Research Article

Iterative Solutions for Solving Variational Inequalities and Fixed-Point Problems

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In this paper, we are interested in variational inequalities and fixed-point problems in Hilbert spaces. We present an iterative algorithm for finding a solution of the studied variational inequalities and fixed-point problems. We show the strong convergence of the suggested algorithm.

1. Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $C \subset H$ be a nonempty closed convex set. Let $f : C \rightarrow H$, $g : C \rightarrow H$, $\varphi : C \rightarrow C$, and $T : C \rightarrow C$ be four nonlinear operators. Use $\text{Fix}(T)$ to denote the fixed-point set of $T$.

In this paper, we will investigate the following variational inequalities and fixed-point problems of finding a point $u^* \in C$ such that

$$
\begin{align*}
&u^* \in \text{GVI}(C, f, \varphi), \\
&\varphi(u^*) \in \text{VI}(C, g) \cap \text{Fix}(T),
\end{align*}
$$

where $\text{GVI}(C, f, \varphi)$ denotes the solution set of the generalized variational inequality (shortly, GVI) which is to find a point $x^* \in C$ such that

$$
\langle f(x^*), \varphi(x) - \varphi(x^*) \rangle \geq 0, \quad \forall x \in C,
$$

and $\text{VI}(C, g)$ means the solution set of the variational inequality (shortly, VI) which is to find a point $x^* \in C$ such that

$$
\langle g(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C.
$$

Throughout, we use $\Theta$ to denote the solution set of problem (1), that is,

$$
\Theta = \text{GVI}(C, f, \varphi) \cap \varphi^{-1}(\text{VI}(C, g) \cap \text{Fix}(T)).
$$

It is well known that variational inequalities play key roles and provide a useful mathematical framework, theory, and method for studying many valuable problems arising in water resources, finance, economics, medical images, and so on ([1–6]). A lot of work and a great deal of algorithms for solving GVI or VI have been introduced and investigated, see, e.g., [7–15]. Among them, a basic and important algorithm is the projected algorithm which generates a sequence $\{x_n\}$ with the form

$$
x_{n+1} = \text{proj}_C[x_n - \kappa_n f(x_n)], \quad n \geq 0,
$$

where $\kappa_n$ is step-size and $\text{proj}_C : H \rightarrow C$ is the orthogonal projection.

At the same time, we are also interested in the fixed-point problem of finding a point $u^*$ such that $Tu^* = u^*$. Iterative
solution for solving a fixed-point problem is an active research field, see, e.g., [16–24]. Recently, iterative algorithms for solving variational inequalities and fixed-point problems have been investigated extensively by many authors [25–33].

Motivated by the work in this direction, in this paper, we devote to research variational inequalities and fixed-point problem (1). We introduce an iterative algorithm for finding a solution of problem (1). We show the strong convergence of the suggested algorithm.

2. Preliminaries

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Recall that an operator $f : C \to H$ is said to be

(i) strongly monotone if

$$\langle f(u) - f(v), u - v \rangle \geq \lambda \| u - v \|^2, \quad \forall u, v \in C$$  \hspace{1cm} (6)

(ii) $\theta$-inverse strongly $\varphi$-monotone if there exists a constant $\theta > 0$ such that

$$\langle f(u) - f(v), \varphi(u) - \varphi(v) \rangle \geq \theta \| f(u) - f(v) \|^2 + \varphi(u) - \varphi(v), \quad \forall u, v \in C$$  \hspace{1cm} (7)

(iii) relaxed $(\mu, \nu)$-cocoercive [34, 35], if there exist two constants $\mu > 0, \nu > 0$ such that

$$\langle f(u) - f(v), u - v \rangle \geq (-\mu) \| f(u) - f(v) \|^2 + \nu \| u - v \|^2, \quad \forall u, v \in C$$  \hspace{1cm} (8)

An operator $T : C \to C$ is said to be

(i) pseudocontractive [36] if

$$\| T(u) - T(v) \|^2 \leq \| u - v \|^2 + \| (I - T)u - (I - T)v \|^2, \quad \forall u, v \in C$$  \hspace{1cm} (9)

(ii) $L$-Lipschitz if

$$\| T(u) - T(v) \| \leq L \| u - v \|, \quad \forall u, v \in C$$  \hspace{1cm} (10)

where $L > 0$ is a constant.

If $L < 1$, then $T$ is said to be $L$-contraction. If $L = 1$, then $T$ is said to be nonexpansive.

An operator $A : H \to 2^H$ is said to be monotone if $\langle x - y, u - v \rangle \geq 0$ for all $x, y \in \text{dom}(A), u \in A(x)$, and $v \in A(y)$. A monotone operator $A$ on $H$ is said to be maximal if its graph is not strictly contained in the graph of any other monotone operator on $H$.

For $\forall x^* \in H$, there exists a unique point in $C$, denoted by $\text{proj}_C(x^*)$ satisfying

$$\| x^* - \text{proj}_C(x^*) \| \leq \| x - x^* \|, \quad \forall x \in C.$$  \hspace{1cm} (11)

Moreover, $\text{proj}_C$ is firmly nonexpansive, that is,

$$\| \text{proj}_C(u^*) - \text{proj}_C(v^*) \|^2 \leq \langle \text{proj}_C(u^*) - \text{proj}_C(v^*), u^* - v^* \rangle, \quad \forall u^*, v^* \in H.$$  \hspace{1cm} (12)

Further, $\text{proj}_C$ has the following property:

$$\langle u^* - \text{proj}_C(u^*), x^* - \text{proj}_C(u^*) \rangle \leq 0, \quad \forall u^* \in H, x^* \in C.$$  \hspace{1cm} (13)

Lemma 1 ([37]). Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ be an $L$-Lipschitz pseudocontractive operator. Then, $\forall x^* \in C$ and $y^* \in \text{Fix}(T)$, we have

$$\| (1 - \zeta)x^* + \zeta T[(1 - \lambda)x^* + \lambda T(x^*)] - y^* \| \|^2 \leq \zeta(c(\zeta - \lambda))\| T[(1 - \lambda)x^* + \lambda T(x^*)] - x^* \|^2 + \| x^* - y^* \|^2,$$  \hspace{1cm} (14)

where $0 < \zeta < \lambda < 1/(\sqrt{1 + L^2} + 1)$.

