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Research Article

Monotonicity and Symmetry of Solutions to Fractional Laplacian in Strips

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In this paper, using the method of moving planes, we study the monotonicity in some directions and symmetry of the Dirichlet problem involving the fractional Laplacian $\begin{cases} (-\Delta)^{\alpha/2}u(x) = f(u(x)), & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^n \setminus \Omega, \end{cases}$ in a slab-like domain $\Omega = \mathbb{R}^{n-1} \times (0, h) \subset \mathbb{R}^n$.

1. Introduction

The fractional Laplacian in $\mathbb{R}^n$ is a nonlocal pseudo-differential operator defined by

$$(-\Delta)^{\alpha/2}u(x) = C_{n,\alpha} \lim_{\varepsilon \longrightarrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(z)}{|x - z|^{n+\alpha}} dz, \quad (1)$$

where $C_{n,\alpha}$ is a normalisation constant and α is any real number between 0 and 2. Let

$$L_\alpha = \left\{ u : \mathbb{R}^n \longrightarrow \mathbb{R}^1 \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} dx < \infty \right\}. \quad (2)$$

Then, it is easy to verify that for $u \in L_\alpha \cap C^{1,1}_{loc}$, the integral on the right-hand side of (1) is well defined. Throughout this paper, we consider the fractional Laplacian in this setting.

Due to applications in physics, chemistry, biology, probability, and finance, differential equations involving the fractional Laplacian $(-\Delta)^{\alpha/2}$ have received growing attention from the mathematical communicity in recent years (see [1–14]). There are many papers devoted to the study of qualitative properties of fractional Laplacian equations in bounded or unbounded domains, but seldom are concerned with slab-like domains. For example, in [15], the authors established the symmetry and monotonicity of positive solutions of the following problem with more general nonlinearity on a bounded domain.

$$\begin{cases} (-\Delta)^{\alpha/2}u(x) = f(u(x)), & x \in B_1(0), \\ u(x) = 0, & x \in \mathbb{R}^n \setminus B_1(0), \end{cases} \quad (3)$$

using a direct method of moving planes. For local elliptic operators, these kinds of approaches were introduced decades ago in the paper [16] and then summarized in the book [17], among which the narrow region principle and the decay at infinity have been applied extensively by many researchers to solve various problems. For more articles concerning the method of moving plans for nonlocal equations, please see [18–20] and the references therein.

However, there are some papers of elliptic second-order boundary value problems concerned with features like monotonicity in some directions and symmetry for positive solutions in slab-like domains. For instance, in [21], using the "sliding method," the authors studied monotonicity in some directions and symmetry of elliptic second-order boundary value problems of the type.

$$\begin{cases} \Delta u + f(u) = 0, & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \quad (4)$$

in a slab $\Omega = \mathbb{R} \times (0, h) \subset \mathbb{R}^2$. For more articles concerning the "sliding method," please see [22, 23] and the references therein.

Motivated by the above work, in this paper, using the direct method of moving planes, we study the monotonicity in some directions and symmetry of fractional Laplacian boundary value problems of the type.

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) = f(u(x)), & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^n \setminus \Omega, \end{cases} \quad (5)$$

in a class of special unbounded domains Ω of $\mathbb{R}^n$: infinite cylinders or more generally, product domains of the form

$$\Omega = \mathbb{R}^{n-j} \times \omega, \quad (6)$$

where ω is a smooth bounded domain in $\mathbb{R}^j$.

We denote the variables in Ω by (x', y), $x' \in \mathbb{R}^{n-j}$, and $y \in \omega \subset \mathbb{R}^j$ with $j \geq 1$. It is not assumed that Ω is bounded. The function f appearing in (5) will always be assumed to be (globally) Lipschitz continuous. We firmly believe that the result introduced here is of great importance, and the ideals and methods can be applied to study a variety of nonlocal problems with more general operators and nonlinearities.

In most of what follows, we consider the case $j = 1$. In this case, the proof of monotonicity and symmetry yields the following statement for $j = 1$.

