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Research Article

Notes on Solutions for Some Systems of Complex Functional Equations in $\mathbb{C}^2$

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The purpose of this article is to give the details of finding the transcendental entire solutions with finite order for the systems of nonlinear partial differential-difference equations $\begin{cases} (((\partial f_1(z_1,z_2))/\partial z_1)+((\partial f_1(z_1,z_2))/\partial z_2))^{n_1}+P_1(z)f_2(z_1+c_1,z_2+c_2)^{m_1}=Q_1(z), \\ (((\partial f_1(z_1,z_2))/\partial z_1)+((\partial f_1(z_1,z_2))/\partial z_2))^{n_2}+P_2(z)f_1(z_1+c_1,z_2+c_2)^{m_2}=Q_2(z), \end{cases}$ where $P_1(z), P_2(z), Q_1(z),$ and $Q_2(z)$ are polynomials in $\mathbb{C}^2$; $n_1, n_2, m_1,$ and m_2 are positive integers, and $c=(c_1,c_2)\in\mathbb{C}^2$. We obtain that there exist some pairs of the transcendental entire solutions of finite order for the above system, which is a very powerful supplement to the previous theorems given by Xu and Cao and Xu and Yang.

1. Introduction

In 1970, Yang [1] proved that the functional equations $f^n+g^m=1$ have no nonconstant entire solutions, if m, n are positive integers satisfying $(1/m)+(1/n)<1$. After this result, with the aid of the Nevanlinna theory and the difference analogues of the Nevanlinna theory (see [2–6]), there were rapid developments on complex differential and difference equations in one and several complex variables. Some classical results and topics in different fields are considered in difference versions, for example, difference Riccati equations, difference Painlevé equations, and difference Fermat equations (see [7–14]). Recently, Cao and Xu [15–17] investigated the existence of the entire and meromorphic solutions for some Fermat-type partial differential-difference equations by utilizing the Nevanlinna theory and difference Nevanlinna theory of several complex variables [18, 19] and obtained the following theorems which is an extension of the previous results given by Liu and his collaborators (see [20–24]).

Theorem 1 (see ([16], Theorem 1.1)). *Let $c=(c_1,c_2)\in\mathbb{C}^2$. Then, the Fermat-type partial differential-difference equation*

$$\left(\frac{\partial f(z_1,z_2)}{\partial z_1}\right)^n+f(z_1+c_1,z_2+c_2)^m=1, \quad (1)$$

does not have any transcendental entire solution with finite order, where m and n are two distinct positive integers.

Theorem 2 (see ([15], Theorem 3.2)). *Let $c=(c_1,c_2)\in\mathbb{C}\setminus\{0\}$. Suppose that f is a nontrivial meromorphic solution of the Fermat type partial difference equations*

$$\frac{1}{f(z_1+c_1,z_2+c_2)^m}+\frac{1}{f(z_1,z_2)^m}=A(z_1,z_2)f(z_1,z_2)^n, \quad (2)$$

or

$$\frac{1}{f(z_1+c_1,z_2+c_2)^m}+\frac{1}{f(z_1+c_1,z_2)^m}+\frac{1}{f(z_1,z_2+c_2)^m}=A(z_1,z_2)f(z_1,z_2)^n, \tag{3}$$

where $m\in\mathbb{N}, n\in\mathbb{N}\cup\{0\}$, and $A(z_1,z_2)$ is a nonzero meromorphic function on $\mathbb{C}^2$ with respect to the solution f, that is $T(r,A)=o(T(r,f))$. If $\delta_f(\infty)>0$, then

$$\limsup_{r\longrightarrow\infty}\frac{\log T(r,f)}{r}>0. \tag{4}$$

Remark 3. Let $n=0$ and $A(z_1,z_2)=1$, then the above equations become

$$\frac{1}{f(z_1+c_1,z_2+c_2)^m}+\frac{1}{f(z_1,z_2)^m}=1,$$
$$\frac{1}{f(z_1+c_1,z_2+c_2)^m}+\frac{1}{f(z_1+c_1,z_2)^m}+\frac{1}{f(z_1,z_2+c_2)^m}=1, \tag{5}$$

which can be called as the partial difference equations of Fermat type.

In 2020, the first author and his coauthors discussed the transcendental entire solutions with finite order for the systems of partial differential difference equations and gave the conditions on the existence of the finite-order transcendental entire solutions for the following systems, which are some extension and improvements of the previous results given by Xu and Cao and Gao [16, 25].

Theorem 4 (see ([26], Theorem 1.2)). *Let $c=(c_1,c_2)\in\mathbb{C}^2$, and $m_j, n_j (j=1,2)$ be positive integers. If the following system of Fermat-type partial differential-difference equations*

$$\begin{cases}\left(\dfrac{\partial f_1(z_1,z_2)}{\partial z_1}\right)^{n_1}+f_2(z_1+c_1,z_2+c_2)^{m_1}=1,\\ \left(\dfrac{\partial f_2(z_1,z_2)}{\partial z_1}\right)^{n_2}+f_1(z_1+c_1,z_2+c_2)^{m_2}=1,\end{cases} \tag{6}$$

satisfies one of the conditions

(i) $m_1m_2>n_1n_2$;

(ii) $m_j>(n_j/(n_j-1))$ for $n_j\geq 2$, $j=1,2$.