Lemma 2 ([24]). Let $C$ be a nonempty, convex, and closed subset of a Hilbert space $H$. Let $T : C \to C$ be a continuous pseudocontractive operator. Then,

(i) $\text{Fix}(T) \subseteq C$ is closed and convex

(ii) $T$ is demiclosedness, i.e., $u_n \to z$ and $T(u_n) \to z^*$ imply that $T(z) = z^*$

Lemma 3 ([23]). Let $\{ \alpha_n \} \subseteq [0, \infty), \{ \theta_n \} \subseteq (0, 1)$, and $\{ \eta_n \}$ be real number sequences. Suppose the following conditions are satisfied:

(i) $\forall n \geq 1, \{ (1 - \theta_n)\alpha_n + \eta_n \} \subseteq [0, 1]$

(ii) $\sum_{n=1}^{\infty} \theta_n = \infty$

(iii) $\limsup_{n \to \infty} (\eta_n/\theta_n) \leq 0$ or $\sum_{n=1}^{\infty} | \eta_n | < \infty$

Then, $\lim_{n \to \infty} \alpha_n \alpha_n = 0$.

Lemma 4 ([38, 39]). Let $\{ x_n \}$ be a real number sequence. Assume there exists at least a subsequence $\{ x_{n_k} \}$ of $\{ x_n \}$ such that

$$x_{n_k} \leq x_{n_{k+1}}, \quad \forall k \geq 0,$$  \hspace{1cm} (15)

for all $k \geq 0$. For every $n \geq N_\omega$, define an integer sequence $\{ \mu_n \}$
\[ \mu(n) = \max \{ i \leq n : x_n < x_{n+1} \} \]  

(16)

Then, \( \mu(n) \to \infty \) as \( n \to \infty \) and for all \( n \geq N_0 \), max \( \{ x_{\mu(n)} , x_n \} \leq x_{\mu(n)+1} \).

3. Main Results

In this section, we present our iterative algorithm and convergence theorem. Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Assume that

(i) \( \phi : C \to C \) is a \( \rho \)-contractive operator

(ii) \( \varphi : C \to C \) is a weakly continuous and \( \lambda \)-strongly monotone operator such that its range \( R(\varphi) = C \)

(iii) \( f : C \to H \) is a \( \Theta \)-inverse strongly \( \varphi \)-monotone operator

(iv) \( g : C \to H \) is an \( L_1 \)-Lipschitz and relaxed \( (\mu, \nu) \)-cocoercive operator

(v) \( T : C \to C \) is an \( L_2 \)-Lipschitz pseudocontractive operator with \( L_2 > 1 \)

Let \( \{ \theta_n \} \), \( \{ \kappa_n \} \), \( \{ \lambda_n \} \), and \( \{ \tau_n \} \) be four real number sequences in \( [0, 1] \) and \( \{ \kappa_n \} \) and \( \{ \gamma_n \} \) be two real number sequences in \( (0, \infty) \).

Now, we present our algorithm for solving problem (1).

Algorithm 5. Let \( x_0 \in C \) be an initial value. Define the sequence \( \{ x_n \} \) by the following form:

\[
\begin{align*}
    u_n &= \theta_n \phi(x_n) + (1 - \theta_n) \text{proj}_C[\varphi(x_n) - \kappa_n f(x_n)], \\
    y_n &= (1 - c_n) u_n + c_n T[(1 - \lambda_n) u_n + \lambda_n T(u_n)], \\
    z_n &= \text{proj}_C[y_n - \gamma_n T(y_n)], \\
    \phi(x_{n+1}) &= (1 - \tau_n) \phi(x_n) + \tau_n z_n, \quad n \geq 0.
\end{align*}
\]

(17)

Theorem 6. Suppose that \( \Theta \neq \emptyset \). Assume that the following conditions are satisfied:

(C1): \( \lim_{n \to \infty} \theta_n = 0 \) and \( \sum_{n=1}^{\infty} \theta_n = \infty \)

(C2): \( 0 < a_1 < c_n < c_1 < \lambda_n < b_1 < 1/(\sqrt{1 + L_2^2 + 1}) \) for all \( n \geq 0 \)

(C3): \( \nu > \mu L_1^2 \) and \( 0 < a_2 \leq \gamma_n \leq b_2 \leq 2(\nu - \mu L_1^2)/L_1^2 \) for all \( n \geq 0 \)

(C4): \( \rho < \lambda < \lambda_{\infty} \) and \( 0 < \lim \inf_{n \to \infty} \tau_n \leq 1 \)

(C5): \( \rho < \lambda < \lambda_{\infty} \) and \( 0 < \lim \inf_{n \to \infty} \kappa_n \leq \lambda_{\infty} \leq \lambda_{\infty} < 2 \beta \)

Then, the sequence \( \{ x_n \} \) generated by (17) converges strongly to \( u^* \in \Theta \) verifying

\[ ||\phi(u^*) - \varphi(u^*)|| > 0, \quad \forall x^* \in \Theta. \]

(18)

Proof. Since \( \varphi \) is \( \lambda \)-strongly monotone, we can get from (6) that

\[ ||\varphi(u) - \varphi(v)|| > \lambda ||u - v||, \quad \forall u, v \in C. \]

Thus, VI (18) has a unique solution, denoted by \( u^* \). It follows that \( u^* \in \text{GVI}(C, f, \varphi) \) and \( \varphi(u^*) \in \text{Fix}(T) \cap \text{VI}(C, g) \).

Using inequality (13), we can obtain that \( \varphi(u^*) = \text{proj}_C[\varphi(u^*) - \kappa_n f(u^*)] \) for all \( n \geq 0 \).