Theorem 1. *Let*

$$\Sigma = \left\{ \left(x', y\right) \mid x' \in \mathbb{R}^{n-1}, 0 < y < h \right\}. \quad (7)$$

Suppose $u \in L_\alpha \cap C^{1,1}_{loc}(\Sigma)$ satisfies

$$\begin{cases} (-\Delta)^{\alpha/2} u = f(u), & in\ \Sigma, \\ u(x) > 0, & in\ \Sigma, \\ u(x) = 0, & in\ \mathbb{R}^n \setminus \Sigma, \end{cases} \quad (8)$$

with $f(\cdot)$ being Lipschitz continuous. Then, for any positive $l < h/2$,

$$u\left(x', y\right) < u\left(x', 2l - y\right), in \sum_l = \left\{ \left(x', y\right) \mid x' \in \mathbb{R}^{n-1}, 0 < y < l \right\}, \quad (9)$$

and u is symmetric in y about $y = h/2$.

If we further assume that $u \in C^3_{loc}(\bar{\Sigma}_{h/2})$, then

$$\frac{\partial u}{\partial y} > 0, in \sum_{h/2} = \left\{ \left(x', y\right) \mid 0 < y < \frac{h}{2} \right\}. \quad (10)$$

Remark 2. Here, the domain Ω is an infinite cylinder, and it is more general than the usual unbounded domains. For instance, if we let $h \longrightarrow \infty$ in Theorem 1, we can get monotonicity of positive solutions of the Dirichlet problem involving the fractional Laplacian in the half space.

2. Preliminaries and Lemmas

Let T_λ be a hyperplane in $\mathbb{R}^n$. Without loss of generality, we may assume that

$$\begin{aligned} T_\lambda &= \left\{ x = \left(x', y\right) \in \mathbb{R}^{n-1} \times (0, h) \mid y = \lambda \right\}, \\ \sum_\lambda &= \left\{ x = \left(x', y\right) \in \mathbb{R}^{n-1} \times (0, h) \mid 0 < y < \lambda \right\}. \end{aligned} \quad (11)$$

And for $(x', y) \in \Sigma_\lambda$, we let $x^\lambda = (x', 2\lambda - y)$ be the reflection of x about the plane T_λ. Denote $w_\lambda(x) = u(x^\lambda) - u(x)$. For simplicity of notation, in the following, we denote w_λ by w and Σ_λ by Σ.

Lemma 3 (Narrow region principle [15]). *Let Ω be a bounded narrow region in Σ, such that it is contained in $\{x \mid \lambda - l < y < \lambda\}$ with small l. Suppose that $w \in L_\alpha \cap C^{1,1}_{loc}(\Omega)$ and is lower semicontinuous on $\bar{\Omega}$. If $c(x)$ is bounded from below in Ω and*

$$\begin{cases} (-\Delta)^{\alpha/2} w(x) + c(x) w(x) \geq 0 & in\ \Omega, \\ w(x) \geq 0 & in\ \Sigma \setminus \Omega, \\ w\left(x^\lambda\right) = -w(x) & in\ \Sigma, \end{cases} \quad (12)$$

then for sufficiently small l, we have

$$w(x) \geq 0\ in\ \Omega. \quad (13)$$

Furthermore, if $w = 0$ at some point in Ω, then

$$w(x) = 0\ almost\ every\ where\ in\ \mathbb{R}^n. \quad (14)$$

These conclusions hold for unbounded region Ω if we further assume that

$$\underline{\lim}_{|x| \longrightarrow \infty} w(x) \geq 0. \quad (15)$$

Lemma 4 (A Hopf type lemma for antisymmetric functions [24]). *Assume that $w \in C^3_{loc}(\bar{\Sigma})$, $\overline{\lim}_{x \longrightarrow \partial\Sigma} c(x) = o(1/[dist(x, \partial\Sigma)]^2)$, and*

$$\begin{cases} (-\Delta)^{\alpha/2} w(x) + c(x) w(x) = 0 & in\,\Sigma, \\ w(x) \geq 0 & in\,\Sigma, \\ w\left(x^{\lambda}\right) = -w(x) & in\,\Sigma. \end{cases} \quad (16)$$

Then,

$$\frac{\partial w}{\partial \nu} < 0, x \in \partial\Sigma. \quad (17)$$

3. Proof of Theorem 1

Proof of Theorem 1. Now we carry on the method of moving planes on the solution u along y direction. □

Step 1. We show that, for sufficiently small $\lambda > 0$,

$$w_{\lambda}(x) > 0, x \in \sum_{\lambda}, \quad (18)$$

where $w_{\lambda}(x) = u(x^{\lambda}) - u(x)$.

