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Research Article

Iterative Solutions for Solving Variational Inequalities and Fixed-Point Problems

Li-Jun Zhu,[1,2] Naseer Shahzad ,[3] and Asim Asiri[3]

[1]*The Key Laboratory of Intelligent Information and Big Data Processing of Ningxia Province, North Minzu University, Yinchuan 750021, China*
[2]*Health Big Data Research Institute of North Minzu University, Yinchuan 750021, China*
[3]*Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*

Correspondence should be addressed to Naseer Shahzad; nshahzad@kau.edu.sa

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In this paper, we are interested in variational inequalities and fixed-point problems in Hilbert spaces. We present an iterative algorithm for finding a solution of the studied variational inequalities and fixed-point problems. We show the strong convergence of the suggested algorithm.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $C \subset H$ be a nonempty closed convex set. Let $f : C \longrightarrow H$, $g : C \longrightarrow H$, $\varphi : C \longrightarrow C$, and $T : C \longrightarrow C$ be four nonlinear operators. Use $\mathrm{Fix}(T)$ to denote the fixed-point set of T.

In this paper, we will investigate the following variational inequalities and fixed-point problems of finding a point $u^\dagger$ such that

$$\begin{aligned} u^\dagger &\in \mathrm{GVI}(C, f, \varphi), \\ \varphi(u^\dagger) &\in \mathrm{VI}(C, g) \cap \mathrm{Fix}(T), \end{aligned} \tag{1}$$

where $\mathrm{GVI}(C, f, \varphi)$ denotes the solution set of the generalized variational inequality (shortly, GVI) which is to find a point $x^\dagger \in C$ such that

$$\left\langle f(x^\dagger), \varphi(x) - \varphi(x^\dagger) \right\rangle \geq 0, \quad \forall x \in C, \tag{2}$$

and $\mathrm{VI}(C, g)$ means the solution set of the variational inequality (shortly, VI) which is to find a point $x^\dagger \in C$ such that

$$\left\langle g(x^\dagger), x - x^\dagger \right\rangle \geq 0, \quad \forall x \in C. \tag{3}$$

Throughout, we use Θ to denote the solution set of problem (1), that is,

$$\Theta = \mathrm{GVI}(C, f, \varphi) \bigcap \varphi^{-1}(\mathrm{VI}(C, g) \cap \mathrm{Fix}(T)). \tag{4}$$

It is well known that variational inequalities play key roles and provide a useful mathematical framework, theory, and method for studying many valuable problems arising in water resources, finance, economics, medical images, and so on ([1–6]). A lot of work and a great deal of algorithms for solving GVI or VI have been introduced and investigated, see, e.g., [7–15]. Among them, a basic and important algorithm is the projected algorithm which generates a sequence $\{x_n\}$ with the form

$$x_{n+1} = \mathrm{proj}_C[x_n - \kappa_n f(x_n)], \quad n \geq 0, \tag{5}$$

where κ_n is step-size and $\mathrm{proj}_C : H \longrightarrow C$ is the orthogonal projection.

At the same time, we are also interested in the fixed-point problem of finding a point $u^\dagger$ such that $Tu^\dagger = u^\dagger$. Iterative solution for solving a fixed-point problem is an active research field, see, e.g., [16–24]. Recently, iterative algorithms for solving variational inequalities and fixed-point problems have been investigated extensively by many authors [25–33].

Motivated by the work in this direction, in this paper, we devote to research variational inequalities and fixed-point problem (1). We introduce an iterative algorithm for finding a solution of problem (1). We show the strong convergence of the suggested algorithm.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that an operator $f : C \longrightarrow H$ is said to be

(i) strongly monotone if

$$\langle f(u) - f(v), u - v\rangle \geq \lambda \|u - v\|^2, \quad \forall u, v \in C \tag{6}$$

(ii) ϑ-inverse strongly φ-monotone if there exists a constant $\vartheta > 0$ such that

$$\langle f(u) - f(v), \varphi(u) - \varphi(v)\rangle \geq \vartheta \|f(u) - f(v)\|^2, \quad \forall u, v \in C \tag{7}$$

(iii) relaxed (μ, ν)-cocoercive [34, 35], if there exist two constants $\mu > 0, \nu > 0$ such that

$$\langle f(u) - f(v), u - v\rangle \geq (-\mu)\|f(u) - f(v)\|^2 + \nu\|u - v\|^2, \quad \forall u, v \in C \tag{8}$$

An operator $T : C \longrightarrow C$ is said to be

(i) pseudocontractive [36] if

$$\|T(u^\dagger) - T(v^\dagger)\|^2 \leq \|u^\dagger - v^\dagger\|^2 + \|(I - T)u^\dagger - (I - T)v^\dagger\|^2, \quad \forall u^\dagger, v^\dagger \in C \tag{9}$$

(ii) L-Lipschitz if

$$\|T(u^\dagger) - T(v^\dagger)\| \leq L\|u^\dagger - u^\dagger\|, \quad \forall u^\dagger, v^\dagger \in C, \tag{10}$$

where $L > 0$ is a constant

If $L < 1$, then T is said to be L-contraction. If $L = 1$, then T is said to be nonexpansive.

An operator $A : H \longrightarrow 2^H$ is said to be monotone if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in \text{dom}(A)$, $u \in A(x)$, and $v \in A(y)$. A monotone operator A on H is said to be maximal if its graph is not strictly contained in the graph of any other monotone operator on H.

For $\forall x^\dagger \in H$, there exists a unique point in C, denoted by $\text{proj}_C[x^\dagger]$ satisfying

$$\|x^\dagger - \text{proj}_C[x^\dagger]\| \leq \|x - x^\dagger\|, \quad \forall x \in C. \tag{11}$$

Moreover, proj_C is firmly nonexpansive, that is,

$$\|\text{proj}_C[u^\dagger] - \text{proj}_C[v^\dagger]\|^2 \leq \langle \text{proj}_C[u^\dagger] - \text{proj}_C[v^\dagger], u^\dagger - v^\dagger\rangle, \quad \forall u^\dagger, v^\dagger \in H. \tag{12}$$

Further, proj_C has the following property:

$$\langle u^\dagger - \text{proj}_C[u^\dagger], x^\dagger - \text{proj}_C[u^\dagger]\rangle \leq 0, \quad \forall u^\dagger \in H, x^\dagger \in C. \tag{13}$$

Lemma 1 ([37]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \longrightarrow C$ be an L -Lipschitz pseudocontractive operator. Then, $\forall x^\dagger \in C$ and $y^\dagger \in Fix(T)$, we have*

$$\begin{aligned} &\|(1 - \varsigma)x^\dagger + \varsigma T[(1 - \lambda)x^\dagger + \lambda T(x^\dagger)] - y^\dagger\|^2 \\ &\leq \varsigma(\varsigma - \lambda)\|T[(1 - \lambda)x^\dagger + \lambda T(x^\dagger)] - x^\dagger\|^2 + \|x^\dagger - y^\dagger\|^2, \end{aligned} \tag{14}$$

where $0 < \varsigma < \lambda < 1/(\sqrt{1 + L^2} + 1)$.