Since \( f \) is \( \Theta \)-inverse strongly \( \varphi \)-monotone, for any \( u \in C \), we have

\[ \begin{align*}
    &||\varphi(u) - \varphi(u')|| - ||\varphi(u') - \varphi(u)||^2 \\
    &= ||\varphi(u) - \varphi(u')||^2 - 2\kappa(f(u) - f(u'), \varphi(u) - \varphi(u')) \\
    &\quad + \kappa^2 ||f(u) - f(u')||^2 \\
    &\leq ||\varphi(u) - \varphi(u')||^2 + \kappa||f(u) - f(u')||^2 \\
    &\leq ||\varphi(u) - \varphi(u')||^2 + \kappa(\kappa - 2\beta)||f(u) - f(u')||^2.
\end{align*} \]

(20)

Based on (20), we deduce

\[ \begin{align*}
    &||\varphi(x_n) - \varphi(u') - \kappa_n f(x_n)||^2 \\
    &\leq ||\varphi(x_n) - \varphi(u')||^2 + \kappa_n(\kappa_n - 2\beta)||f(x_n) - f(u')||^2 \\
    &\leq ||\varphi(x_n) - \varphi(u')||^2,
\end{align*} \]

(21)

\[ \begin{align*}
    &||\varphi(x_{n+1}) - \varphi(u') - \kappa_{n+1} f(x_{n+1})||^2 \\
    &\leq ||\varphi(x_{n+1}) - \varphi(u')||^2 + \kappa_n(\kappa_n - 2\beta)||f(x_{n+1}) - f(x_n)||^2.
\end{align*} \]

(22)

By (17), (19), and (21), we derive

\[ \begin{align*}
    &||u_n - \varphi(u')|| = ||\theta_n \phi(x_n) + (1 - \theta_n) \text{proj}_C[\varphi(x_n) - \kappa_n f(x_n)] \\
    &\quad - \text{proj}_C[\varphi(u') - \kappa_n f(u')]|| \\
    &\quad \leq ||\theta_n \phi(x_n) - \varphi(u') + \kappa_n f(u')|| \\
    &\quad + (1 - \theta_n) ||\varphi(x_n) - \kappa_n f(x_n) - (\varphi(u') - \kappa_n f(u'))|| \\
    &\quad \leq \theta_n ||\phi(x_n) - \varphi(u')|| + \theta_n ||\varphi(u') - \varphi(u')|| \\
    &\quad + \kappa_n(\kappa_n - 2\beta)||f(x_n) - f(u')|| \\
    &\quad - \theta_n \kappa_n f(x_n) - (\varphi(u') - \kappa_n f(u'))|| \\
    &\quad + (1 - \theta_n) \kappa_n f(u') ||f(x_n) - f(u')|| \\
    &\quad \leq \theta_n \rho ||f(x_n) - f(u')|| \\
    &\quad + \theta_n ||f(x_n) - f(u')|| \\
    &\quad - \varphi(u') ||f(x_n) - f(u')|| + \theta_n ||\varphi(u') - \varphi(u') + \kappa_n f(u')|| \\
    &\quad + (1 - \theta_n) ||\varphi(u') - \varphi(u')|| \\
    &\quad = [1 - (1 - \theta_n) \rho] \kappa_n f(x_n) - \varphi(u') + \theta_n ||\varphi(u') - \varphi(u')|| + \theta_n ||\varphi(u') - \varphi(u')|| \\
    &\quad - \varphi(u') + \kappa_n f(u') ||f(x_n) - f(u')|| + \theta_n ||\varphi(u') - \varphi(u')|| + 2\beta ||f(u')||^2.
\end{align*} \]

(23)
According to (21) and (23), we obtain
\[
\|u_n - \varphi(u')\|^2 \leq \| \varphi(x_n) - \varphi(u') + k_n f(u') \| + (1 - \theta_n) \\
\cdot \left[ \| \varphi(x_n) - k_n f(x_n) - (\varphi(u') - k_n f(u')) \| \right]^2 \\
\leq \theta_n \| \varphi(x_n) - \varphi(u') + k_n f(u') \|^2 \\
+ (1 - \theta_n) \| \varphi(x_n) - k_n f(x_n) - (\varphi(u') - k_n f(u')) \|^2 \\
\leq \theta_n \| \varphi(x_n) - \varphi(u') + k_n f(u') \|^2 + (1 - \theta_n) \\
\cdot \left[ \| \varphi(x_n) - \varphi(u') \| + k_n (k_n - 2\theta) \| f(x_n) - f(u') \| \right]^2.
\] (24)

Applying Lemma 1 to (17), we have
\[
\|v_n - \varphi(u')\|^2 = \|(1 - c_n)u_n + c_n T((1 - \lambda_n)u_n + \lambda_n T(u_n)) - \varphi(u')\|^2 \\
\leq \|u_n - \varphi(u')\|^2 + c_n(1 - \lambda_n)u_n + \lambda_n T(u_n) - u_n\|^2 \\
\leq \|u_n - \varphi(u')\|^2. 
\] (25)

Since $g$ is relaxed $(\mu, v)$-cocoercive and $L_1$-Lipschitz, for all $u, v \in C$, we have
\[
\|(I - g_\gamma g)u - (I - g_\gamma g)v\|^2 \\
= \|u - v\|^2 - 2g_\gamma g_\gamma g(u) - g(v), u - v\| + y_\gamma^2\gamma g(u) - g(v)\|^2 \\
\leq \|u - v\|^2 - 2g_\gamma [\mu_\gamma g_\gamma g(u) - g(v)] + vu - v^2 + y_\gamma^2\gamma g(u) - g(v)\|^2 \\
\leq \|u - v\|^2 + 2\mu_\gamma y_\gamma^2\| u - v \|^2 - 2v - v^2 + y_\gamma^2\gamma g(u) - g(v)\|^2 \\
= (1 + 2\mu_\gamma y_\gamma^2\gamma + 2\gamma_\gamma^2\gamma)\| u - v \|^2. 
\] (26)

Since $0 < y_\gamma < 2(\nu - \mu y_\gamma^2\gamma)/L_\gamma^2$, $1 + 2\mu_\gamma y_\gamma^2\gamma + 2\gamma_\gamma^2\gamma \leq 1$. Thus, from (26), we obtain
\[
\|(I - g_\gamma g)u - (I - g_\gamma g)v\|^2 \leq \|u - v\|^2, \quad \forall u, v \in C. 
\] (27)

Hence,
\[
\|z_n - \varphi(u')\| = \| \text{proj}_{C}(I - g_\gamma g)y_n - \text{proj}_{C}(I - g_\gamma g)\varphi(u') \| \\
\leq \|(I - g_\gamma g)y_n - (I - g_\gamma g)\varphi(u') \| \\
\leq \|y_n - \varphi(u') \|. 
\] (28)