Then, system (6) does not have any pair of transcendental entire solution with finite order.

Remark 5. Here, (f,g) is called as a pair of finite-order transcendental entire solutions for system

$$\begin{cases}f^{n_1}+g^{m_1}=1,\\ f^{n_2}+g^{m_2}=1,\end{cases} \tag{7}$$

if f,g are transcendental entire functions and $\rho=\max\{\rho(f),\rho(g)\}<\infty$.

Remark 6. The condition $m_j>(n_j/(n_j-1))$ implies $m_j>1$. Thus, a question rises naturally: what will happen on the existence of transcendental entire solutions with finite order when $m_j=1, j=1,2$ in system (6)?

In fact, we give the following example to explain that system (6) has a pair of transcendental entire solutions with finite order when $m_1=m_2=1$ and $n_1=n_2=2$, that is,

$$\begin{cases}\left(\dfrac{\partial f_1(z_1,z_2)}{\partial z_1}\right)^{2}+f_2(z_1+c_1,z_2+c_2)=1,\\ \left(\dfrac{\partial f_2(z_1,z_2)}{\partial z_1}\right)^{2}+f_1(z_1+c_1,z_2+c_2)=1[rgb]0.00,0.00,1.00.\end{cases} \tag{8}$$

Example 1. Let

$$f_1(z)=\quad 1-\frac{1}{4}\pi^2-\frac{1}{4}z_1^2+\frac{1}{2}z_1z_2-\frac{\pi i}{2}z_2+(z_1-\pi i)e^{z_2}-\left[e^{z_2}+\frac{1}{2}(z_2-\pi i)\right]^2,$$
$$f_2(z)=\quad 1-\frac{1}{4}\pi^2-\frac{1}{4}z_1^2+\frac{1}{2}z_1z_2-\frac{\pi i}{2}z_2-(z_1-\pi i)e^{z_2}-\left[e^{z_2}-\frac{1}{2}(z_2-\pi i)\right]^2. \tag{9}$$

Then, $f=(f_1,f_2)$ is a pair of transcendental entire solutions of system (8) with $(c_1,c_2)=(\pi i,\pi i)$ and $\rho(f)=1$.

Corresponding to system (6), we further consider the following system of the partial differential difference equation

$$\begin{cases}\left(\dfrac{\partial f_1(z_1,z_2)}{\partial z_1}+\dfrac{\partial f_1(z_1,z_2)}{\partial z_2}\right)^{n_1}+P_1(z)f_2(z_1+c_1,z_2+c_2)^{m_1}=Q_1(z),\\ \left(\dfrac{\partial f_2(z_1,z_2)}{\partial z_1}+\dfrac{\partial f_2(z_1,z_2)}{\partial z_2}\right)^{n_2}+P_2(z)f_1(z_1+c_1,z_2+c_2)^{m_2}=Q_2(z),\end{cases} \tag{10}$$

where $P_1(z), P_2(z)$ are two nonzero polynomials in $\mathbb{C}^2$ and obtained.

Theorem 7. *Let $c=(c_1,c_2)\in\mathbb{C}^2$, $m_j, n_j (j=1,2)$ be positive integers satisfies one of the conditions*

(i) $m_1m_2>n_1n_2$;

(ii) $m_j>n_j/(n_j-1)$ for $n_j\geq 2$, $j=1,2$.

Then, system (10) does not have any pair of transcendental entire solutions with finite order.

The following example shows that the conditions $m_j>(n_j/n_j-1)$ for $n_j\geq 2$ and $j=1,2$ are precise and the existence of finite-order transcendental entire solutions for the system (10) when $n_1=n_2=2$, $m_1=m_2=1$ and $P_1(z)=P_2(z)=Q_1(z)=Q_2(z)=1$, that is,

$$\begin{cases}\left(\dfrac{\partial f_1(z_1,z_2)}{\partial z_1}+\dfrac{\partial f_1(z_1,z_2)}{\partial z_2}\right)^2+f_2(z_1+c_1,z_2+c_2)=1,\\ \left(\dfrac{\partial f_2(z_1,z_2)}{\partial z_1}+\dfrac{\partial f_2(z_1,z_2)}{\partial z_2}\right)^2+f_1(z_1+c_1,z_2+c_2)=1.\end{cases}\tag{11}$$

Example 2. Let

$$\begin{aligned}f_1(z)&=1-\frac{1}{4}\pi^2-\frac{1}{4}z_1^2+\frac{1}{2}(z_2-z_1)(z_1-\pi i)+(z_1-\pi i)e^{z_2-z_1}-\left[e^{z_2-z_1}+\frac{1}{2}(z_2-z_1-\pi i)\right]^2,\\ f_2(z)&=1-\frac{1}{4}\pi^2-\frac{1}{4}z_1^2+\frac{1}{2}(z_2-z_1)(z_1-\pi i)-(z_1-\pi i)e^{z_2-z_1}-\left[e^{z_2-z_1}-\frac{1}{2}(z_2-z_1-\pi i)\right]^2.\end{aligned}\tag{12}$$

Then, $f=(f_1,f_2)$ is a pair of transcendental entire solutions of system (11) with $(c_1,c_2)=(\pi i,2\pi i)$ and $\rho(f)=1$.

Remark 8. In Sections 3 and 4, we give the details proceeding for obtaining a class of finite-order transcendental entire solutions for systems (8) and (11).

