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Research Article

Exponential Stabilization of a Swelling Porous-Elastic System with Microtemperature Effect and Distributed Delay

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The swelling porous thermoelastic system with the presence of temperatures, microtemperature effect, and distributed delay terms is considered. We will establish the well posedness of the system, and we prove the exponential stability result.

1. Introduction and Preliminaries

Eringen was the first to present a theory in which a mixture of viscous liquid and solids mixed with gas [1]. Then, after studying this heat-resistant mixture, you get to the field equations [2].

Expansive (swelling) soils have also been classified under porous media theory which studies this type of problem. This is why this field is considered fertile for study, as there are many studies to reduce the damage caused by swelling soil, especially in civil engineering and architecture, for more depth (see [3–8]).

The basic field equations of the linear theory of swelling porous elastic soils were presented by

$$\rho_u u_{tt} = P_{1x} + G_1 + H_1, \tag{1}$$

$$\rho_\phi \phi_{tt} = P_{2x} + G_2 + H_2, \tag{2}$$

where u, ϕ are the displacement of the fluid and the elastic solid material and $\rho_u, \rho_\phi > 0$ are the densities of each constituent. And (P_1, G_1, H_1) are the partial tension, internal body forces, and eternal forces acting on the displacement, respectively, similarly (P_2, G_2, H_2), but acting on the elastic solid. In addition, the constitutive equations of partial tensions are given by

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \underbrace{\begin{pmatrix} a_1, a_2 \\ a_2, a_3 \end{pmatrix}}_{A} \cdot \begin{pmatrix} u_x \\ \phi_x \end{pmatrix}, \tag{3}$$

where $a_1, a_3 > 0$ and $a_2 \neq 0$ are real numbers. A is matrix positive definite with $a_1 a_3 > a_2^2$.

Quintanilla [8] investigated (2) by taking

$$G_1 = G_2 = \xi(u_t - \phi_t), H_1 = a_3 u_{xxt}, H_2 = 0, \tag{4}$$

where $\xi > 0$; they obtained the stability is exponentially.

Similarly, in [9], the authors are considered (2) with a different conditions

$$G_1 = G_2 = 0, \quad H_1 = -\rho_u \gamma(x) u_t, \quad H_2 = 0, \tag{5}$$

where $\gamma(x)$ is an internal viscous damping function with positive mean. By the spectral method, they obtained the exponential stability result. For more details, see [8–15].

Time delays are very important in most natural phenomena and industrial devices, where the time lag is a source of instability, and it is a problem worthy of attention.

Also, there are many works that have studied this type of problems, of which [11, 15–24].

The basic evolution equations for one-dimensional theories of swelling porous materials with temperature and microtemperature [25–28] are given by

$$\rho_u u_{tt} = T_x, \tag{6}$$

$$\rho_\varphi \phi_{tt} = H_x + G, \tag{7}$$

$$\rho \eta_t = q_x, \tag{8}$$

$$\rho E_t = P_x^* + q - Q. \tag{9}$$

Here, $T, H, G, q, \eta, P^*, Q,$ and E represent the stress, the equilibrated stress, the equilibrated body force, the heat flux vector, the entropy, the first heat flux moment, the mean heat flux, and the first moment of energy. The constitutive equations are

$$T = P_1 + G_1 + H_1 P^* = -k_2 w_x, \tag{10}$$

$$H = P_2 + P_3 \rho \eta = \gamma u_x + c_0 \theta + m\phi, \tag{11}$$

$$G = G_2 + H_2 Q = -k_3 w - k_1 \theta_x, \tag{12}$$

$$q = \kappa \theta_x \rho E = -\alpha w - d\phi_x, \tag{13}$$

where w is the microtemperature vector, $\rho_u, \rho_\phi, k_1, k_2, k_3, a_1, a_2, a_3, \alpha, \kappa, c_0, \mu_1 > 0$. As coupling is considered, $a_2 \neq 0$ and satisfies

$$a = a_3 - \frac{a_2^2}{a_1} > 0. \tag{14}$$

And the coefficients $\gamma, m, d > 0$.

The goal of this work is the thermal effects, so we suppose that the heat capacity $c_0 > 0$, and for more excitement in posing the problem, we suppose that the thermal conductivity is nonexistent $\kappa \neq 0$.

And by introducing the distributed delay term, form a new problem different from previous studies. Under appropriate suppositions, the well posedness of the system is established, and we prove the exponential stability result by the energy method.