Combining (17), (23), (25), and (28), we obtain
\[
\|\varphi(x_{n+1}) - \varphi(u')\| \leq (1 - \tau_n)\|\varphi(x_n) - \varphi(u')\| + \tau_n\|z_n - \varphi(u')\| \\
\leq (1 - \tau_n)\|\varphi(x_n) - \varphi(u')\| + \tau_n\|u_n - \varphi(u')\| \\
\leq (1 - \tau_n)\|\varphi(x_n) - \varphi(u')\| + \tau_n \\cdot \left[ 1 - \left(1 - \frac{\rho}{\lambda}\right)\|g_\gamma g_\gamma g - \varphi(u')\| + \tau_n \|g_\gamma g_\gamma g - \varphi(u')\| \right] \\
= \left[1 - \left(1 - \frac{\rho}{\lambda}\right)\|g_\gamma g_\gamma g - \varphi(u')\| \right] \|\varphi(x_n) - \varphi(u')\| \\
+ \left(1 - \frac{\rho}{\lambda}\right)\|g_\gamma g_\gamma g - \varphi(u')\| \|\varphi(x_{n+1}) - \varphi(u')\| + 2\|g_\gamma g_\gamma g\| \|\varphi(u')\| \|u_n - \varphi(u')\| \\
\leq \left[1 - \left(1 - \frac{\rho}{\lambda}\right)\|g_\gamma g_\gamma g - \varphi(u')\| \right] \|\varphi(x_n) - \varphi(u')\| \\
+ \left(1 - \frac{\rho}{\lambda}\right)\|g_\gamma g_\gamma g - \varphi(u')\| \|\varphi(u')\| l \|u_n - \varphi(u')\|. 
\] (29)

By induction, we have
\[
\|\varphi(x_n) - \varphi(u')\| \leq \max \left\{ \|\varphi(x_n) - \varphi(u')\|, \frac{\|\varphi(u') - \varphi(u')\| + 2\|f(u')\|}{1 - \rho/\lambda} \right\}. 
\] (30)

It follows that
\[
\|x_n - u\|^2 \leq \frac{1}{\lambda} \|\varphi(x_n) - \varphi(u')\|^2 \\
\leq \frac{1}{\lambda} \max \left\{ \|\varphi(x_n) - \varphi(u')\|, \frac{\|\varphi(u') - \varphi(u')\| + 2\|f(u')\|}{1 - \rho/\lambda} \right\}. 
\] (31)

So, $\{\varphi(x_n)\}, \{x_n\}, \{y_n\}, \{z_n\}$, and $\{u_n\}$ are bounded. From (17), we have
\[
\varphi(x_{n+1}) - \varphi(x_n) = \tau_n(z_n - \varphi(x_n)), \quad n \geq 0. 
\] (32)

It follows that
\[
\langle \varphi(x_{n+1}) - \varphi(x_n), \varphi(x_n) - \varphi(u') \rangle = \tau_n(z_n - \varphi(x_n), \varphi(x_n) - \varphi(u')). 
\] (33)

Thanks to (33), we deduce
\[
\|\varphi(x_{n+1}) - \varphi(u')\|^2 - \|\varphi(x_n) - \varphi(u')\|^2 - \|\varphi(x_{n+1}) - \varphi(x_n)\|^2 \\
= \tau_n \|z_n - \varphi(u')\|^2 - \|\varphi(x_n) - \varphi(u')\|^2 - \|z_n - \varphi(x_n)\|^2. 
\] (34)

Combining (32) and (34), we obtain
\[
\|\varphi(x_{n+1}) - \varphi(u')\|^2 - \|\varphi(x_n) - \varphi(u')\|^2 \\
= \tau_n \|z_n - \varphi(u')\|^2 - \|\varphi(x_n) - \varphi(u')\|^2 - \|z_n - \varphi(x_n)\|^2 \\
= \tau_n \|z_n - \varphi(u')\|^2 - \|\varphi(x_n) - \varphi(u')\|^2 - \tau_n (1 - \tau_n) \|z_n - \varphi(x_n)\|^2 \\
\leq \tau_n \|u_n - \varphi(u')\|^2 - \|\varphi(x_n) - \varphi(u')\|^2 - \tau_n (1 - \tau_n) \|z_n - \varphi(x_n)\|^2. 
\] (35)

By virtue of (23), we get
\[
\|u_n - \varphi(u')\|^2 \leq \left[ 1 - \left(1 - \frac{\rho}{\lambda}\right)\|g_\gamma g_\gamma g - \varphi(u')\| \right] \|\varphi(x_n) - \varphi(u')\|^2 \\
+ \left(1 - \frac{\rho}{\lambda}\right)\|g_\gamma g_\gamma g - \varphi(u')\| \|\varphi(u')\| l \|u_n - \varphi(u')\|. 
\] (36)

Now, we consider two cases.

Case 1. There exists some integer $N_0 > 0$ such that $\{\|\varphi(x_n) - \varphi(u')\|\}$ is decreasing when $n \geq N_0$. Then, $\lim_{n \to \infty} \|\varphi(x_n) - \varphi(u')\|$ exists. According to (35), (36), and (C1), we
have
\[
\begin{align*}
\tau_n(1-t_n)\|z_n - \phi(x_n)\|^2 &\leq \|\phi(x_n) - \phi(u^*)\|^2 - \|\phi(x_{n+1}) - \phi(u^*)\|^2 \\
&\quad + \tau_n(1-t_n)\|u_n - \phi(u^*)\|^2 - \|\phi(x_n) - \phi(u^*)\|^2 \\
&\leq \|\phi(x_n) - \phi(u^*)\|^2 - \|\phi(x_{n+1}) - \phi(u^*)\|^2 \\
&\quad + \left(1 - \frac{\rho_n}{\rho}\right)\|\phi(x_n) - \phi(u^*)\|^2 + \frac{2\theta_n\|\phi(x_n) - \phi(u^*)\|^2}{1 - \rho\lambda} \longrightarrow 0. \tag{37}
\end{align*}
\]

This together with (C4) implies that
\[
\lim_{n \to \infty} \|z_n - \phi(x_n)\| = 0. \tag{38}
\]

Therefore, by (32), we have
\[
\lim_{n \to \infty} \|\phi(x_{n+1}) - \phi(x_n)\| = 0. \tag{39}
\]