Next, we continue to discuss the existence of the finite-order transcendental entire solutions for several systems including both the difference operator and the partial differential such as

$$\begin{cases}\left(\dfrac{\partial f_1}{\partial z_1}\right)^{n_1}+[f_2(z_1+c_1,z_2+c_2)-f_1(z_1,z_2)]^{m_1}=1,\\ \left(\dfrac{\partial f_2}{\partial z_1}\right)^{n_1}+[f_1(z_1+c_1,z_2+c_2)-f_2(z_1,z_2)]^{m_1}=1,\end{cases}\tag{13}$$

$$\begin{cases}\left(\dfrac{\partial f_1}{\partial z_1}+\dfrac{\partial f_1}{\partial z_2}\right)^{n_1}+[f_2(z_1+c_1,z_2+c_2)-f_1(z_1,z_2)]^{m_1}=1,\\ \left(\dfrac{\partial f_2}{\partial z_1}+\dfrac{\partial f_2}{\partial z_2}\right)^{n_1}+[f_1(z_1+c_1,z_2+c_2)-f_2(z_1,z_2)]^{m_1}=1,\end{cases}\tag{14}$$

where c_1, c_2 are constants in $\mathbb{C}$. It is easy to find the finite-order transcendental entire solutions for systems (13) and (14). For $c_2\neq 0$, system (5) has a pair of finite-order transcendental entire solutions (f_1,f_2) of the forms

$$\begin{cases}f_1=az_1+\dfrac{b+d-2ac_1}{2c_2}z_2+\dfrac{d}{2}+e^{(\pi i/c_2)z_2},\\ f_2=az_1+\dfrac{b+d-2ac_1}{2c_2}z_2+\dfrac{b}{2}-e^{(\pi i/c_2)z_2},\end{cases}\tag{15}$$

and for $c_2\neq c_1$, system (14) has a pair of finite-order transcendental entire solutions (f_1,f_2) of the forms

$$\begin{cases}f_1=az_1+\dfrac{b+d-2ac_1}{2(c_2-c_1)}(z_2-z_1)+\dfrac{d}{2}+e^{(\pi i/(c_2-c_1))(z_2-z_1)},\\ f_2=az_1+\dfrac{b+d-2ac_1}{2(c_2-c_1)}(z_2-z_1)+\dfrac{b}{2}-e^{(\pi i/(c_2-c_1))(z_2-z_1)},\end{cases}\tag{16}$$

where $a,b,d\in\mathbb{C}$ satisfy $1-a^{n_1}=b^{m_1}$ and $1-a^{n_2}=b^{m_2}$. Furthermore, we can give the finite-order transcendental entire solutions for systems (13) and (14) when $n_1=n_2=2$ and $m_1=m_2=1$ easily.

Example 3. The function

$$f=(f_1,f_2)=\left(z_1-z_2+e^{\pi i z_2}, z_1-z_2-e^{\pi i z_2}\right),\tag{17}$$

is a pair of transcendental entire solutions with $\rho(f)=1$ for system (13) when $(c_1,c_2)=(1,1)$, $n_1=n_2=2$, and $m_1=m_2=1$.

Example 4. The function

$$f = (f_1, f_2) = \left(2z_1 - z_2 + e^{\pi i(z_2 - z_1)}, 2z_1 - z_2 - e^{\pi i(z_2 - z_1)}\right), \tag{18}$$

is a pair of transcendental entire solutions with $\rho(f) = 1$ for system (14) when $(c_1, c_2) = (1, 2)$, $n_1 = n_2 = 2$ and $m_1 = m_2 = 1$.

Corresponding to systems (13) and (14), we can also obtain the solutions of the following systems

$$\begin{cases} \left(\dfrac{\partial f_1}{\partial z_1}\right)^{n_1} + [f_2(z_1 + c_1, z_2 + c_2) - f_2(z_1, z_2)]^{m_1} = 1, \\ \left(\dfrac{\partial f_2}{\partial z_1}\right)^{n_1} + [f_1(z_1 + c_1, z_2 + c_2) - f_1(z_1, z_2)]^{m_1} = 1, \end{cases} \tag{19}$$

$$\begin{cases} \left(\dfrac{\partial f_1}{\partial z_1} + \dfrac{\partial f_1}{\partial z_2}\right)^{n_1} + [f_2(z_1 + c_1, z_2 + c_2) - f_2(z_1, z_2)]^{m_1} = 1, \\ \left(\dfrac{\partial f_2}{\partial z_1} + \dfrac{\partial f_2}{\partial z_2}\right)^{n_1} + [f_1(z_1 + c_1, z_2 + c_2) - f_1(z_1, z_2)]^{m_1} = 1, \end{cases} \tag{20}$$

where c_1, c_2 are constants in $\mathbb{C}$. In fact, for $c_2 \neq 0$, then systems (19) has a pair of solutions with the forms

$$(f_1, f_2) = \left(a_1 z_1 + \frac{b_2 - a_1 c_1}{c_2} z_2 + G_1(z_2), a_2 z_1 + \frac{b_1 - a_2 c_1}{c_2} z_2 + G_2(z_2)\right), \tag{21}$$

where $G_1(z_2), G_2(z_2)$ are two period functions with period c_2, and for $s := z_2 - z_1$ and $s_0 := c_2 - c_1 \neq 0$, then system (20) has a pair of solutions with the forms

$$(f_1, f_2) = \left(a_1 z_1 + \frac{b_2 - a_1 c_1}{c_2 - c_1} s + G_1(s), a_2 z_1 + \frac{b_1 - a_2 c_1}{c_2 - c_1} s + G_2(s)\right), \tag{22}$$

where $G_1(s), G_2(s)$ are two period functions with period s_0, and $a_1, a_2, c_1, c_2, d_1,$ and d_2 are constants satisfying

$$a_1^{n_1} + b_1^{m_1} = 1, a_2^{n_2} + b_2^{m_2} = 1. \tag{23}$$

2. Proof of Theorem 7

The following lemmas will be used in this paper.