Lemma 2 ([24]). *Let C be a nonempty, convex, and closed subset of a Hilbert space H. Let $T : C \longrightarrow C$ be a continuous pseudocontractive operator. Then,*

(i) $Fix(T) \subset C$ is closed and convex

(ii) T is demiclosedness, i.e., $u_n \rightharpoonup \tilde{z}$ and $T(u_n) \longrightarrow z^\dagger$ imply that $T(\tilde{z}) = z^\dagger$

Lemma 3 ([23]). *Let $\{\varpi_n\} \subset [0,\infty)$, $\{\vartheta_n\} \subset (0, 1)$, and $\{\eta_n\}$ be real number sequences. Suppose the following conditions are satisfied:*

(i) $\varpi_{n+1} \leq (1 - \vartheta_n)\varpi_n + \eta_n, \forall n \geq 1$

(ii) $\sum_{n=1}^{\infty} \vartheta_n = \infty$

(iii) $\limsup_{n\longrightarrow\infty}(\eta_n/\vartheta_n) \leq 0$ or $\sum_{n=1}^{\infty} |\eta_n| < \infty$

Then, $\lim_{n\longrightarrow\infty}\varpi_n = 0$.

Lemma 4 ([38, 39]). *Let $\{x_n\}$ be a real number sequence. Assume there exists at least a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that*

$$x_{n_k} \leq x_{n_k+1}, \tag{15}$$

for all $k \geq 0$. For every $n \geq N_0$, define an integer sequence $\{\mu$

$(n)\}$ *as*

$$\mu(n) = \max\left\{i \le n : x_{n_i} < x_{n_i+1}\right\}. \tag{16}$$

Then, $\mu(n) \longrightarrow \infty$ as $n \longrightarrow \infty$ and for all $n \ge N_0$, max $\{x_{\mu(n)}, x_n\} \le x_{\mu(n)+1}$.

3. Main Results

In this section, we present our iterative algorithm and convergence theorem. Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that

(i) $\phi : C \longrightarrow C$ is a ρ-contractive operator

(ii) $\varphi : C \longrightarrow C$ is a weakly continuous and λ-strongly monotone operator such that its rang $R(\varphi) = C$

(iii) $f : C \longrightarrow H$ is a ϑ-inverse strongly φ-monotone operator

(iv) $g : C \longrightarrow H$ is an L_1-Lipschitz and relaxed (μ, ν)-cocoercive operator

(v) $T : C \longrightarrow C$ is an L_2-Lipschitz pseudocontractive operator with $L_2 > 1$

Let $\{\vartheta_n\}$, $\{\varsigma_n\}$, $\{\lambda_n\}$, and $\{\tau_n\}$ be four real number sequences in $[0, 1]$ and $\{\kappa_n\}$ and $\{\gamma_n\}$ be two real number sequences in $(0, \infty)$.

Now, we present our algorithm for solving problem (1).

Algorithm 5. Let $x_0 \in C$ be an initial value. Define the sequence $\{x_n\}$ by the following form:

$$\begin{cases} u_n = \vartheta_n\phi(x_n) + (1 - \vartheta_n)\mathrm{proj}_C[\varphi(x_n) - \kappa_n f(x_n)], \\ y_n = (1 - \varsigma_n)u_n + \varsigma_n T[(1 - \lambda_n)u_n + \lambda_n T(u_n)], \\ z_n = \mathrm{proj}_C[y_n - \gamma_n g(y_n)], \\ \varphi(x_{n+1}) = (1 - \tau_n)\varphi(x_n) + \tau_n z_n, \quad n \ge 0. \end{cases} \tag{17}$$

Theorem 6. *Suppose that $\Theta \ne \emptyset$. Assume that the following conditions are satisfied:*

(C1):$\lim_{n\longrightarrow\infty}\vartheta_n = 0$ *and* $\sum_{n=1}^{\infty}\vartheta_n = \infty$

(C2):$0 < a_1 < \varsigma_n < c_1 < \lambda_n < b_1 < 1/(\sqrt{1 + L_2^2} + 1)$ *for all* $n \ge 0$

(C3):$\nu > \mu L_1^2$ *and* $0 < a_2 \le \gamma_n \le b_2 < 2(\nu - \mu L_1^2)/L_1^2$ *for all* $n \ge 0$

(C4):$0 < \liminf_{n\longrightarrow\infty}\tau_n \le \limsup_{n\longrightarrow\infty}\tau_n < 1$

(C5):$0 < \rho < \lambda < 2\vartheta$ *and* $0 < \liminf_{n\longrightarrow\infty}\kappa_n \le \limsup_{n\longrightarrow\infty}\kappa_n < 2\vartheta$

Then, the sequence $\{x_n\}$ generated by (17) converges strongly to $u^\dagger \in \Theta$ verifying

$$\left\langle \phi(u^\dagger) - \varphi(u^\dagger), \varphi(x^\dagger) - \varphi(u^\dagger)\right\rangle \le 0, \quad \forall x^\dagger \in \Theta. \tag{18}$$

Proof. Since φ is λ-strongly monotone, we can get from (6) that

$$\|\varphi(u) - \varphi(v)\| \ge \lambda\|u - v\|, \quad \forall u, v \in C. \tag{19}$$

Thus, VI (18) has a unique solution, denoted by $u^\dagger$. It follows that $u^\dagger \in \mathrm{GVI}(C, f, \varphi)$ and $\varphi(u^\dagger) \in \mathrm{Fix}(T) \cap \mathrm{VI}(C, g)$. Using inequality (13), we can obtain that $\varphi(u^\dagger) = \mathrm{proj}_C[\varphi(u^\dagger) - \kappa_n f(u^\dagger)]$ for all $n \ge 0$.

Since f is ϑ-inverse strongly φ-monotone, for any $u \in C$, we have

$$\begin{aligned} &\left\|(\varphi(u) - \kappa f(u)) - \left(\varphi(u^\dagger) - \kappa f(u^\dagger)\right)\right\|^2 \\ &\quad = \left\|\varphi(u) - \varphi(u^\dagger)\right\|^2 - 2\kappa\left\langle f(u) - f(u^\dagger), \varphi(u) - \varphi(u^\dagger)\right\rangle \\ &\qquad + \kappa^2\left\|f(u) - f(u^\dagger)\right\|^2 \le \left\|\varphi(u) - \varphi(u^\dagger)\right\|^2 \\ &\qquad - 2\kappa\vartheta\left\|f(u) - f(u^\dagger)\right\|^2 + \kappa^2\left\|f(u) - f(u^\dagger)\right\|^2 \\ &\quad \le \left\|\varphi(u) - \varphi(u^\dagger)\right\|^2 + \kappa(\kappa - 2\vartheta)\left\|f(u) - f(u^\dagger)\right\|^2. \end{aligned} \tag{20}$$

Based on (20), we deduce

$$\begin{aligned} &\left\|(\varphi(x_n) - \kappa_n f(x_n)) - \left(\varphi(u^\dagger) - \kappa_n f(u^\dagger)\right)\right\|^2 \\ &\quad \le \left\|\varphi(x_n) - \varphi(u^\dagger)\right\|^2 + \kappa_n(\kappa_n - 2\vartheta)\left\|f(x_n) - f(u^\dagger)\right\|^2 \\ &\quad \le \left\|\varphi(x_n) - \varphi(u^\dagger)\right\|^2, \end{aligned} \tag{21}$$

$$\begin{aligned} &\|\varphi(x_{n+1}) - \kappa_{n+1}f(x_{n+1}) - (\varphi(x_n) - \kappa_{n+1}f(x_n))\|^2 \\ &\quad \le \|\varphi(x_{n+1}) - \varphi(x_n)\|^2 + \kappa_{n+1}(\kappa_{n+1} - 2\vartheta)\|f(x_{n+1}) - f(x_n)\|^2. \end{aligned} \tag{22}$$