By (24), we have
\[
\begin{align*}
\|\phi(x_{n+1}) - \phi(u^*)\|^2 &= \|(1 - \tau_n)(\phi(x_n) - \phi(u^*)) + \tau_n(z_n - \phi(u^*))\|^2 \\
&\leq (1 - \tau_n)\|\phi(x_n) - \phi(u^*)\|^2 + \tau_n\|z_n - \phi(u^*)\|^2 \\
&\leq (1 - \tau_n)\|\phi(x_n) - \phi(u^*)\|^2 + \tau_n\|u_n - \phi(u^*)\|^2 \\
&\leq (1 - \tau_n)\|\phi(x_n) - \phi(u^*)\|^2 + \tau_n\|\phi(x_n) - \phi(u^*)\|^2 \\
&\quad - \|\phi(u^*) + \kappa_n f(u^*)\|^2 + \tau_n(1 - \delta_n)\|\phi(x_n) - \phi(u^*)\|^2 \\
&\leq \|\phi(x_n) - \phi(u^*)\|^2 + \tau_n\|\phi(x_n) - \phi(u^*)\|^2 \\
&\quad + \kappa_n f(u^*)\|^2 + \tau_n(1 - \delta_n)\|\phi(x_n) - \phi(u^*)\|^2 \\
&\quad - \|\phi(u^*)\|^2. \tag{40}
\end{align*}
\]

It results in that
\[
\begin{align*}
\tau_n(1 - \delta_n)\kappa_n(2\theta_n - \kappa_n)\|f(x_n) - f(u^*)\|^2 \\
&\leq \|\phi(x_n) - \phi(u^*)\|^2 - \|\phi(x_{n+1}) - \phi(u^*)\|^2 + \tau_n\|\phi(x_n) - \phi(u^*)\|^2 \\
&\quad - \|\phi(u^*) + \kappa_n f(u^*)\|^2 \\
&\leq (\|\phi(x_n) - \phi(u^*)\|^2 + \|\phi(x_{n+1}) - \phi(u^*)\|^2)\|\phi(x_n) - \phi(u^*)\|^2 \\
&\quad + \tau_n\|\phi(x_n) - \phi(u^*) + \kappa_n f(u^*)\|^2 \longrightarrow 0. \tag{41}
\end{align*}
\]

Hence,
\[
\lim_{n \to \infty} \|f(x_n) - f(u^*)\| = 0. \tag{42}
\]

Set \(v_n = \phi(x_n) - \kappa_n f(x_n) - (\phi(u^*) - \kappa_n f(u^*))\) for all \(n \geq 0\).

Using (13) and (21), we have
\[
\begin{align*}
\|\text{proj}_C[\phi(x_n) - \kappa_n f(x_n)] - \phi(u^*)\|^2 \\
&= \|\text{proj}_C[\phi(x_n) - \kappa_n f(x_n)] - \text{proj}_C[\phi(u^*) - \kappa_n f(u^*)]\|^2 \\
&\leq \langle v_n, \text{proj}_C[\phi(x_n) - \kappa_n f(x_n)] - \phi(u^*) \rangle \\
&= \frac{1}{2} \{\|v_n\|^2 + \|\text{proj}_C[\phi(x_n) - \kappa_n f(x_n)]\|^2 \}
\end{align*}
\]

\[
\begin{align*}
&\quad - \|\phi(u^*)\|^2 - \|\phi(x_n) - \kappa_n f(x_n)\| - \kappa_n(f(x_n) - f(u^*))\|^2 \\
&\leq \frac{1}{2} \{\|\phi(x_n) - \phi(u^*)\|^2 + \|\text{proj}_C[\phi(x_n) - \kappa_n f(x_n)]\|^2 \}
\end{align*}
\]

\[
\begin{align*}
&\quad - \|\phi(u^*)\|^2 - \|\phi(x_n) - \kappa_n f(x_n)\| - \kappa_n(f(x_n) - f(u^*))\|^2 \\
&\quad - \kappa_n\|f(x_n) - f(u^*)\|^2 + 2\kappa_n(f(x_n) - \text{proj}_C[\phi(x_n) - \kappa_n f(x_n)])\|f(x_n) - \text{proj}_C[\phi(x_n) - \kappa_n f(x_n)]\|f(x_n) - f(u^*)\rangle.
\end{align*}
\]

In the light of (17) and (44), we have
\[
\begin{align*}
\|u_n - \phi(u^*)\|^2 &\leq \delta_n\|\phi(x_n) - \phi(u^*)\|^2 + (1 - \delta_n)\|\phi(x_n) - \phi(u^*)\|^2 \\
&\leq \delta_n\|\phi(x_n) - \phi(u^*)\|^2 + (1 - \delta_n)\|\phi(x_n) - \phi(u^*)\|^2 \\
&\quad - \|\phi(u^*)\|^2 + 2\kappa_n\|\phi(x_n) - \text{proj}_C[\phi(x_n) - \kappa_n f(x_n)]\|^2 \\
&\quad - \|\phi(x_n) - \text{proj}_C[\phi(x_n) - \kappa_n f(x_n)]\|^2 \|f(x_n) - \text{proj}_C[\phi(x_n) - \kappa_n f(x_n)]\|^2. \tag{45}
\end{align*}
\]

Based on (40) and (45), we obtain
\[
\begin{align*}
\|\phi(x_{n+1}) - \phi(u^*)\|^2 \leq (1 - \tau_n)\|\phi(x_n) - \phi(u^*)\|^2 + \tau_n\|u_n - \phi(u^*)\|^2 \\
&\leq \|\phi(x_n) - \phi(u^*)\|^2 + \tau_n\|\phi(x_n) - \phi(u^*)\|^2 \\
&\quad - \|\phi(u^*)\|^2 + \tau_n(1 - \delta_n)\|\phi(x_n) - \phi(u^*)\|^2 \\
&\quad + 2\tau_n\kappa_n\|\phi(x_n) - \text{proj}_C[\phi(x_n) - \kappa_n f(x_n)]\|^2 \|f(x_n) - \text{proj}_C[\phi(x_n) - \kappa_n f(x_n)]\|^2. \tag{46}
\end{align*}
\]