Lemma 9 ([27, 28]). *Let f be a nonconstant meromorphic function on $\mathbb{C}^n$ and let $I = (i_1, \cdots, i_n)$ be a multi-index with length $|I| = \sum_{j=1}^{n} i_j$. Assume that $T(r_0, f) \geq e$ for some r_0. Then,*

$$m\left(r, \frac{\partial^I f}{f}\right) = S(r, f), \tag{24}$$

holds for all $r \geq r_0$ outside a set $E \subset (0, +\infty)$ of finite logarithmic measure $\int_E (dt/t) < \infty$, where $\partial^I f = (\partial^{|I|} f)/(\partial z_1^{i_1} \cdots \partial z_n^{i_n})$.

Lemma 10 ([18, 19]). *Let f be a nonconstant meromorphic function with finite order on $\mathbb{C}^n$ such that $f(0) \neq 0, \infty$, and let $\varepsilon > 0$. Then, for $c \in \mathbb{C}^n$,*

$$m\left(r, \frac{f(z)}{f(z+c)}\right) + m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f), \tag{25}$$

holds for all $r \geq r_0$ outside a set $E \subset (0, +\infty)$ of finite logarithmic measure $\int_E (dt/t) < \infty$.

Lemma 11 (see [29]). *Let f be a nonconstant meromorphic function on $\mathbb{C}^n$. Take a positive integer m and take polynomials of f and its partial derivatives:*

$$\begin{aligned} P(f) &= \sum_{\mathbf{p} \in I} a_{\mathbf{p}} f^{p_0} \left(\partial^{i_1} f\right)^{p_1} \cdots \left(\partial^{i_l} f\right)^{p_l}, (p) = (p_0, \cdots, p_l), \\ Q(f) &= \sum_{\mathbf{q} \in I} c_{\mathbf{q}} f^{q_0} \left(\partial^{j_1} f\right)^{q_1} \cdots \left(\partial^{j_l} f\right)^{q_l}, (q) = (q_0, \cdots, q_l), \\ B(f) &= \sum_{k=0}^{m} b_k f^k, \end{aligned} \tag{26}$$

where I, J are finite sets of distinct elements and $a_{\mathbf{p}}, c_{\mathbf{q}}$, and b_k are meromorphic functions on $\mathbb{C}^n$ with $b_m \not\equiv 0$. Assume that f satisfies the equation

$$B(f) Q(f) = P(f), \tag{27}$$

such that $P(f), Q(f)$, and $B(f)$ are differential polynomials, that is, their coefficients a satisfy $m(r, a) = S(r, f)$. If $\deg(P(f)) \leq m = \deg(B(f))$, then

$$m(r, Q(f)) = S(r, f), \tag{28}$$

holds for all r possibly outside of a set E with finite logarithmic measure.

Proof. Let (f_1, f_2) be a pair of transcendental entire functions with finite-order satisfying system (10). Here, we will discuss two following cases.

Case 1. $n_1 n_2 > m_1 m_2$. In view of Lemma 10, the following conclusions that

$$m\left(r, \frac{f_j(z_1, z_2)}{f_j(z_1 + c_1, z_2 + c_2)}\right) = S\left(r, f_j\right), \quad j = 1, 2, \tag{29}$$

holds for all $r > 0$ outside of a possible exceptional set $E_j \subset [1, +\infty)$ of finite logarithmic measure $\int_{E_j} (dt/t) < \infty$. Thus, we can deduce from (29) that

$$T\left(r,f_j(z_1,z_2)\right)=m\left(r,f_j(z_1,z_2)\right)\le m\left(r,\frac{f_j(z_1,z_2)}{f(z_1+c_1,z_2+c_2)}\right)+m\left(r,f_j(z_1+c_1,z_2+c_2)\right)+\log 2$$
$$=m\left(r,f_j(z_1+c_1,z_2+c_2)\right)+S\left(r,f_j\right)=T\left(r,f_j(z_1+c_1,z_2+c_2)\right)+S\left(r,f_j\right),\quad j=1,2, \tag{30}$$

for all $r\in E=:E_1\cup E_2$. By using Lemma 9 and Lemma 11, it follows from (30) that