By (17), (19), and (21), we derive

$$\begin{aligned} \left\|u_n - \varphi(u^\dagger)\right\| &= \|\vartheta_n\phi(x_n) + (1 - \vartheta_n)\mathrm{proj}_C[\varphi(x_n) - \kappa_n f(x_n)] \\ &\quad - \mathrm{proj}_C\left[\varphi(u^\dagger) - \kappa_n f(u^\dagger)\right]\| \\ &\le \left\|\vartheta_n\left(\phi(x_n) - \varphi(u^\dagger) + \kappa_n f(u^\dagger)\right)\right. \\ &\quad + (1 - \vartheta_n)\left((\varphi(x_n) - \kappa_n f(x_n)) - \left(\varphi(u^\dagger)\right.\right. \\ &\quad \left.\left.\left. - \kappa_n f(u^\dagger)\right)\right)\right\| \\ &\le \vartheta_n\left\|\phi(x_n) - \phi(u^\dagger)\right\| + \vartheta_n\left\|\phi(u^\dagger) - \varphi(u^\dagger)\right. \\ &\quad \left. + \kappa_n f(u^\dagger)\right\| + (1 - \vartheta_n)\|(\varphi(x_n) - \kappa_n f(x_n)) \\ &\quad - \left(\varphi(u^\dagger) - \kappa_n f(u^\dagger)\right)\| \le \vartheta_n\rho\left\|x_n - u^\dagger\right\| \\ &\quad + \vartheta_n\left\|\phi(u^\dagger) - \varphi(u^\dagger) + \kappa_n f(u^\dagger)\right\| \\ &\quad + (1 - \vartheta_n)\left\|\varphi(x_n) - \varphi(u^\dagger)\right\| \le \vartheta_n\frac{\rho}{\lambda}\|\varphi(x_n) \\ &\quad - \varphi(u^\dagger)\| + \vartheta_n\left\|\phi(u^\dagger) - \varphi(u^\dagger) + \kappa_n f(u^\dagger)\right\| \\ &\quad + (1 - \vartheta_n)\left\|\varphi(x_n) - \varphi(u^\dagger)\right\| \\ &= \left[1 - \left(1 - \frac{\rho}{\lambda}\right)\vartheta_n\right]\left\|\varphi(x_n) - \varphi(u^\dagger)\right\| + \vartheta_n\|\phi(u^\dagger) \\ &\quad - \varphi(u^\dagger) + \kappa_n f(u^\dagger)\| \le \left[1 - \left(1 - \frac{\rho}{\lambda}\right)\vartheta_n\right]\|\varphi(x_n) \\ &\quad - \varphi(u^\dagger)\| + \vartheta_n\left(\left\|\phi(u^\dagger) - \varphi(u^\dagger)\right\| + 2\vartheta\left\|f(u^\dagger)\right\|\right). \end{aligned} \tag{23}$$

According to (21) and (23), we obtain

$$\begin{aligned}\|u_n - \varphi(u^\dagger)\|^2 &\le \|\vartheta_n(\phi(x_n) - \varphi(u^\dagger) + \kappa_n f(u^\dagger)) + (1-\vartheta_n) \\ &\quad \cdot ((\varphi(x_n) - \kappa_n f(x_n)) - (\varphi(u^\dagger) - \kappa_n f(u^\dagger)))\|^2 \\ &\le \vartheta_n \|\phi(x_n) - \varphi(u^\dagger) + \kappa_n f(u^\dagger)\|^2 \\ &\quad + (1-\vartheta_n)\|(\varphi(x_n) - \kappa_n f(x_n)) - (\varphi(u^\dagger) - \kappa_n f(u^\dagger))\|^2 \\ &\le \vartheta_n \|\phi(x_n) - \varphi(u^\dagger) + \kappa_n f(u^\dagger)\|^2 + (1-\vartheta_n) \\ &\quad \cdot \left[\|\varphi(x_n) - \varphi(u^\dagger)\|^2 + \kappa_n(\kappa_n - 2\vartheta)\|f(x_n) - f(u^\dagger)\|^2\right].\end{aligned} \tag{24}$$

Applying Lemma 1 to (17), we have

$$\begin{aligned}\|y_n - \varphi(u^\dagger)\|^2 &= \|(1-\varsigma_n)u_n + \varsigma_n T[(1-\lambda_n)u_n + \lambda_n T(u_n)] - \varphi(u^\dagger)\|^2 \\ &\le \|u_n - \varphi(u^\dagger)\|^2 + \varsigma_n(\varsigma_n - \lambda_n)\|T[(1-\lambda_n)u_n + \lambda_n T(u_n)] - u_n\|^2 \\ &\le \|u_n - \varphi(u^\dagger)\|^2.\end{aligned} \tag{25}$$

Since g is relaxed (μ, ν)-cocoercive and L_1-Lipschitz, for all $u, \nu \in C$, we have

$$\begin{aligned}&\|(I - \gamma_n g)u - (I - \gamma_n g)\nu\|^2 \\ &\quad = \|u - \nu\|^2 - 2\gamma_n\langle g(u) - g(\nu), u - \nu\rangle + \gamma_n^2\|g(u) - g(\nu)\|^2 \\ &\quad \le \|u - \nu\|^2 - 2\gamma_n\left[-\mu\|g(u) - g(\nu)\|^2 + \nu\|u - \nu\|^2\right] + \gamma_n^2\|g(u) - g(\nu)\|^2 \\ &\quad \le \|u - \nu\|^2 + 2\gamma_n\mu L_1^2\|u - \nu\|^2 - 2\gamma_n\nu\|u - \nu\|^2 + \gamma_n^2 L_1^2\|u - \nu\|^2 \\ &\quad = \left(1 + 2\gamma_n\mu L_1^2 - 2\gamma_n\nu + \gamma_n^2 L_1^2\right)\|u - \nu\|^2.\end{aligned} \tag{26}$$

Since $0 < \gamma_n < 2(\nu - \mu L_1^2)/L_1^2$, $1 + 2\gamma_n\mu L_1^2 - 2\gamma_n\nu + \gamma_n^2 L_1^2 \le 1$. Thus, from (26), we obtain

$$\|(I - \gamma_n g)u - (I - \gamma_n g)\nu\| \le \|u - \nu\|, \quad \forall u, \nu \in C. \tag{27}$$

Hence,

$$\begin{aligned}\|z_n - \varphi(u^\dagger)\| &= \|\mathrm{proj}_C(I - \gamma_n g)y_n - \mathrm{proj}_C(I - \gamma_n g)\varphi(u^\dagger)\| \\ &\le \|(I - \gamma_n g)y_n - (I - \gamma_n g)\varphi(u^\dagger)\| \\ &\le \|y_n - \varphi(u^\dagger)\|.\end{aligned} \tag{28}$$

Combining (17), (23), (25), and (28), we obtain

$$\begin{aligned}\|\varphi(x_{n+1}) - \varphi(u^\dagger)\| &\le (1-\tau_n)\|\varphi(x_n) - \varphi(u^\dagger)\| + \tau_n\|z_n - \varphi(u^\dagger)\| \\ &\le (1-\tau_n)\|\varphi(x_n) - \varphi(u^\dagger)\| + \tau_n\|u_n - \varphi(u^\dagger)\| \\ &\le (1-\tau_n)\|\varphi(x_n) - \varphi(u^\dagger)\| + \tau_n \\ &\quad \cdot \left[1 - \left(1 - \frac{\rho}{\lambda}\right)\vartheta_n\right]\|\varphi(x_n) - \varphi(u^\dagger)\| + \tau_n\vartheta_n \\ &\quad \cdot (\|\phi(u^\dagger) - \varphi(u^\dagger)\| + 2\vartheta\|f(u^\dagger)\|) \\ &= \left[1 - \left(1 - \frac{\rho}{\lambda}\right)\tau_n\vartheta_n\right]\|\varphi(x_n) - \varphi(u^\dagger)\| \\ &\quad + \left(1 - \frac{\rho}{\lambda}\right)\tau_n\vartheta_n\frac{\|\phi(u^\dagger) - \varphi(u^\dagger)\| + 2\vartheta\|f(u^\dagger)\|}{1 - \rho/\lambda}.\end{aligned} \tag{29}$$