As usual, we can easily verify that w_{λ} satisfies the following linear equation

$$(-\Delta)^{\alpha/2} w_{\lambda} + c_{\lambda}(x) w_{\lambda} = 0, x \in \sum_{\lambda}. \quad (19)$$

Indeed, $u(x^{\lambda})$ satisfies the same equation in (8) as $u(x)$; thus, (19) is obtained by subtracting one from the other and letting

$$c_{\lambda}(x) = \begin{cases} \dfrac{f\left(u\left(x^{\lambda}\right)\right) - f(u(x))}{u(x) - u\left(x^{\lambda}\right)}, & u(x) \neq u\left(x^{\lambda}\right), \\ 0, & u(x) = u\left(x^{\lambda}\right). \end{cases} \quad (20)$$

By the assumption that f is (globally) Lipschitz continuous, with some Lipschitz constant b, we have

$$\|c_{\lambda}\|_{L^{\infty}(\Sigma_{\lambda})} \leq b, \forall \lambda \in \left(0, \frac{h}{2}\right). \quad (21)$$

From the narrow region principle, we can easily know that for sufficiently small $\sigma > 0$,

$$w_{\lambda}(x) \geq 0, \forall x \in \sum_{\lambda}, \lambda \in (0, \sigma). \quad (22)$$

Furthermore, it follows from $w_{\lambda}(x', 0) > 0$ that we have

$$w_{\lambda}(x) > 0, \forall x \in \sum_{\lambda}, \lambda \in (0, \sigma). \quad (23)$$

Step 2. The proof in Step 1 provides a starting point, from which we can now move the plane T_{λ} to the right as long as (18) holds to its limiting position.

Let

$$\lambda_0 = \sup\left\{\lambda \in \left(0, \frac{h}{2}\right) \mid w_{\mu}(x) > 0, \forall x \in \sum_{\mu}, \mu \leq \lambda\right\}. \quad (24)$$

In this part, we show that

$$\begin{gathered} \lambda_0 = \frac{h}{2}, \\ w_{\lambda_0}(x) \equiv 0, x \in \sum_{\lambda_0}. \end{gathered} \quad (25)$$

Suppose that $\lambda_0 < h/2$, we show that the plane T_{λ} can be moved further. To be more rigorous, we only need to prove that there exists $\varepsilon > 0$, such that for any $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$, we have

$$w_{\lambda_0}(x) > 0, x \in \sum_{\lambda_0}. \quad (26)$$

This is a contradiction with the definition of λ_0. Hence, we have $\lambda_0 = h/2$.

Now we prove (26) by the narrow region principle (Lemma 3). By the definition of λ_0, we can easily have

$$w_{\lambda_0}(x) \geq 0, x \in \sum_{\lambda_0}. \quad (27)$$

In fact, when $\lambda_0 < h/2$, we have

$$(x) > 011 w_{\lambda_0}(x) > 0, x \in \sum_{\lambda_0}. \quad (28)$$

If not, there exists $\hat{x}$ such that

$$w_{\lambda_0}(\hat{x}) = \min_{\Sigma_{\lambda_0}} w_{\lambda_0}(x) = 0. \quad (29)$$