$$n_1T(r,f_2(z_1,z_2))\le n_1T(r,f_2(z_1+c_1,z_2+c_2))+S(r,f_2)\le T(r,P_1(z)f_2(z_1+c_1,z_2+c_2)^{n_1})+S(r,f_2)$$
$$=T\left(r,\left(\frac{\partial f_1}{\partial z_1}+\frac{\partial f_1}{\partial z_2}\right)^{m_1}-Q_1(z)\right)+S(r,f_2)=m_1T\left(r,\frac{\partial f_1}{\partial z_1}+\frac{\partial f_1}{\partial z_2}\right)+O(\log r)+S(r,f_2)+S(r,f_1)$$
$$=m_1m\left[r,\frac{\partial f_1}{\partial z_1}+\frac{\partial f_1}{\partial z_2}\right)+O(\log r)+S(r,f_2)+S(r,f_1)\le m_1\left[m\left(r,\frac{(\partial f_1/\partial z_1)+(\partial f_1/\partial z_2)}{f_1}\right)+m(r,f_1)\right]$$
$$+O(\log r)+S(r,f_1)+S(r,f_2)=m_1T(r,f_1)+O(\log r)+S(r,f_1)+S(r,f_2), \tag{31}$$

for all $r\in E$. Similarly, we have

$$n_2T(r,f_1)\le m_2T(r,f_2)+O(\log r)+S(r,f_1)+S(r,f_2),\quad r\in E. \tag{32}$$

In view of (31) and (32), it yields

$$(n_1n_2-m_1m_2)T\left(r,f_j\right)\le O(\log r)+S(r,f_1)+S(r,f_2),\quad r\in E. \tag{33}$$

In view of $n_1n_2>m_1m_2$, this is impossible since f_1,f_2 are transcendental entire functions.

Case 2. $m_j>(n_j/(n_j-1))$, $n_j\ge 2$, $j=1,2$. In view of the Nevanlinna second fundamental theorem concerning small functions, Lemma 10, and system (11), we can deduce that

$$m_1T\left(r,\frac{\partial f_1}{\partial z_1}+\frac{\partial f_1}{\partial z_2}\right)=T\left(r,\left(\frac{\partial f_1}{\partial z_1}+\frac{\partial f_1}{\partial z_2}\right)^{m_1}\right)+S(r,f_1)\le\bar N\left(r,\frac{1}{((\partial f_1/\partial z_1)+(\partial f_1/\partial z_2))^{m_1}}\right)+\bar N\left(r,\frac{1}{((\partial f_1/\partial z_1)+(\partial f_1/\partial z_2))^{m_1}-Q_1(z)}\right)$$
$$+S(r,f_1)\le\bar N\left(r,\frac{1}{((\partial f_1/\partial z_1)+(\partial f_1/\partial z_2))^{m_1}}\right)+\bar N\left(r,\frac{1}{P_1(z)f_2(z_1+c_1,z_2+c_2)^{n_1}}\right)+S(r,f_1)\le\bar N\left(r,\frac{1}{(\partial f_1/\partial z_1)+(\partial f_1/\partial z_2)}\right)$$
$$+\bar N\left(r,\frac{1}{f_2(z_1+c_1,z_2+c_2)}\right)+O(\log r)+S(r,f_1)\le T\left(r,\frac{\partial f_1}{\partial z_1}+\frac{\partial f_1}{\partial z_2}\right)+T(r,f_2(z_1+c_1,z_2+c_2))+O(\log r)+S(r,f_1)+S(r,f_2), \tag{34}$$

that is,

$$(m_1-1)T\left(r,\frac{\partial f_1}{\partial z_1}+\frac{\partial f_1}{\partial z_2}\right)\le T(r,f_2(z+c))+O(\log r)$$
$$+S(r,f_1)+S(r,f_2). \tag{35}$$

Similarly, we have

$$(m_2-1)T\left(r,\frac{\partial f_2}{\partial z_1}+\frac{\partial f_2}{\partial z_2}\right)\le T(r,f_1(z+c))+O(\log r)$$
$$+S(r,f_1)+S(r,f_2). \tag{36}$$

On the other hand, in view of system (10) and Lemma 10, it follows that

$$\begin{aligned}n_1T(r,f_2(z_1+c_1,z_2+c_2))+O(\log r)&=T(r,P_1(z)f_2(z_1+c_1,z_2+c_2)^{n_1})\\&+S(r,f_2)=T\left(r,\left(\frac{\partial f_1}{\partial z_1}+\frac{\partial f_1}{\partial z_2}\right)^{m_1}-Q_1(z)\right)\\&+S(r,f_2)=m_1T\left(r,\frac{\partial f_1}{\partial z_1}+\frac{\partial f_1}{\partial z_2}\right)+O(\log r)\\&+S(r,f_1)+S(r,f_2).\end{aligned} \tag{37}$$

Similarly, we have

$$\begin{aligned}n_2T(r,f_1(z_1+c_1,z_2+c_2))=m_2T\left(r,\frac{\partial f_2}{\partial z_1}+\frac{\partial f_2}{\partial z_2}\right)+O(\log r)\\+S(r,f_1)+S(r,f_2).\end{aligned} \tag{38}$$

In view of (35)–(38), we obtain that

$$\begin{aligned}\left(n_1-\frac{m_1}{m_1-1}\right)T(r,f_2(z_1+c_1,z_2+c_2))\le O(\log r)+S(r,f_1)+S(r,f_2),\\\left(n_2-\frac{m_2}{m_2-1}\right)T(r,f_1(z_1+c_1,z_2+c_2))\le O(\log r)+S(r,f_1)+S(r,f_2).\end{aligned} \tag{39}$$

The fact that $m_j>(n_j/(n_j-1))$ can lead to a contradiction since f_1,f_2 are transcendental entire functions.