By induction, we have

$$\|\varphi(x_n) - \varphi(u^\dagger)\| \le \max\left\{\|\varphi(x_0) - \varphi(u^\dagger)\|, \frac{\|\phi(u^\dagger) - \varphi(u^\dagger)\| + 2\vartheta\|f(u^\dagger)\|}{1 - \rho/\lambda}\right\}. \tag{30}$$

It follows that

$$\begin{aligned}\|x_n - u^\dagger\| &\le \frac{1}{\lambda}\|\varphi(x_n) - \varphi(u^\dagger)\| \\ &\le \frac{1}{\lambda}\max\left\{\|\varphi(x_0) - \varphi(u^\dagger)\|, \frac{\|\phi(u^\dagger) - \varphi(u^\dagger)\| + 2\vartheta\|f(u^\dagger)\|}{1 - \rho/\lambda}\right\}.\end{aligned} \tag{31}$$

So, $\{\varphi(x_n)\}$, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ are bounded. From (17), we have

$$\varphi(x_{n+1}) - \varphi(x_n) = \tau_n(z_n - \varphi(x_n)), \quad n \ge 0. \tag{32}$$

It follows that

$$\langle \varphi(x_{n+1}) - \varphi(x_n), \varphi(x_n) - \varphi(u^\dagger)\rangle = \tau_n\langle z_n - \varphi(x_n), \varphi(x_n) - \varphi(u^\dagger)\rangle. \tag{33}$$

Thanks to (33), we deduce

$$\begin{aligned}&\|\varphi(x_{n+1}) - \varphi(u^\dagger)\|^2 - \|\varphi(x_n) - \varphi(u^\dagger)\|^2 - \|\varphi(x_{n+1}) - \varphi(x_n)\|^2 \\ &\quad = \tau_n\left[\|z_n - \varphi(u^\dagger)\|^2 - \|\varphi(x_n) - \varphi(u^\dagger)\|^2 - \|z_n - \varphi(x_n)\|^2\right].\end{aligned} \tag{34}$$

Combining (32) and (34), we obtain

$$\begin{aligned}&\|\varphi(x_{n+1}) - \varphi(u^\dagger)\|^2 - \|\varphi(x_n) - \varphi(u^\dagger)\|^2 \\ &\quad = \tau_n\left[\|z_n - \varphi(u^\dagger)\|^2 - \|\varphi(x_n) - \varphi(u^\dagger)\|^2 - \|z_n - \varphi(x_n)\|^2\right] \\ &\qquad + \tau_n^2\|z_n - \varphi(x_n)\|^2 \\ &\quad = \tau_n\left[\|z_n - \varphi(u^\dagger)\|^2 - \|\varphi(x_n) - \varphi(u^\dagger)\|^2\right] - \tau_n(1-\tau_n)\|z_n \\ &\qquad - \varphi(x_n)\|^2 \\ &\quad \le \tau_n\left[\|u_n - \varphi(u^\dagger)\|^2 - \|\varphi(x_n) - \varphi(u^\dagger)\|^2\right] - \tau_n(1-\tau_n)\|z_n \\ &\qquad - \varphi(x_n)\|^2.\end{aligned} \tag{35}$$

By virtue of (23), we get

$$\begin{aligned}\|u_n - \varphi(u^\dagger)\|^2 &\le \left[1 - \left(1 - \frac{\rho}{\lambda}\right)\vartheta_n\right]\|\varphi(x_n) - \varphi(u^\dagger)\|^2 \\ &\quad + \left(1 - \frac{\rho}{\lambda}\right)\vartheta_n\left(\frac{\|\phi(u^\dagger) - \varphi(u^\dagger)\| + 2\vartheta\|f(u^\dagger)\|}{1 - \rho/\lambda}\right)^2.\end{aligned} \tag{36}$$

Now, we consider two cases.

Case 1. There exists some integer $N_0 > 0$ such that $\{\|\varphi(x_n) - \varphi(u^\dagger)\|\}$ is decreasing when $n \ge N_0$. Then, $\lim_{n\longrightarrow\infty}\|\varphi(x_n) - \varphi(u^\dagger)\|$ exists. According to (35), (36), and $(C1)$, we have

$$\begin{aligned}\tau_n(1-\tau_n)\|z_n-\varphi(x_n)\|^2 &\le \|\varphi(x_n)-\varphi(u^\dagger)\|^2-\|\varphi(x_{n+1})-\varphi(u^\dagger)\|^2\\&\quad+\tau_n\left[\|u_n-\varphi(u^\dagger)\|^2-\|\varphi(x_n)-\varphi(u^\dagger)\|^2\right]\\&\le \|\varphi(x_n)-\varphi(u^\dagger)\|^2-\|\varphi(x_{n+1})-\varphi(u^\dagger)\|^2\\&\quad+\left(1-\frac{\rho}{\lambda}\right)\vartheta_n\left(\frac{\|\phi(u^\dagger)-\varphi(u^\dagger)\|+2\vartheta\|f(u^\dagger)\|}{1-\rho/\lambda}\right)^2\longrightarrow 0.\end{aligned}\tag{37}$$

This together with $(C4)$ implies that

$$\lim_{n\longrightarrow\infty}\|z_n-\varphi(x_n)\|=0.\tag{38}$$

Therefore, by (32), we have

$$\lim_{n\longrightarrow\infty}\|\varphi(x_{n+1})-\varphi(x_n)\|=0.\tag{39}$$

By (24), we have

$$\begin{aligned}\|\varphi(x_{n+1})-\varphi(u^\dagger)\|^2 &= \|(1-\tau_n)(\varphi(x_n)-\varphi(u^\dagger))+\tau_n(z_n-\varphi(u^\dagger))\|^2\\&\le (1-\tau_n)\|\varphi(x_n)-\varphi(u^\dagger)\|^2+\tau_n\|z_n-\varphi(u^\dagger)\|^2\\&\le (1-\tau_n)\|\varphi(x_n)-\varphi(u^\dagger)\|^2+\tau_n\|u_n-\varphi(u^\dagger)\|^2\\&\le (1-\tau_n)\|\varphi(x_n)-\varphi(u^\dagger)\|^2+\tau_n\vartheta_n\|\phi(x_n)\\&\quad-\varphi(u^\dagger)+\kappa_n f(u^\dagger)\|^2+\tau_n(1-\vartheta_n)\kappa_n(\kappa_n-2\vartheta)\\&\quad\cdot\|f(x_n)-f(u^\dagger)\|^2+\tau_n(1-\vartheta_n)\|\varphi(x_n)-\varphi(u^\dagger)\|^2\\&\le \|\varphi(x_n)-\varphi(u^\dagger)\|^2+\tau_n\vartheta_n\|\phi(x_n)-\varphi(u^\dagger)\\&\quad+\kappa_n f(u^\dagger)\|^2+\tau_n(1-\vartheta_n)\kappa_n(\kappa_n-2\vartheta)\|f(x_n)\\&\quad-f(u^\dagger)\|^2.\end{aligned}\tag{40}$$

It results in that

$$\begin{aligned}&\tau_n(1-\vartheta_n)\kappa_n(2\vartheta-\kappa_n)\|f(x_n)-f(u^\dagger)\|^2\\&\quad\le \|\varphi(x_n)-\varphi(u^\dagger)\|^2-\|\varphi(x_{n+1})-\varphi(u^\dagger)\|^2+\tau_n\vartheta_n\|\phi(x_n)\\&\qquad-\varphi(u^\dagger)+\kappa_n f(u^\dagger)\|^2\\&\quad\le \left(\|\varphi(x_n)-\varphi(u^\dagger)\|+\|\varphi(x_{n+1})-\varphi(u^\dagger)\|\right)\|\varphi(x_{n+1})-\varphi(x_n)\|\\&\qquad+\tau_n\vartheta_n\|\phi(x_n)-\varphi(u^\dagger)+\kappa_n f(u^\dagger)\|^2\longrightarrow 0.\end{aligned}\tag{41}$$

Hence,

$$\lim_{n\longrightarrow\infty}\|f(x_n)-f(u^\dagger)\|=0.\tag{42}$$

Set $\nu_n=\varphi(x_n)-\kappa_n f(x_n)-(\varphi(u^\dagger)-\kappa_n f(u^\dagger))$ for all $n\ge 0$.

