omega)} \le C(k-l)2^{n(l-k)/p'_0}\|F\chi_l\|_{L^p(\omega)}. \tag{57}$$

This implies that

$$\begin{aligned}
&\left\|\left\|\left\{[b,H]\left(f_j\right)\right\}_j\right\|_{\ell^r}\right\|_{\dot K_p^{\alpha,q}(\omega)} \\
&\le C\left\{\sum_{k=-\infty}^{\infty}2^{\alpha qk}\left(\sum_{l=-\infty}^{k}(k-l)2^{n(l-k)/p_0'}\|F\chi_l\|_{L^p(\omega)}\right)^q\right\}^{1/q} \\
&= C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{k}2^{\alpha l}(k-l)2^{(l-k)\left(\left(n/p_0'\right)-\alpha\right)}\|F\chi_l\|_{L^p(\omega)}\right)^q\right\}^{1/q}.
\end{aligned} \tag{58}$$

If $0 < q \le 1$, then we obtain

$$\begin{aligned}
&\left\|\left\|\left\{[b,H]\left(f_j\right)\right\}_j\right\|_{\ell^r}\right\|_{\dot K_p^{\alpha,q}(\omega)} \\
&\le C\left\{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{k}2^{\alpha ql}\|F\chi_l\|_{L^p(\omega)}^q2^{(l-k)\left(\left(n/p_0'\right)-\alpha\right)q}(k-l)^q\right\}^{1/q} \\
&\le C\left\{\sum_{l=-\infty}^{\infty}2^{\alpha ql}\|F\chi_l\|_{L^p(\omega)}^q\sum_{k=l}^{\infty}2^{(l-k)\left(\left(n/p_0'\right)-\alpha\right)q}(k-l)^q\right\}^{1/q} \\
&\le C\left(\sum_{l=-\infty}^{\infty}2^{\alpha ql}\|F\chi_l\|_{L^p(\omega)}^q\right)^{1/q} \le C\|F\|_{\dot K_p^{\alpha,q}(\omega)}.
\end{aligned} \tag{59}$$

If $1 < q < \infty$, then we use Hölder's inequality and obtain

$$\begin{aligned}
&\left\|\left\|\left\{[b,H]\left(f_j\right)\right\}_j\right\|_{\ell^r}\right\|_{\dot K_p^{\alpha,q}(\omega)} \\
&\le C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{k}2^{\alpha ql}\|F\chi_l\|_{L^p(\omega)}^q2^{(l-k)\left(\left(n/p_0'\right)-\alpha\right)q/2}\right)\right. \\
&\quad\left.\times\left(\sum_{l=-\infty}^{k}(k-l)^{q'}2^{(l-k)\left(\left(n/p_0'\right)-\alpha\right)q'/2}\right)^{q/q'}\right\}^{1/q} \\
&\le C\left(\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{k}2^{\alpha ql}\|F\chi_l\|_{L^p(\omega)}^q2^{(l-k)\left(n/p'_0-\alpha\right)q/2}\right)^{1/q} \\
&\le C\left(\sum_{l=-\infty}^{\infty}2^{\alpha ql}\|F\chi_l\|_{L^p(\omega)}^q\sum_{k=l}^{\infty}2^{(l-k)\left(n/p'_0-\alpha\right)q/2}\right)^{1/q} \\
&\le C\left(\sum_{l=-\infty}^{\infty}2^{\alpha ql}\|F\chi_l\|_{L^p(\omega)}^q\right)^{1/q} \le C\|F\|_{\dot K_p^{\alpha,q}(\omega)}.
\end{aligned} \tag{60}$$

This completes the proof of Theorem 10.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The research was supported by the National Natural Science Foundation of China (11671397, 12071473).

References

[1] B. Muckenhoupt, "Weighted norm inequalities for the Hardy maximal function," *Transactions of the American Mathematical Society*, vol. 165, pp. 207–226, 1972.

[2] R. Hunt, B. Muckenhoupt, and R. L. Wheeden, "Weighted norm inequalities for the conjugate function and Hilbert transform," *Transactions of the American Mathematical Society*, vol. 176, pp. 227–251, 1973.

[3] R. R. Coifman and C. Fefferman, "weighted norm inequalities for maximal functions and singular integrals," *Studia Mathematica*, vol. 51, no. 3, pp. 241–250, 1974.

[4] Z. Fu and S. Lu, "Commutators of generalized Hardy operators," *Mathematische Nachrichten*, vol. 282, no. 6, pp. 832–845, 2009.

[5] Y. Chen and K. S. Lau, "Some new classes of Hardy spaces," *Journal of Functional Analysis*, vol. 84, no. 2, pp. 255–278, 1989.

[6] J. García-Cuerva, "Hardy spaces and Beurling algebras," *Journal of the London Mathematical Society*, vol. s2-39, no. 3, pp. 499–513, 1989.

[7] V. S. Guliyev, M. N. Omarova, M. A. Ragusa, and A. Scapellato, "Regularity of solutions of elliptic equations in divergence form in modified local generalized Morrey spaces," *Analysis and Mathematical Physics*, vol. 11, no. 1, 2021.

[8] S. Lu and D. Yang, "Some new Hardy spaces associated with the Herz spaces and their applications (in chinese)," *Journal of Beijing Normal University*, vol. 29, pp. 10–19, 1993.

[9] N. Sarfraz and A. Hussain, "Estimates for the commutators of p-adic Hausdorff operator on Herz-Morrey spaces," *Mathematics*, vol. 7, no. 2, p. 127, 2019.

[10] D. Wang and J. Zhou, "Some new class of BMO spaces," *Acta Mathematica Sinica*, vol. 60, pp. 833–846, 2017.

[11] Z. Fu, Z. G. Liu, S. Z. Lu, and H. B. Wang, "Characterization for commutators of n-dimensional fractional Hardy operators," *Science in China Series A: Mathematics*, vol. 50, no. 10, pp. 1418–1426, 2007.

[12] L. Diening, P. Harjulehto, P. Hästö, and M. Ruzicka, "Lebesgue and Sobolev spaces with variable exponents," in *Lecture Notes in Mathematics, vol. 2017*, Springer, Berlin, 2011.

[13] S. Lu, Y. Ding, and D. Yan, *Singular Integrals and Related Topics*, World Scientific, 2007.

[14] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 2nd edition, 1952.

[15] L. Zhang, J. Sun, and D. Liu, "Dual spaces and imbeddings of weighted Lebesgue spaces," *Applied Mathematics*, vol. 12, pp. 56–60, 1999.

[16] S. Lu and D. Yang, "The decomposition of the weighted Herz spaces on R^n and its applications," *Science In China Series A*, vol. 38, pp. 147–158, 1995.