Then,
\[
\begin{align*}
\tau_n(1 - \delta_n)\|\phi(x_n) - \text{proj}_C[\phi(x_n) - \kappa_n f(x_n)]\|^2 \\
&\leq (\|\phi(x_n) - \phi(u^*)\|^2 + \|\phi(x_{n+1}) - \phi(u^*)\|^2)\|\phi(x_n) - \phi(u^*)\|^2 \\
&\quad - \|\phi(x_n)\|^2 + \tau_n(1 - \delta_n)\|\phi(x_n) - \phi(u^*)\|^2 \\
&\quad + 2\tau_n\kappa_n\|\phi(x_n) - \text{proj}_C[\phi(x_n) - \kappa_n f(x_n)]\|^2 \|f(x_n) - \text{proj}_C[\phi(x_n) - \kappa_n f(x_n)]\|^2. \tag{47}
\end{align*}
\]
According to (C1), (C4), (39), (42), and (47), we deduce

\[ \lim_{n \to \infty} \| \varphi(x_n) - \text{proj}_C[\varphi(x_n) - \kappa_n f(x_n)] \| = 0. \]  

(48)

Since \( u_n - \varphi(x_n) = (1 - \beta_n)(\text{proj}_C[\varphi(x_n) - \kappa_n f(x_n)] - \varphi(x_n)) \), from (38), (39), and (48), we have

\[ \lim_{n \to \infty} \| \varphi(x_n) - u_n \| = \lim_{n \to \infty} \| \varphi(x_{n+1}) - u_n \| = \lim_{n \to \infty} \| z_n - u_n \| = 0. \]  

(49)

From (26) and (28), we get

\[ \| z_n - \varphi(u^\tau) \|^2 \leq \| y_n - \varphi(u^\tau) \|^2 - 2 \gamma_n \langle \mu \|g(y_n) - g(\varphi(u^\tau)) \|, \| y_n - \varphi(u^\tau) \| \rangle + \| y_n - \varphi(u^\tau) \|^2 \]

\[ + \nu \| y_n - \varphi(u^\tau) \|^2 \]

\[ = \| y_n - \varphi(u^\tau) \|^2 + \left( 2 \gamma_n \mu + \gamma_n^2 \right) \| g(y_n) - g(\varphi(u^\tau)) \|^2 \]

\[ - \| \varphi(u^\tau) \|^2 + \| y_n - \varphi(u^\tau) \|^2 \]

\[ - \| y_n - \varphi(u^\tau) \|^2 + \left( 2 \gamma_n \mu + \gamma_n^2 \right) \| g(y_n) - g(\varphi(u^\tau)) \|^2. \]

(50)

It follows that

\[ \| \varphi(x_{n+1}) - \varphi(u^\tau) \|^2 \leq (1 - \tau_n) \| \varphi(x_n) - \varphi(u^\tau) \|^2 + \tau_n \| u_n - \varphi(u^\tau) \|^2 + \tau_n \left( 2 \gamma_n \mu + \gamma_n^2 \right) \| g(y_n) - g(\varphi(u^\tau)) \|^2 \]

\[ \leq (1 - \tau_n) \| \varphi(x_n) - \varphi(u^\tau) \|^2 - \| \varphi(x_{n+1}) - \varphi(u^\tau) \|^2 \]

\[ + \tau_n \left( 2 \gamma_n \mu + \gamma_n^2 \right) \| g(y_n) - g(\varphi(u^\tau)) \|^2 \to 0. \]  

(51)

which together with (49) implies that

\[ \lim_{n \to \infty} \| g(y_n) - g(\varphi(u^\tau)) \| = 0. \]  

(53)

Since proj\(_C\) is firmly nonexpansive, from (12) and (28), we have

\[ \| z_n - \varphi(u^\tau) \|^2 = \| \text{proj}_C[(1 - \gamma_n)u_n - \lambda_n T(u_n)] - u_n \|^2 \]

\[ \leq \| (1 - \gamma_n)u_n - \lambda_n T(u_n) - u_n \|^2 \]

\[ \leq \| (1 - \gamma_n)u_n - \varphi(u^\tau) \|^2 + 2 \| y_n - \varphi(u^\tau) \| \| y_n - \varphi(u^\tau) \| + \| y_n - \varphi(u^\tau) \|^2 \]

(60)

In view of (25) and (59), we get

\[ \| y_n - u_n \|^2 \leq \| u_n - \varphi(u^\tau) \|^2 + \| y_n - \varphi(u^\tau) \|^2 \]

\[ \leq \| y_n - \varphi(u^\tau) \|^2 + \| y_n - \varphi(u^\tau) \|^2 + \| y_n - \varphi(u^\tau) \|^2 \to 0. \]  

(61)
Since $T$ is $L_2$-Lipschitz, we have
\[
\|T(u_n) - u_n\| \leq \|T(u_n) - T[(1 - \lambda_n)u_n + \lambda_n T(u_n)]\| + \|T[(1 - \lambda_n)u_n + \lambda_n T(u_n)] - u_n\| \leq \lambda_n L_2\|T(u_n) - u_n\|.
\]

Hence,
\[
\|T(u_n) - u_n\| \leq \frac{1}{1 - \lambda_n L_2}\|T[(1 - \lambda_n)u_n + \lambda_n T(u_n)] - u_n\|.
\]

Owing to (C2), (61) and (63), we deduce
\[
\lim_{n \to \infty} \|T(u_n) - u_n\| = 0.
\]

Next, we show that \(\limsup_{n \to \infty} (\phi(u') - \phi(y'), u_n - \phi(u')) \leq 0\). Let \(u_n\) be a subsequence of \(\{u_n\}\) such that
\[
\limsup_{n \to \infty} (\phi(u') - \phi(y'), u_n - \phi(u')) = \lim_{n \to \infty} (\phi(u') - \phi(y'), u_n - \phi(u')).
\]

Since \(\{x_n\}\) is bounded, there exists a subsequence \(\{x_{n_i}\}\) of \(\{x_n\}\) which converges weakly to some point \(z \in C\). Without loss of generality, we may assume that \(x_{n_i} \to z\). This implies that \(\phi(z) = \phi(z_{n_i})\) due to the weak continuity of \(\phi\). Thus, \(u_{n_i} - \phi(z), y_{n_i} - \phi(z)\), and \(z_{n_i} - \phi(z)\). Applying Lemma 2 to (64) to deduce \(\phi(z) \in \text{Fix}(T)\).