Using (13) and (21), we have

$$\begin{aligned}&\|\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]-\varphi(u^\dagger)\|^2\\&\quad=\|\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]-\mathrm{proj}_C\left[\varphi(u^\dagger)-\kappa_n f(u^\dagger)\right]\|^2\\&\quad\le\left\langle \nu_n,\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]-\varphi(u^\dagger)\right\rangle\\&\quad=\frac{1}{2}\left\{\|\nu_n\|^2+\|\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\right.\\&\qquad-\varphi(u^\dagger)\|^2-\|\varphi(x_n)-\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\\&\qquad\left.-\kappa_n(f(x_n)-f(u^\dagger))\|^2\right\}\\&\quad\le\frac{1}{2}\left\{\|\varphi(x_n)-\varphi(u^\dagger)\|^2+\|\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\right.\\&\qquad-\varphi(u^\dagger)\|^2-\|\varphi(x_n)-\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\|^2\\&\qquad-\kappa_n^2\|f(x_n)-f(u^\dagger)\|+2\kappa_n\langle\varphi(x_n)-\mathrm{proj}_C\\&\qquad\left.\times[\varphi(x_n)-\kappa_n f(x_n)],f(x_n)-f(u^\dagger)\rangle\right\}.\end{aligned}\tag{43}$$

It yields

$$\begin{aligned}&\|\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]-\varphi(u^\dagger)\|^2\\&\quad\le\|\varphi(x_n)-\varphi(u^\dagger)\|^2-\kappa_n^2\|f(x_n)-f(u^\dagger)\|-\|\varphi(x_n)\\&\qquad-\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\|^2+2\kappa_n\langle\varphi(x_n)\\&\qquad-\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)],f(x_n)-f(u^\dagger)\rangle.\end{aligned}\tag{44}$$

In the light of (17) and (44), we have

$$\begin{aligned}\|u_n-\varphi(u^\dagger)\|^2&\le\vartheta_n\|\phi(x_n)-\varphi(u^\dagger)\|^2+(1-\vartheta_n)\|\mathrm{proj}_C\\&\quad\cdot[\varphi(x_n)-\kappa_n f(x_n)]-\varphi(u^\dagger)\|^2\\&\le\vartheta_n\|\phi(x_n)-\varphi(u^\dagger)\|^2+(1-\vartheta_n)\|\varphi(x_n)\\&\quad-\varphi(u^\dagger)\|^2+2\kappa_n\|\varphi(x_n)-\mathrm{proj}_C\\&\quad\cdot[\varphi(x_n)-\kappa_n f(x_n)]\|\|f(x_n)-f(u^\dagger)\\&\quad\cdot\|-(1-\vartheta_n)\|\varphi(x_n)-\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\|^2.\end{aligned}\tag{45}$$

Based on (40) and (45), we obtain

$$\begin{aligned}\|\varphi(x_{n+1})-\varphi(u^\dagger)\|^2&\le(1-\tau_n)\|\varphi(x_n)-\varphi(u^\dagger)\|^2+\tau_n\|u_n-\varphi(u^\dagger)\|^2\\&\le\|\varphi(x_n)-\varphi(u^\dagger)\|^2+\tau_n\vartheta_n\|\phi(x_n)-\varphi(u^\dagger)\|^2\\&\quad-\tau_n(1-\vartheta_n)\|\varphi(x_n)-\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\|^2\\&\quad+2\tau_n\kappa_n\|\varphi(x_n)-\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\|\|f(x_n)\\&\quad-f(u^\dagger)\|.\end{aligned}\tag{46}$$

Then,

$$\begin{aligned}&\tau_n(1-\vartheta_n)\|\varphi(x_n)-\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\|^2\\&\quad\le\left(\|\varphi(x_n)-\varphi(u^\dagger)\|+\|\varphi(x_{n+1})-\varphi(u^\dagger)\|\right)\|\varphi(x_{n+1})\\&\qquad-\varphi(x_n)\|+\tau_n\vartheta_n\|\phi(x_n)-\varphi(u^\dagger)\|^2+2\tau_n\kappa_n\|\varphi(x_n)\\&\qquad-\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\|\|f(x_n)-f(u^\dagger)\|.\end{aligned}\tag{47}$$

According to $(C1)$, $(C4)$, (39), (42), and (47), we deduce

$$\lim_{n\longrightarrow\infty}\|\varphi(x_n)-\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\|=0. \quad (48)$$

Since $u_n-\phi(x_n)=(1-\vartheta_n)(\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]-\phi(x_n))$, from (38), (39), and (48), we have

$$\lim_{n\longrightarrow\infty}\|\varphi(x_n)-u_n\|=\lim_{n\longrightarrow\infty}\|\varphi(x_{n+1})-u_n\|=\lim_{n\longrightarrow\infty}\|z_n-u_n\|=0. \quad (49)$$

From (26) and (28), we get

$$\begin{aligned}\|z_n-\varphi(u^\dagger)\|^2 &\le \|y_n-\varphi(u^\dagger)\|^2-2\gamma_n\left[-\mu\|g(y_n)-g(\varphi(u^\dagger))\|^2\right.\\&\quad\left.+\nu\|y_n-\varphi(u^\dagger)\|^2\right]+\gamma_n^2\|g(y_n)-g(\varphi(u^\dagger))\|^2\\&\le \|y_n-\varphi(u^\dagger)\|^2+\left(2\gamma_n\mu+\gamma_n^2-\frac{2\gamma_n\nu}{L_1^2}\right)\|g(y_n)\\&\quad-g(\varphi(u^\dagger))\|^2.\end{aligned} \quad (50)$$

It follows that

$$\begin{aligned}\|\varphi(x_{n+1})-\varphi(u^\dagger)\|^2 &\le (1-\tau_n)\|\varphi(x_n)-\varphi(u^\dagger)\|^2+\tau_n\|u_n\\&\quad-\varphi(u^\dagger)\|^2+\tau_n\left(2\gamma_n\mu+\gamma_n^2-\frac{2\gamma_n\nu}{L_1^2}\right)\\&\quad\cdot\|g(y_n)-g(\varphi(u^\dagger))\|^2,\end{aligned} \quad (51)$$

which together with (49) implies that

$$\begin{aligned}&-\tau_n\left(2\gamma_n\mu+\gamma_n^2-\frac{2\gamma_n\nu}{L_1^2}\right)\|g(y_n)-g(\varphi(u^\dagger))\|^2\\&\quad\le(1-\tau_n)\left(\|\varphi(x_n)-\varphi(u^\dagger)\|^2-\|\varphi(x_{n+1})-\varphi(u^\dagger)\|^2\right)\\&\qquad+\tau_n\left(\|u_n-\varphi(u^\dagger)\|^2-\|\varphi(x_{n+1})-\varphi(u^\dagger)\|^2\right)\longrightarrow 0.\end{aligned} \quad (52)$$