Therefore, this completes the proof of Theorem 7.

3. Entire Solutions for System (8)

Now, the details that we obtain a pair of finite-order transcendental entire solutions for system (8) will be given below.

Let (f_1,f_2) be a pair of finite-order transcendental entire solutions for system (8). Differentiating both equations in system (8) for z_1, we deduce

$$\begin{cases}2\dfrac{\partial f_1(z_1,z_2)}{\partial z_1}\dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}+\dfrac{\partial f_2(z_1+c_1,z_2+c_2)}{\partial z_1}=0,\\2\dfrac{\partial f_2(z_1,z_2)}{\partial z_1}\dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1^2}+\dfrac{\partial f_1(z_1+c_1,z_2+c_2)}{\partial z_1}=0.\end{cases} \tag{40}$$

Let $F_1(z_1,z_2)=(\partial f_1(z_1,z_2))/\partial z_1$ and $F_2(z_1,z_2)=(\partial f_2(z_1,z_2))/\partial z_1$, then it follows from (18) that

$$\begin{cases}2F_1(z_1,z_2)\dfrac{\partial F_1(z_1,z_2)}{\partial z_1}=-F_2(z_1+c_1,z_2+c_2),\\2F_2(z_1,z_2)\dfrac{\partial F_2(z_1,z_2)}{\partial z_1}=-F_1(z_1+c_1,z_2+c_2)[rgb]0.00,0.00,1.00.\end{cases} \tag{41}$$

By Lemmas 9–11, it yields that $(\partial F_j(z_1,z_2))/\partial z_1=S(r,f_j)$ for $j=1,2$. Thus, we can assume that

$$\frac{\partial F_1(z_1,z_2)}{\partial z_1}=a_1,\ \frac{\partial F_2(z_1,z_2)}{\partial z_1}=a_2, \tag{42}$$

where $a_1,a_2\in\mathbb{C}$. Solving Equation (42), we have

$$F_1(z_1,z_2)=a_1z_1+\varphi_1(z_2),\ F_2(z_1,z_2)=a_2z_1+\varphi_2(z_2), \tag{43}$$

where $\varphi_1(z_2),\varphi_2(z_2)$ are finite-order transcendental entire functions in z_2. Due to Equations (41) and (42), we obtain that

$$F_1(z)=-\frac{1}{2a_1}F_2(z+c),\ F_2(z)=-\frac{1}{2a_2}F_1(z+c). \tag{44}$$

Substituting (43) into (44), we can deduce that

$$\begin{cases}a_1z_1+\varphi_1(z_2)=-\dfrac{1}{2a_1}(a_2z_1+a_2c_1)-\dfrac{1}{2a_1}\varphi_2(z_2+c_2),\\a_2z_1+\varphi_2(z_2)=-\dfrac{1}{2a_2}(a_1z_1+a_1c_1)-\dfrac{1}{2a_2}\varphi_1(z_2+c_2),\end{cases} \tag{45}$$

which implies that $a_1^3=a_2^3=-(1/8)$. It would be well if $a_1=a_2=-(1/2)$. So, it follows that

$$\begin{aligned}&F_1(z_1,z_2)=-\frac{1}{2}z_1+\varphi_1(z_2),\ F_2(z_1,z_2)=-\frac{1}{2}z_1+\varphi_2(z_2),\\&\varphi_1(z_2+c_2)=\varphi_2(z_2)+\frac{1}{2}c_1,\ \varphi_2(z_2+c_2)=\varphi_1(z_2)+\frac{1}{2}c_1.\end{aligned} \tag{46}$$

This means that

$$\varphi_1(z_2+2c_2)-\varphi_1(z_2)=c_1,\ \varphi_1(z_2+2c_2)-\varphi_1(z_2)=c_1, \tag{47}$$

which imply

$$\varphi_1(z_2)=G_1(z_2)+\frac{c_1}{2c_2}z_2,\ \varphi_2(z_2)=G_2(z_2)+\frac{c_1}{2c_2}z_2, \tag{48}$$

where $G_1(z_2),G_2(z_2)$ are finite-order entire period function with period $2c_2$ satisfying $G_2(z_2+c_2)=G_1(z_2)$.

Solving the following system

$$\begin{cases}\dfrac{\partial f_1(z_1,z_2)}{\partial z_1}=F_1(z_1,z_2)=-\dfrac{1}{2}z_1+\varphi_1(z_2),\\\dfrac{\partial f_2(z_1,z_2)}{\partial z_1}=F_2(z_1,z_2)=-\dfrac{1}{2}z_1+\varphi_2(z_2),\end{cases} \tag{49}$$

we obtain that

$$\begin{cases}f_1(z_1,z_2)=-\dfrac{1}{4}z_1^2+z_1\varphi_1(z_2)+\psi_1(z_2),\\f_2(z_1,z_2)=-\dfrac{1}{4}z_1^2+z_1\varphi_2(z_2)+\psi_2(z_2),\end{cases} \tag{50}$$

where $\psi_1(z_2), \psi_2(z_2)$ are finite-order entire functions in z_2. Substituting (50) into (8), and combining with the periodicity of $\varphi_1(z_2)$ and $\varphi_2(z_2)$, it yields that

$$\left(-\frac{1}{2}z_1+\frac{c_1}{2c_2}z_2+G_1(z_2)\right)^2-\frac{1}{4}(z_1+c_1)^2+(z_1+c_1)\varphi_2(z_2+c_2)+\psi_2(z_2+c_2)=1,$$
$$\left(-\frac{1}{2}z_1+\frac{c_1}{2c_2}z_2+G_2(z_2)\right)^2-\frac{1}{4}(z_1+c_1)^2+(z_1+c_1)\varphi_1(z_2+c_2)+\psi_1(z_2+c_2)=1. \quad (51)$$