Now, we show that \(\phi(z) \in \text{VI}(C, g)\). Let
\[
S_1(v) = \begin{cases} 
g(v) + N_C v, & v \in C, \\
\& \& \\
\emptyset, & v \notin C.
\end{cases}
\]

Since \(g\) is relaxed \((\mu, v)\)-cocoercive, for all \(x, y \in C\), we have
\[
\langle g(x) - g(y), x - y \rangle \geq (-\mu)\|g(x) - g(y)\|^2 + v\|x - y\|^2 \geq (v - \mu L^2)\|x - y\|^2 \geq 0,
\]

which implies that \(g\) is monotone and so \(S_1\) is maximal monotone. Let \((v, u) \in G(S_1)\). Owing to \(u - g(v) \in N_C v\) and \(z \in C\), we get
\[
\langle v - z_n, u - g(v) \rangle \geq 0.
\]

According to \(z_n = \text{proj}_C(I - y_n g) y_n\), we obtain
\[
\langle v - z_n, z_n - (I - y_n g) y_n \rangle \geq 0.
\]

Then,
\[
\langle v - z_n, \frac{z_n - y_n}{y_n} + g(y_n) \rangle \geq 0.
\]

It follows that
\[
\langle v - z_n, u \rangle \geq \langle v - z_n, g(v) - g(z_n) \rangle + \langle v - z_n, g(z_n) - g(y_n) \rangle - \langle v - z_n, \frac{z_n - y_n}{y_n} \rangle \leq 0.
\]

Since \(z_n \to \phi(z)\), \(\|z_n - y_n\| \to 0\), it follows from (71) that \(\langle v - \phi(z), u \rangle \geq 0\). Therefore, \(\phi(z) \in S_{1_i}(0)\) and \(\phi(z) \in \text{VI}(C, g)\).

Next, we prove \(z \in \text{GVI}(C, f, \phi)\). Let
\[
S_2(v) = \begin{cases} 
f(v) + N_C v, & v \in C, \\
\emptyset, & v \notin C.
\end{cases}
\]

It is known that \(S_2\) is maximal \(\varphi\)-monotone. Let \((v, w) \in G(S_2)\). Since \(w - f(v) \in N_C(v)\) and \(x_n \in C\), we have \(\langle \varphi(v) - \varphi(x_n), w - f(v) \rangle \geq 0\). Set \(w_n = \text{proj}_C(\varphi(x_n) - \kappa_n f(x_n))\). Then,
\[
\langle \varphi(v) - w_n, w_n - [\varphi(x_n) - \kappa_n f(x_n)] \rangle \geq 0.
\]

It follows that
\[
\langle \varphi(v) - w_n, \frac{w_n - \varphi(x_n)}{\kappa_n} + f(x_n) \rangle \geq 0.
\]

Thus,
\[
\langle \varphi(v) - \varphi(x_n), w \rangle \geq \langle \varphi(v) - \varphi(x_n), f(v) - f(x_n) \rangle + \langle \varphi(v) - \varphi(x_n), f(x_n) \rangle - \langle \varphi(v) - w_n, \frac{w_n - \varphi(x_n)}{\kappa_n} \rangle - \langle \varphi(v) - w_n, f(x_n) \rangle \geq 0.
\]

Since \(\|\varphi(x_n) - w_n\| \to 0\) and \(\varphi(x_n) \to \phi(z)\), we deduce that \(\langle \varphi(v) - \varphi(z), w \rangle \geq 0\) by taking \(i \to \infty\) in (75). Thus, \(z \in S_{1_i}(0)\) by the maximal \(\varphi\)-monotonicity of \(S_2\). Hence, \(z \in \text{GVI}(C, f, \phi)\). Therefore, \(z \in \varphi^{-1}(\text{Fix}(T) \cap \text{VI}(C, g)) \cap \text{GVI}(C, f, \phi) = \emptyset\).
From (49) and (65), we obtain
\[
\limsup_{n \to \infty} (\phi(u^n) - \phi(u^0), u_n - \phi(u^0)) = \lim_{i \to \infty} (\phi(u^i) - \phi(u^0), \varphi(x_n) - \phi(u^0)) = (\phi(u^i) - \phi(u^0), \varphi(z) - \phi(u^0)) \leq 0.
\] (76)

By (17), we have
\[
\|u_n - \phi(u^0)\|^2 = \|\theta_n(\varphi(x_n) - \phi(u^0)) + (1 - \theta_n) \cdot (\pi_x(\varphi(x_n) - \kappa_n f(x_n)) - \phi(u^0))\|^2 \\
\leq (1 - \theta_n)^2 \|\pi_x(\varphi(x_n) - \kappa_n f(x_n)) - \phi(u^0)\|^2 + 2\theta_n (\varphi(x_n) - \phi(u^0), u_n - \phi(u^0)) \\
\leq (1 - \theta_n)^2 \|\varphi(x_n) - \phi(u^0)\|^2 + 2\theta_n (\varphi(x_n) - \phi(u^0), u_n - \phi(u^0)) \\
\leq (1 - \theta_n)^2 \|\varphi(x_n) - \phi(u^0)\|^2 + 2\theta_n \|\varphi(x_n) - \phi(u^0), u_n - \phi(u^0)\|^2 + 2\theta_n \|\varphi(x_n) - \phi(u^0), u_n - \phi(u^0)\|^2 \\
\leq (1 - \theta_n)^2 \|\varphi(x_n) - \phi(u^0)\|^2 + 2\theta_n \|\varphi(x_n) - \phi(u^0), u_n - \phi(u^0)\|^2 + 2\theta_n \|\varphi(x_n) - \phi(u^0), u_n - \phi(u^0)\|^2.
\] (77)

It follows that
\[
\|u_n - \phi(u^0)\|^2 \leq \left[ 1 - \frac{2(1 - \rho/l)\theta_n}{1 - \theta_n \rho/l} \right] \|\varphi(x_n) - \phi(u^0)\|^2 \\
+ \frac{\theta_n}{1 - \theta_n \rho/l} \|\varphi(x_n) - \phi(u^0)\|^2 \\
+ \frac{2\theta_n}{1 - \theta_n \rho/l} (\varphi(u^0) - \phi(u^0), u_n - \phi(u^0)).
\] (78)