Therefore,

$$\lim_{n\longrightarrow\infty}\|g(y_n)-g(\varphi(u^\dagger))\|=0. \quad (53)$$

Since proj_C is firmly nonexpansive, from (12) and (28), we have

$$\begin{aligned}\|z_n-\varphi(u^\dagger)\|^2 &= \|\mathrm{proj}_C(I-\gamma_n g)y_n-\mathrm{proj}_C(I-\gamma_n g)\varphi(u^\dagger)\|^2\\&\le\left\langle(I-\gamma_n g)y_n-(I-\gamma_n g)\varphi(u^\dagger),z_n-\varphi(u^\dagger)\right\rangle\\&=\frac{1}{2}\left\{\|(I-\gamma_n g)y_n-(I-\gamma_n g)\varphi(u^\dagger)\|^2+\|z_n-\varphi(u^\dagger)\|^2\right.\\&\quad\left.-\|(I-\gamma_n g)y_n-(I-\gamma_n g)\varphi(u^\dagger)-(z_n-\varphi(u^\dagger))\|^2\right\}\\&\le\frac{1}{2}\left\{\|y_n-\varphi(u^\dagger)\|^2+\|z_n-\varphi(u^\dagger)\|^2-\|y_n-z_n\right.\\&\quad\left.-\gamma_n(g(y_n)-g(\varphi(u^\dagger)))\|^2\right\}\\&\le\frac{1}{2}\left\{\|u_n-\varphi(u^\dagger)\|^2+\|z_n-\varphi(u^\dagger)\|^2-\|y_n-z_n\|^2\right.\\&\quad-\gamma_n^2\|g(y_n)-g(\varphi(u^\dagger))\|^2\\&\quad\left.+2\gamma_n\left\langle g(y_n)-g(\varphi(u^\dagger)),y_n-z_n\right\rangle\right\},\end{aligned} \quad (54)$$

which yields

$$\begin{aligned}\|z_n-\varphi(u^\dagger)\|^2 &\le \|u_n-\varphi(u^\dagger)\|^2-\|y_n-z_n\|^2+2\gamma_n\|g(y_n)\\&\quad-g(\varphi(u^\dagger))\|\|y_n-z_n\|.\end{aligned} \quad (55)$$

This together with (40) implies that

$$\begin{aligned}\|\varphi(x_{n+1})-\varphi(u^\dagger)\|^2 &\le (1-\tau_n)\|\varphi(x_n)-\varphi(u^\dagger)\|^2+\tau_n\|u_n\\&\quad-\varphi(u^\dagger)\|^2-\tau_n\|y_n-z_n\|^2+2\tau_n\gamma_n\|g(y_n)\\&\quad-g(\varphi(u^\dagger))\|\|y_n-z_n\|.\end{aligned} \quad (56)$$

It follows that

$$\begin{aligned}\tau_n\|y_n-z_n\|^2 &\le (1-\tau_n)\|\varphi(x_n)-\varphi(u^\dagger)\|^2+\tau_n\|u_n-\varphi(u^\dagger)\|^2\\&\quad-\|\varphi(x_{n+1})-\varphi(u^\dagger)\|^2+2\tau_n\gamma_n\|g(y_n)\\&\quad-g(\varphi(u^\dagger))\|\|y_n-z_n\|\longrightarrow 0(\text{by (49) and (53)}).\end{aligned} \quad (57)$$

So,

$$\lim_{n\longrightarrow\infty}\|y_n-z_n\|=0. \quad (58)$$

By (49) and (58), we have

$$\lim_{n\longrightarrow\infty}\|y_n-u_n\|=0. \quad (59)$$

In view of (25) and (59), we get

$$\begin{aligned}&\varsigma_n(\lambda_n-\varsigma_n)\|T[(1-\lambda_n)u_n+\lambda_n T(u_n)]-u_n\|^2\\&\quad\le\|u_n-\varphi(u^\dagger)\|^2-\|y_n-\varphi(u^\dagger)\|^2\\&\quad\le\|u_n-y_n\|\left(\|u_n-\varphi(u^\dagger)\|+\|y_n-\varphi(u^\dagger)\|\right)\longrightarrow 0.\end{aligned} \quad (60)$$

It follows from $(C2)$ and (60) that

$$\lim_{n\longrightarrow\infty}\|T[(1-\lambda_n)u_n+\lambda_n T(u_n)]-u_n\|=0. \quad (61)$$

Since T is L_2-Lipschitz, we have

$$\begin{aligned}\|T(u_n)-u_n\| &\le \|T(u_n)-T[(1-\lambda_n)u_n+\lambda_n T(u_n)]\|+\|T\\ &\quad\cdot[(1-\lambda_n)u_n+\lambda_n T(u_n)]-u_n\| \le \lambda_n L_2\|T(u_n)\\ &\quad-u_n\|+\|T[(1-\lambda_n)u_n+\lambda_n T(u_n)]-u_n\|.\end{aligned} \tag{62}$$

Hence,

$$\|T(u_n)-u_n\| \le \frac{1}{1-\lambda_n L_2}\|T[(1-\lambda_n)u_n+\lambda_n T(u_n)]-u_n\|. \tag{63}$$

Owing to $(C2)$, (61) and (63), we deduce

$$\lim_{n\longrightarrow\infty}\|T(u_n)-u_n\|=0. \tag{64}$$

Next, we show that $\limsup_{n\longrightarrow\infty}\langle \phi(u^\dagger)-\varphi(u^\dagger), u_n-\varphi(u^\dagger)\rangle \le 0$. Let $\{u_{n_i}\}$ be a subsequence of $\{u_n\}$ such that

$$\limsup_{n\longrightarrow\infty}\langle \phi(u^\dagger)-\varphi(u^\dagger), u_n-\varphi(u^\dagger)\rangle = \lim_{i\longrightarrow\infty}\langle \phi(u^\dagger)-\varphi(u^\dagger), u_{n_i}-\varphi(u^\dagger)\rangle. \tag{65}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to some point $z\in C$. Without loss of generality, we may assume that $x_{n_i}\rightharpoonup z$. This implies that $\varphi(x_{n_i})\rightharpoonup\varphi(z)$ due to the weak continuity of φ. Thus, $u_{n_i}\rightharpoonup\varphi(z)$, $y_{n_i}\rightharpoonup\varphi(z)$, and $z_{n_i}\rightharpoonup\varphi(z)$. Applying Lemma 2 to (64) to deduce $\varphi(z)\in \mathrm{Fix}(T)$.