Thus, we have

$$\psi_2(z_2+c_2)=1-\frac{1}{4}c_1^2-\frac{c_1^2}{2c_2}z_2-c_1G_1(z_2)-\left(\frac{c_1}{2c_2}z_2+G_1(z_2)\right)^2,$$
$$\psi_1(z_2+c_2)=1-\frac{1}{4}c_1^2-\frac{c_1^2}{2c_2}z_2-c_1G_2(z_2)-\left(\frac{c_1}{2c_2}z_2+G_2(z_2)\right)^2, \quad (52)$$

which mean that

$$\psi_2(z_2)=1+\frac{1}{4}c_1^2-\frac{c_1^2}{2c_2}z_2-c_1G_2(z_2)-\left(\frac{c_1}{2c_2}(z_2-c_2)+G_2(z_2)\right)^2, \quad (53)$$

$$\psi_1(z_2)=1+\frac{1}{4}c_1^2-\frac{c_1^2}{2c_2}z_2-c_1G_1(z_2)-\left(\frac{c_1}{2c_2}(z_2-c_2)+G_1(z_2)\right)^2. \quad (54)$$

In view of (48)–(54), it follows that

$$f_1(z_1,z_2)=1+\frac{1}{4}c_1^2-\frac{1}{4}z_1^2+\frac{c_1}{2c_2}z_1z_2-\frac{c_1^2}{2c_2}z_2+(z_1-c_1)G_1(z_2), -\left[\frac{c_1}{2c_2}(z_2-c_2)+G_1(z_2)\right]^2,$$
$$f_2(z_1,z_2)=1+\frac{1}{4}c_1^2-\frac{1}{4}z_1^2+\frac{c_1}{2c_2}z_1z_2-\frac{c_1^2}{2c_2}z_2+(z_1-c_1)G_2(z_2), -\left[\frac{c_1}{2c_2}(z_2-c_2)+G_2(z_2)\right]^2, \quad (55)$$

where $G_1(z_2), G_2(z_2)$ are finite-order transcendental entire period functions with period $2c_2$ satisfying $G_2(z_2+c_2) = G_1(z_2)$. Substituting (f_1,f_2) into system (2), it is easy to confirm that (f_1,f_2) is a solution of system (8).

4. Entire Solutions for System (11)

Let (f_1,f_2) be a pair of finite-order transcendental entire solutions of system (13). Next, the detail that we obtain one form of (f_1,f_2) is listed as follows. Differentiating system (13) for z_1, z_2, respectively, we have

$$\begin{cases} 2F_1(z_1,z_2)\left(\dfrac{\partial F_1(z_1,z_2)}{\partial z_1}+\dfrac{\partial F_1(z_1,z_2)}{\partial z_2}\right)+F_2(z_1+c_1,z_2+c_2)=0,\\ 2F_2(z_1,z_2)\left(\dfrac{\partial F_2(z_1,z_2)}{\partial z_1}+\dfrac{\partial F_2(z_1,z_2)}{\partial z_2}\right)+F_1(z_1+c_1,z_2+c_2)=0, \end{cases} \quad (56)$$

where

$$F_1(z_1,z_2)=\frac{\partial f_1(z_1,z_2)}{\partial z_1}+\frac{\partial f_1(z_1,z_2)}{\partial z_2},\quad \text{for } j=1,2. \quad (57)$$

In view of Lemmas 9–11, it follows that $(\partial F_j(z_1,z_2))/\partial z_1 = S(r,f_j)$ for $j=1,2$. For the convenience, assume that

$$\frac{\partial F_j(z_1,z_2)}{\partial z_1}+\frac{\partial F_j(z_1,z_2)}{\partial z_2}=b_j,\quad j=1,2, \quad (58)$$

where $b_j \in \mathbb{C}$. The characteristic equations for Equation (58) are

$$\frac{dz_1}{dt}=1,\ \frac{dz_2}{dt}=1,\ \frac{dF_j}{dt}=b_j. \quad (59)$$

In view of the initial conditions: $z_1=0, z_2=s$, and $F_j = F_j(0,s) := F_j(s)$ with a parameter s, we thus obtain the following parametric representation for the solutions of the characteristic equations: $z_1=t$, $z_2=t+s$,

$$F_j=\int_0^t b_j dt+\mu_j(s)=b_jt+\mu_j(s), \quad (60)$$

where $\mu_j(s), j=1,2$ are entire functions with finite order in s. Thus, it follows that

$$F_j(z_1,z_2)=b_jz_1+\mu_j(z_2-z_1),\quad j=1,2. \quad (61)$$

In view of (56) and (58), it follows that

$$\begin{cases} 2b_1F_1(z_1,z_2)=-F_2(z_1+c_1,z_2+c_2),\\ 2b_2F_2(z_1,z_2)=-F_1(z_1+c_1,z_2+c_2). \end{cases} \quad (62)$$