Set \( M = \sup_n \|\varphi(x_n) - \phi(u^0)\|^2 \). Therefore,
\[
\|\varphi(x_{n+1}) - \phi(u^0)\|^2 \leq (1 - r_n) \|\varphi(x_n) - \phi(u^0)\|^2 + r_n \|\varphi(u^0) - \phi(u^0)\|^2 \\
\leq \left[ 1 - \frac{2(1 - \rho/l)\theta_n}{1 - \theta_n \rho/l} \right] \|\varphi(x_n) - \phi(u^0)\|^2 \\
+ \frac{\theta_n}{1 - \theta_n \rho/l} \|\varphi(x_n) - \phi(u^0)\|^2 \\
+ \frac{2\theta_n}{1 - \theta_n \rho/l} (\varphi(u^0) - \phi(u^0), u_n - \phi(u^0)) \\
= \left[ 1 - \frac{2(1 - \rho/l)\theta_n}{1 - \theta_n \rho/l} \right] \|\varphi(x_n) - \phi(u^0)\|^2 + \frac{2(1 - \rho/l)\theta_n}{1 - \theta_n \rho/l} \|\varphi(u^0) - \phi(u^0), u_n - \phi(u^0)\|^2 \\
\times \left\{ \frac{\theta_n}{2(1 - \rho/l) M + \frac{1}{1 - \rho/l} (\varphi(u^0) - \phi(u^0), u_n - \phi(u^0))} \right\}.
\] (79)

By Lemma 3 and (79), we conclude that \( \varphi(x_n) \to \varphi(u^0) \) and \( x_n \to u^0 \).

Case 2. For any \( N \), there exists an integer \( n_0 > N \) such that \( \|\varphi(x_{n_0}) - \phi(u^0)\| \leq \|\varphi(x_{n_0+1}) - \phi(u^0)\| \). Let \( \psi_n = \|\varphi(x_n) - \phi(u^0)\|^2 \). Then, we have \( \psi_n \leq \psi_{n+1} \). Let \( \{\mu_n\} \) be an integer sequence defined by, for all \( n \geq n_0 \),
\[
\mu(n) = \max \{ l \in \mathbb{N} | n \leq l \leq n, \psi_l \leq \psi_{l+1} \}.
\] (80)

Note that \( \mu(n) \) is nondecreasing and satisfies \( \lim_{n \to \infty} \mu(n) = \infty \) and \( \psi_{\mu(n)} \leq \psi_{\mu(n)+1} \) \( \forall n \geq n_0 \).

Similarly, we can deduce
\[
\limsup_{n \to \infty} (\phi(u^0) - \phi(u^0), u_{\mu(n)} - \phi(u^0)) \leq 0 \] (81)
\[
\psi_{\mu(n)+1} \leq \left[ 1 - \frac{2(1 - \rho/l)\theta_n}{1 - \theta_n \rho/l} \right] \|\varphi(x_n) - \phi(u^0)\|^2 + \frac{2(1 - \rho/l)\theta_n}{1 - \theta_n \rho/l} \|\varphi(u^0) - \phi(u^0), u_n - \phi(u^0)\|^2 \\
\times \left\{ \frac{\theta_n}{2(1 - \rho/l) M + \frac{1}{1 - \rho/l} (\varphi(u^0) - \phi(u^0), u_n - \phi(u^0))} \right\}.
\] (82)

Note that \( \psi_{\mu(n)} \leq \psi_{\mu(n)+1} \). By (82), we have
\[
\psi_{\mu(n)} \leq \frac{\theta_n}{2(1 - \rho/l) M + \frac{1}{1 - \rho/l} (\varphi(u^0) - \phi(u^0), u_n - \phi(u^0))}.
\] (83)

Based on (81) and (83), we derive
\[
\limsup_{n \to \infty} \psi_{\mu(n)} \leq 0,
\] and thus,
\[
\lim_{n \to \infty} \psi_{\mu(n)} = 0.
\] (85)

From (81) and (82), we can deduce
\[
\limsup_{n \to \infty} \psi_{\mu(n)+1} \leq \limsup_{n \to \infty} \psi_{\mu(n)}.
\] (86)

This together with (85) implies that
\[
\lim_{n \to \infty} \psi_{\mu(n)+1} = 0.
\] (87)

By Lemma 4, we obtain
\[
0 \leq \psi_{\mu(n)} \leq \max \left\{ \psi_{\mu(n)}, \psi_{\mu(n)+1} \right\}.
\] (88)

Therefore, \( \psi_{\mu(n)} \to 0 \). That is, \( \varphi(x_n) \to \varphi(u^0) \) and thus \( x_n \to u^0 \). This completes the proof.

In Algorithm 5, choose \( \varphi = I \), identity operator, and \( f : C \to H \) is a \( \Phi \)-inverse strongly monotone operator. Then, we have the following algorithm and corollary.
Algorithm 7. Let \( x_0 \in C \) be an initial value. Define the sequence \( \{x_n\} \) by the following form:

\[
\begin{align*}
u_n &= \theta_n \phi(x_n) + (1 - \theta_n) \text{proj}_C[x_n - \kappa_n f(x_n)], \\
y_n &= (1 - c_n) u_n + c_n T[(1 - \lambda_n) u_n + \lambda_n T(u_n)], \\
z_n &= \text{proj}_C[y_n - \gamma_n g(y_n)], \\
x_{n+1} &= (1 - \tau_n) x_n + \tau_n z_n, \quad n \geq 0.
\end{align*}
\]

(89)

Corollary 8. Suppose that \( \Theta_1 = VI(C, f) \cap VI(C, g) \cap \text{Fix}(T) \neq \emptyset \). Assume that conditions (C1)-(C5) are satisfied. Then, the sequence \( \{x_n\} \) generated by (89) converges strongly to \( v^* = \text{proj}_\Theta \phi(v^*) \).

Algorithm 9. Let \( x_0 \in C \) be an initial value. Define the sequence \( \{x_n\} \) by the following form:

\[
\begin{align*}
u_n &= \theta_n \phi(x_n) + (1 - \theta_n) \text{proj}_C[\phi(x_n) - \kappa_n f(x_n)], \\
z_n &= \text{proj}_C[\nu_n - \gamma_n g(u_n)], \\
\phi(x_{n+1}) &= (1 - \tau_n) \phi(x_n) + \tau_n z_n, \quad n \geq 0.
\end{align*}
\]

(90)

Corollary 10. Suppose that \( \Theta_2 := GVI(C, f, g) \cap \phi^{-1}(VI(C, g)) \neq \emptyset \). Assume that conditions (C1) and (C3)-(C5) are satisfied. Then, the sequence \( \{x_n\} \) generated by (90) converges strongly to \( u^* \in \Theta_2 \), verifying

\[
\langle \phi(u^*), \phi(u^*) - \phi(x^*) \rangle \leq 0, \quad \forall x^* \in \Theta_2.
\]

(91)

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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