Then, we have

$$\begin{aligned} (-\Delta)^{\alpha/2} w_{\lambda_0}(\hat{x}) &= C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{-w_{\lambda_0}(z)}{|\hat{x} - z|^{n+\alpha}} dz \\ &= C_{n,\alpha} PV \int_{\Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(z)}{|\hat{x} - z|^{n+\alpha}} dz + \int_{\mathbb{R}^n \setminus \Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(z)}{|\hat{x} - z|^{n+\alpha}} dz \\ &= C_{n,\alpha} PV \int_{\Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(z)}{|\hat{x} - z|^{n+\alpha}} dz + \int_{\Sigma_{\lambda_0}} \frac{w_{\lambda_0}(z)}{\left|\hat{x} - z^{\lambda}\right|^{n+\alpha}} dz \\ &= C_{n,\alpha} PV \int_{\Sigma_{\lambda_0}} \left(\frac{1}{\left|\hat{x} - z^{\lambda}\right|^{n+\alpha}} dz - \frac{1}{|\hat{x} - z|^{n+\alpha}}\right) \\ &\quad \cdot w_{\lambda_0}(z) dz < 0. \end{aligned} \quad (30)$$

On the other hand,

$$(-\Delta)^{\alpha/2} w_{\lambda_0}(\hat{x}) = \left(-\Delta^{\alpha/2}\right) u\left(\hat{x}^{\lambda_0}\right) - \left(-\Delta^{\alpha/2}\right) u(\hat{x}) = f\left(u\left(\hat{x}^{\lambda_0}\right)\right) - f(u(\hat{x})) = 0. \quad (31)$$

This is a contradiction with (30). Thus, (28) holds.

Then, it follows from (28) that there exists a constant $c_0 > 0$ and $\delta > 0$, such that

$$w_{\lambda_0}(x) \geq c_0, x \in \bar{\Sigma}_{\lambda_0 - \delta}. \quad (32)$$

Since w_λ depends on λ continuously, there exists $\varepsilon \in (0, \delta)$, such that for all $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$, we have

$$w_\lambda(x) > 0, x \in \bar{\Sigma}_{\lambda_0 - \delta}. \quad (33)$$

Then, from the narrow region principle (Lemma 3), we conclude that for all $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$,

$$w_\lambda(x) > 0, x \in \bar{\Sigma}_\lambda. \quad (34)$$

This is a contradiction with the definition of λ_0. Therefore, we must have $\lambda_0 = h/2$, and

$$w_{\lambda_0}(x) \equiv 0, x \in \sum_{\lambda_0}. \quad (35)$$

Consequently, for all λ: $0 < \lambda < h/2$, we have $w_\lambda > 0$ in Σ_λ. Therefore, (9) holds, and u is symmetric in y about $y = h/2$.

Further, if we assume $u \in C^3_{loc}(\bar{\Sigma}_{h/2})$, we now prove (10) holds. Indeed, w_λ satisfies the following linear equation

$$(-\Delta)^{\alpha/2} w_\lambda + c_\lambda(x) w_\lambda = 0, x \in \sum_\lambda, \quad (36)$$

with $w_\lambda(x', \lambda) = 0$. Also, by the former proof, we know that $w_\lambda > 0$ in Σ_λ. Here, we consider the distance from x to the upper boundary $\{y = \lambda\}$ of Σ_λ, denoted by $\operatorname{dist}(x, \partial\Sigma_\lambda) =: d$. Then, $d(x, \partial\Sigma_\lambda) = \lambda - y$. Thus, by (20) we know that

$$\overline{\lim_{x \longrightarrow \partial\Sigma_\lambda}} c(x) \left[d\left(x, \partial \sum_\lambda\right)\right]^2 = \overline{\lim_{x \longrightarrow \partial\Sigma_\lambda}} c(x) [d(\lambda - x_2)]^2 = 0. \quad (37)$$

Therefore,

$$\overline{\lim_{x \longrightarrow \partial\Sigma_\lambda}} c(x) = o\left(\frac{1}{[d(x, \partial\Sigma_\lambda)]^2}\right). \quad (38)$$

Consequently, the Hopf type lemma for antisymmetric functions (Lemma 4) leads to

$$-2\frac{\partial u}{\partial y}\left(x', \lambda\right) \equiv \frac{\partial w_\lambda}{\partial y}\left(x', \lambda\right) < 0, \forall x' \in \mathbb{R}^{n-1}, \lambda \in \left(0, \frac{h}{2}\right), \quad (39)$$

which implies that (10) holds. This completes the proof.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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