Substituting (61) into (62), we have that

$$\begin{cases} 2b_1^2 z_1 + 2b_1\mu_1(s) = -b_2(z_1+c_1) - \mu_2(s+s_0), \\ 2b_2^2 z_1 + 2b_1\mu_1(s) = -b_1(z_1+c_1) - \mu_1(s+s_0), \end{cases} \tag{63}$$

where $s = z_2 - z_1$ and $s_0 := c_2 - c_1$. This implies that $b_1^3 = b_2^3 = -(1/8)$. Let us assume that $b_1 = b_2 = -(1/8)$. Thus, it yields that

$$\begin{cases} \mu_2(s+s_0) = \mu_1(s) + \frac{1}{2}c_1, \\ \mu_1(s+s_0) = \mu_2(s) + \frac{1}{2}c_1. \end{cases} \tag{64}$$

This means

$$\mu_j(s) = G_j(s) + \tau s, \quad j = 1, 2, \tag{65}$$

$$G_2(s+s_0) = G_1(s), \tag{66}$$

where $G_1(s), G_2(s)$ are finite-order transcendental entire period functions with period $2s_0$, and $\tau = c_1/(2(c_2-c_1))$. Then, in view of (61) and (65), we deduce

$$F_j(z_1, z_2) = -\frac{1}{2}z_1 + G_j(z_2 - z_1) + \tau(z_2 - z_1), \quad j = 1, 2, \tag{67}$$

that is,

$$\frac{\partial f_j(z_1,z_2)}{\partial z_1} + \frac{\partial f_j(z_1,z_2)}{\partial z_2} = -\frac{1}{2}z_1 + G_j(z_2 - z_1) + \tau(z_2 - z_1). \tag{68}$$

By making use of the characteristic equations for Equation (68) again, let

$$\frac{dz_1}{dt} = 1, \frac{dz_2}{dt} = 1, \frac{df_j}{dt} = -\frac{1}{2}z_1 + G_j(z_2 - z_1) + \tau(z_2 - z_1). \tag{69}$$

In view of the initial conditions: $z_1 = 0, z_2 = s$, and $f_j = f_j(0, s) := f_j(s)$ with a parameter s, we can deduce that the parametric representation for the solutions of the characteristic equations: $z_1 = t$, $z_2 = t + s$, and

$$f_j = \int_0^t \left(-\frac{1}{2}t + G_j(s) + \tau s\right) dt + \nu_j(s) = -\frac{1}{4}t^2 + t\left(G_j(s) + \tau s\right) + \nu_j(s), \quad j = 1, 2, \tag{70}$$

where $\nu_j(s)$ is an entire function with finite order in s. Substituting $t = z_1$ and $s = z_2 - z_1$ into the above form, we have that

$$f_j(z_1, z_2) = -\frac{1}{4}z_1^2 + z_1\left[G_j(z_2 - z_1) + a_3(z_2 - z_1)\right] + \nu_j(z_2 - z_1), \quad j = 1, 2. \tag{71}$$

Substituting (71) into (13), and combining with the periodicity of $G_j(s)$, it follows that

$$\nu_1(s) = 1 - \frac{1}{4}c_1^2 - c_1G_2(s - s_0) - \tau c_1(s - s_0) - [G_2(s - s_0) + \tau(s - s_0)]^2, \tag{72}$$

$$\nu_2(s) = 1 - \frac{1}{4}c_1^2 - c_1G_1(s - s_0) - \tau c_1(s - s_0) - [G_1(s - s_0) + \tau(s - s_0)]^2. \tag{73}$$

Thus, in view of (66) and (71)–(73), we obtain that a pair of entire solutions of system (13) are of the forms

$$\begin{aligned} f_1(z_1,z_2) &= 1 + \frac{1}{4}c_1^2 - \frac{1}{4}z_1^2 + \frac{c_1}{2(c_2-c_1)}(z_2-z_1)(z_1-c_1) + (z_1-c_1)G_1(z_2-z_1) - \left[G_1(z_2-z_1) + \frac{c_1}{2(c_2-c_1)}(z_2-z_1-(c_2-c_1))\right]^2, \\ f_2(z_1,z_2) &= 1 + \frac{1}{4}c_1^2 - \frac{1}{4}z_1^2 + \frac{c_1}{2(c_2-c_1)}(z_2-z_1)(z_1-c_1) + (z_1-c_1)G_2(z_2-z_1) - \left[G_2(z_2-z_1) + \frac{c_1}{2(c_2-c_1)}(z_2-z_1-(c_2-c_1))\right]^2, \end{aligned} \tag{74}$$

where $G_1(s), G_2(s)$ are finite-order transcendental entire period functions with period $2s_0$ and satisfy (66). Let

$$G_1(s) = e^{(\pi i/(c_2-c_1))s}, G_2(s) = -e^{(\pi i/(c_2-c_1))s}. \tag{75}$$

Thus, (f_1, f_2) is a pair of finite-order transcendental entire solutions of system (13).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflict of interest.

Authors' Contributions

H. Y. Xu is responsible for the conceptualization; H.Y. Xu and H. Li for the writing—original draft preparation; H. Li and H. Y. Xu for the writing—review and editing; and H. Y. Xu and H. Li for the funding acquisition.

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