Now, we show that $\varphi(z)\in \mathrm{VI}(C,g)$. Let

$$S_1(v)=\begin{cases} g(v)+N_C v, & v\in C,\\ \varnothing, & v\notin C.\end{cases} \tag{66}$$

Since g is relaxed (μ,v)-cocoercive, for all $x,y\in C$, we have

$$\begin{aligned}\langle g(x)-g(y),x-y\rangle &\ge (-\mu)\|g(x)-g(y)\|^2+v\|x-y\|^2\\ &\ge (v-\mu L^2)\|x-y\|^2\ge 0,\end{aligned} \tag{67}$$

which implies that g is monotone and so S_1 is maximal monotone. Let $(v,u)\in G(S_1)$. Owing to $u-g(v)\in N_C v$ and $z_n\in C$, we get

$$\langle v-z_n, u-g(v)\rangle \ge 0. \tag{68}$$

According to $z_n=\mathrm{proj}_C(I-\gamma_n g)y_n$, we obtain

$$\langle v-z_n, z_n-(I-\gamma_n g)y_n\rangle \ge 0. \tag{69}$$

Then,

$$\left\langle v-z_n, \frac{z_n-y_n}{\gamma_n}+g(y_n)\right\rangle \ge 0. \tag{70}$$

It follows that

$$\begin{aligned}\langle v-z_{n_i},u\rangle &\ge \langle v-z_{n_i}, g(v)-g(z_{n_i})\rangle\\ &\quad+\langle v-z_{n_i}, g(z_{n_i})-g(y_{n_i})\rangle-\left\langle v-z_{n_i}, \frac{z_{n_i}-y_{n_i}}{\gamma_{n_i}}\right\rangle\\ &\ge \langle v-z_{n_i}, g(z_{n_i})-g(y_{n_i})\rangle-\left\langle v-z_{n_i}, \frac{z_{n_i}-y_{n_i}}{\gamma_{n_i}}\right\rangle.\end{aligned} \tag{71}$$

Since $z_{n_i}\rightharpoonup\varphi(z)$, $\|z_{n_i}-y_{n_i}\|\longrightarrow 0$, it follows from (71) that $\langle v-\varphi(z),u\rangle\ge 0$. Therefore, $\varphi(z)\in S_1^{-1}(0)$ and $\varphi(z)\in \mathrm{VI}(C,g)$.

Next, we prove $z\in \mathrm{GVI}(C,f,\varphi)$. Let

$$S_2(v)=\begin{cases} f(v)+N_C(v), & v\in C,\\ \varnothing, & v\in C.\end{cases} \tag{72}$$

It is known that S_2 is maximal φ-monotone. Let $(v,w)\in G(S_2)$. Since $w-f(v)\in N_C(v)$ and $x_n\in C$, we have $\langle\varphi(v)-\varphi(x_n), w-f(v)\rangle\ge 0$. Set $w_n=\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]$. Then,

$$\langle \varphi(v)-w_n, w_n-[\varphi(x_n)-\kappa_n f(x_n)]\rangle \ge 0. \tag{73}$$

It follows that

$$\left\langle \varphi(v)-w_n, \frac{w_n-\varphi(x_n)}{\kappa_n}+f(x_n)\right\rangle \ge 0. \tag{74}$$

Thus,

$$\begin{aligned}\langle\varphi(v)-\varphi(x_{n_i}),w\rangle &\ge \langle\varphi(v)-\varphi(x_{n_i}), f(v)-f(x_{n_i})\rangle\\ &\quad+\langle\varphi(v)-\varphi(x_{n_i}), f(x_{n_i})\rangle\\ &\quad-\left\langle\varphi(v)-w_{n_i}, \frac{w_{n_i}-\varphi(x_{n_i})}{\kappa_{n_i}}\right\rangle\\ &\quad-\langle\varphi(v)-w_{n_i}, f(x_{n_i})\rangle\\ &\ge-\left\langle\varphi(v)-w_{n_i}, \frac{w_{n_i}-\varphi(x_{n_i})}{\kappa_{n_i}}\right\rangle\\ &\quad-\langle\varphi(x_{n_i})-w_{n_i}, f(x_{n_i})\rangle.\end{aligned} \tag{75}$$

Since $\|\varphi(x_{n_i})-w_{n_i}\|\longrightarrow 0$ and $\varphi(x_{n_i})\rightharpoonup\varphi(z)$, we deduce that $\langle\varphi(v)-\varphi(z),w\rangle\ge 0$ by taking $i\longrightarrow\infty$ in (75). Thus, $z\in S_2^{-1}(0)$ by the maximal φ-monotonicity of S_2. Hence, $z\in \mathrm{GVI}(C,f,\varphi)$. Therefore, $z\in\varphi^{-1}(\mathrm{Fix}(T)\cap \mathrm{VI}(C,g))\bigcap \mathrm{GVI}(C,f,\varphi)=\Theta$.

From (49) and (65), we obtain

$$\begin{aligned}\limsup_{n\longrightarrow\infty}\left\langle\phi\left(u^{\dagger}\right)-\varphi\left(u^{\dagger}\right),u_n-\varphi\left(u^{\dagger}\right)\right\rangle\\ =\lim_{i\longrightarrow\infty}\left\langle\phi\left(u^{\dagger}\right)-\varphi\left(u^{\dagger}\right),\varphi\left(x_{n_i}\right)-\varphi\left(u^{\dagger}\right)\right\rangle\\ =\left\langle\phi\left(u^{\dagger}\right)-\varphi\left(u^{\dagger}\right),\varphi(z)-\varphi\left(u^{\dagger}\right)\right\rangle\le 0.\end{aligned}\tag{76}$$

By (17), we have

$$\begin{aligned}\|u_n-\varphi(u^{\dagger})\|^2&=\|\vartheta_n(\phi(x_n)-\varphi(u^{\dagger}))+(1-\vartheta_n)\\&\quad\cdot(\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]-\varphi(u^{\dagger}))\|^2\\&\le(1-\vartheta_n)^2\|\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]-\varphi(u^{\dagger})\|^2\\&\quad+2\vartheta_n\langle\phi(x_n)-\varphi(u^{\dagger}),u_n-\varphi(u^{\dagger})\rangle\\&\le(1-\vartheta_n)^2\|\varphi(x_n)-\varphi(u^{\dagger})\|^2\\&\quad+2\vartheta_n\langle\phi(x_n)-\phi(u^{\dagger}),u_n-\varphi(u^{\dagger})\rangle\\&\quad+2\vartheta_n\langle\phi(u^{\dagger})-\varphi(u^{\dagger}),u_n-\varphi(u^{\dagger})\rangle\\&\le(1-\vartheta_n)^2\|\varphi(x_n)-\varphi(u^{\dagger})\|^2+2\vartheta_n\frac{\rho}{\lambda}\|\varphi(x_n)\\&\quad-\varphi(u^{\dagger})\|\|u_n-\varphi(u^{\dagger})\|+2\vartheta_n\langle\phi(u^{\dagger})\\&\quad-\varphi(u^{\dagger}),u_n-\varphi(u^{\dagger})\rangle\\&\le(1-\vartheta_n)^2\|\varphi(x_n)-\varphi(u^{\dagger})\|^2+\vartheta_n\frac{\rho}{\lambda}\|\varphi(x_n)\\&\quad-\varphi(u^{\dagger})\|^2+\vartheta_n\frac{\rho}{\lambda}\|u_n-\varphi(u^{\dagger})\|^2+2\vartheta_n\\&\quad\cdot\langle\phi(u^{\dagger})-\varphi(u^{\dagger}),u_n-\varphi(u^{\dagger})\rangle.\end{aligned}\tag{77}$$

It follows that

$$\begin{aligned}\|u_n-\varphi(u^{\dagger})\|^2&\le\left[1-\frac{2(1-\rho/\lambda)\vartheta_n}{1-\vartheta_n\rho/\lambda}\right]\|\varphi(x_n)-\varphi(u^{\dagger})\|^2\\&\quad+\frac{\vartheta_n^2}{1-\vartheta_n\rho/\lambda}\|\varphi(x_n)-\varphi(u^{\dagger})\|^2\\&\quad+\frac{2\vartheta_n}{1-\vartheta_n\rho/\lambda}\langle\phi(u^{\dagger})-\varphi(u^{\dagger}),u_n-\varphi(u^{\dagger})\rangle.\end{aligned}\tag{78}$$

Set $M=\sup_n\|\varphi(x_n)-\varphi(u^{\dagger})\|^2$. Therefore,

$$\begin{aligned}\|\varphi(x_{n+1})-\varphi(u^{\dagger})\|^2&\le(1-\tau_n)\|\varphi(x_n)-\varphi(u^{\dagger})\|^2+\tau_n\|u_n-\varphi(u^{\dagger})\|^2\\&\le\left[1-\frac{2(1-\rho/\lambda)\vartheta_n\tau_n}{1-\vartheta_n\rho/\lambda}\right]\|\varphi(x_n)-\varphi(u^{\dagger})\|^2\\&\quad+\frac{\vartheta_n^2\tau_n}{1-\vartheta_n\rho/\lambda}\|\varphi(x_n)-\varphi(u^{\dagger})\|^2\\&\quad+\frac{2\vartheta_n\tau_n}{1-\vartheta_n\rho/\lambda}\langle\phi(u^{\dagger})-\varphi(u^{\dagger}),u_n-\varphi(u^{\dagger})\rangle\\&=\left[1-\frac{2(1-\rho/\lambda)\vartheta_n\tau_n}{1-\vartheta_n\rho/\lambda}\right]\|\varphi(x_n)-\varphi(u^{\dagger})\|^2+\frac{2(1-\rho/\lambda)\vartheta_n\tau_n}{1-\vartheta_n\rho/\lambda}\\&\quad\times\left\{\frac{\vartheta_n}{2(1-\rho/\lambda)}M+\frac{1}{1-\rho/\lambda}\langle\phi(u^{\dagger})-\varphi(u^{\dagger}),u_n-\varphi(u^{\dagger})\rangle\right\}.\end{aligned}\tag{79}$$

By Lemma 3 and (79), we conclude that $\varphi(x_n)\longrightarrow\varphi(u^{\dagger})$ and $x_n\longrightarrow u^{\dagger}$.

Case 2. For any N, there exists an integer $n_0>N$ such that $\|\varphi(x_{n_0})-\varphi(u^{\dagger})\|\le\|\varphi(x_{n_0+1})-\varphi(u^{\dagger})\|$. Let $\psi_n=\{\|\varphi(x_n)-\varphi(u^{\dagger})\|^2\}$. Then, we have $\psi_{n_0}\le\psi_{n_0+1}$. Let $\{\mu_n\}$ be an integer sequence defined by, for all $n\ge n_0$,

$$\mu(n)=\max\{l\in\mathbb{N}\mid n_0\le l\le n,\psi_l\le\psi_{l+1}\}.\tag{80}$$

Note that $\mu(n)$ is nondecreasing and satisfies $\lim_{n\longrightarrow\infty}\mu(n)=\infty$ and $\psi_{\mu(n)}\le\psi_{\mu(n)+1},\forall n\ge n_0$.

Similarly, we can deduce

$$\limsup_{n\longrightarrow\infty}\left\langle\phi(u^{\dagger})-\varphi(u^{\dagger}),u_{\mu(n)}-\varphi(u^{\dagger})\right\rangle\le 0\tag{81}$$

$$\begin{aligned}\psi_{\mu(n)+1}&\le\left[1-\frac{2(1-\rho/\lambda)\vartheta_\mu(n)\tau_\mu(n)}{1-\vartheta_\mu(n)\rho/\lambda}\right]\psi_{\mu(n)}+\frac{2(1-\rho/\lambda)\vartheta_\mu(n)\tau_\mu(n)}{1-\vartheta_\mu(n)\rho/\lambda}\\&\quad\times\left\{\frac{\vartheta_\mu(n)}{2(1-\rho/\lambda)}M+\frac{1}{1-\rho/\lambda}\langle\phi(u^{\dagger})-\varphi(u^{\dagger}),u_\mu(n)-\varphi(u^{\dagger})\rangle\right\}.\end{aligned}\tag{82}$$

Note that $\psi_{\mu(n)}\le\psi_{\mu(n)+1}$. By (82), we have

$$\psi_{\mu(n)}\le\frac{\vartheta_\mu(n)}{2(1-\rho/\lambda)}M+\frac{1}{1-\rho/\lambda}\langle\phi(u^{\dagger})-\varphi(u^{\dagger}),u_\mu(n)-\varphi(u^{\dagger})\rangle.\tag{83}$$

Based on (81) and (83), we derive

$$\limsup_{n\longrightarrow\infty}\psi_{\mu(n)}\le 0,\tag{84}$$

and thus,

$$\lim_{n\longrightarrow\infty}\psi_{\mu(n)}=0.\tag{85}$$

From (81) and (82), we can deduce

$$\limsup_{n\longrightarrow\infty}\psi_{\mu(n)+1}\le\limsup_{n\longrightarrow\infty}\psi_{\mu(n)}.\tag{86}$$

This together with (85) implies that

$$\lim_{n\longrightarrow\infty}\psi_{\mu(n)+1}=0.\tag{87}$$

By Lemma 4, we obtain

$$0\le\psi_n\le\max\left\{\psi_{\mu(n)},\psi_{\mu(n)+1}\right\}.\tag{88}$$

Therefore, $\psi_n\longrightarrow 0$. That is, $\varphi(x_n)\longrightarrow\varphi(u^{\dagger})$ and thus $x_n\longrightarrow u^{\dagger}$. This completes the proof.

In Algorithm 5, choose $\varphi=I$, identity operator, and $f:C\longrightarrow H$ is a ϑ-inverse strongly monotone operator. Then, we have the following algorithm and corollary.

Algorithm 7. Let $x_0 \in C$ be an initial value. Define the sequence $\{x_n\}$ by the following form:

$$\begin{cases} u_n = \vartheta_n \phi(x_n) + (1-\vartheta_n)\mathrm{proj}_C[x_n - \kappa_n f(x_n)], \\ y_n = (1-\varsigma_n)u_n + \varsigma_n T[(1-\lambda_n)u_n + \lambda_n T(u_n)], \\ z_n = \mathrm{proj}_C[y_n - \gamma_n g(y_n)], \\ x_{n+1} = (1-\tau_n)x_n + \tau_n z_n, \quad n \geq 0. \end{cases} \tag{89}$$

Corollary 8. *Suppose that $\Theta_1 := VI(C, f) \cap VI(C, g) \cap Fix(T) \neq \emptyset$. Assume that conditions (C1)-(C5) are satisfied. Then, the sequence $\{x_n\}$ generated by (89) converges strongly to $v^\dagger = proj_{\Theta_1}\phi(v^\dagger)$.*

Algorithm 9. Let $x_0 \in C$ be an initial value. Define the sequence $\{x_n\}$ by the following form:

$$\begin{cases} u_n = \vartheta_n \phi(x_n) + (1-\vartheta_n)\mathrm{proj}_C[\varphi(x_n) - \kappa_n f(x_n)], \\ z_n = \mathrm{proj}_C[u_n - \gamma_n g(u_n)], \\ \varphi(x_{n+1}) = (1-\tau_n)\varphi(x_n) + \tau_n z_n, \quad n \geq 0. \end{cases} \tag{90}$$

Corollary 10. *Suppose that $\Theta_2 := GVI(C, f, \varphi) \cap \varphi^{-1}(VI(C, g)) \neq \emptyset$. Assume that conditions (C1) and (C3)-(C5) are satisfied. Then, the sequence $\{x_n\}$ generated by (90) converges strongly to $u^\dagger \in \Theta_2$ verifying*

$$\left\langle \phi\left(u^\dagger\right) - \varphi\left(u^\dagger\right), \varphi\left(x^\dagger\right) - \varphi\left(u^\dagger\right) \right\rangle \leq 0, \quad \forall x^\dagger \in \Theta_2. \tag{91}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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