We consider in this work:

$$G_1 = G_2 = 0, \quad P_3 = -dw, \tag{15}$$

$$H_1 = -\gamma\theta, \tag{16}$$

$$H_2 = m\theta - \mu_1 \phi_t - \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \phi_t(x, t-\sigma) d\sigma. \tag{17}$$

Now, by substituting (13)–(17) into (9), we arrive at the following problem:

$$\begin{cases} \rho_u u_{tt} - a_1 u_{xx} - a_2 \phi_{xx} - \gamma \theta_x = 0, \\ \rho_\phi \phi_{tt} - a_3 \phi_{xx} - a_2 u_{xx} - d w_x + m\theta + \mu_1 \phi_t + \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \phi_t(x, t-\sigma) d\sigma = 0, \\ c_0 \theta_t = -\gamma u_{tx} - m\phi_t - k_1 w_x, \\ \alpha w_t = k_2 w_{xx} - k_3 w - k_1 \theta_x - d\phi_{tx}, \end{cases} \tag{18}$$

where

$$(x, \sigma, t) \in \mathscr{H} = (0,1) \times (\tau_1, \tau_2) \times (0, \infty), \tag{19}$$

under the initial and boundary conditions

$$\begin{aligned} & u(x,0) = u_0(x), u_t(x,0) = u_1(x), \theta(x,0) = \theta_0(x) \\ & \phi(x,0) = \phi_0(x), \phi_t(x,0) = \phi_1(x), w(x,0) = w_0(x), \quad x \in (0,1), \\ & \phi_t(x,-t) = f_0(x,t), \quad (x,t) \in (0,1) \times (0, \tau_2), \\ & u(0,t) = u(1,t) = \phi(0,t) = \phi(1,t) = 0, \\ & \theta(0,t) = \theta(1,t) = w_x(0,t) = w_x(1,t) = 0, \quad t \geq 0. \end{aligned} \tag{20}$$

First, as in [24], we introduce the new variable

$$\mathscr{Y}(x, \rho, \sigma, t) = \phi_t(x, t - \sigma\rho), \tag{21}$$

Then, we get

$$\begin{cases} \sigma \mathscr{Y}_t(x, \rho, \sigma, t) + \mathscr{Y}_\rho(x, \rho, \sigma, t) = 0, \\ \mathscr{Y}(x, 0, \sigma, t) = \phi_t(x, t). \end{cases} \tag{22}$$

Consequently, our problem is written in the form

$$\begin{cases} \rho_u u_{tt} - a_1 u_{xx} - a_2 \phi_{xx} - \gamma \theta_x = 0, \\ \rho_\phi \phi_{tt} - a_3 \phi_{xx} - a_2 u_{xx} - d w_x + m\theta + \mu_1 \phi_t + \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \mathscr{Y}(x, 1, \sigma, t) d\sigma = 0, \\ c_0 \theta_t = -\gamma u_{tx} - m\phi_t - k_1 w_x, \\ \alpha w_t = k_2 w_{xx} - k_3 w - k_1 \theta_x - d\phi_{tx}, \\ \sigma \mathscr{Y}_t(x, \rho, \sigma, t) + \mathscr{Y}_\rho(x, \rho, \sigma, t) = 0. \end{cases} \tag{23}$$

where

$$(x, \rho, \sigma, t) \in (0,1) \times \mathscr{H}, \tag{24}$$

with the initial data

$$\begin{cases} u(x,0)=u_0(x), u_t(x,0)=u_1(x), \theta(x,0)=\theta_0(x), \\ \phi(x,0)=\phi_0(x), \phi_t(x,0)=\phi_1(x), w(x,0)=w_0(x)x\in(0,1), \\ \mathscr{Y}(x,\rho,\sigma,0)=\mathrm{f}_0(x,\rho\sigma), (x,\rho,\sigma)\in(0,1)\times(0,1)\times(0,\tau_2), \end{cases} \tag{25}$$

and the boundary conditions

$$\begin{cases} u(0,t)=u(1,t)=\phi(0,t)=\phi(1,t)=0, \\ \theta(0,t)=\theta(1,t)=w_x(0,t)=w_x(1,t)=0, \quad t\geq 0. \end{cases} \tag{26}$$

Here, the integral represent the distributed delay terms with $\tau_1, \tau_2 > 0$ are a time delay; μ_2 is an L^∞ function satisfying:

(H1) $\mu_2 : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| d\sigma < \mu_1. \tag{27}$$

Meanwhile, from $(23)_4$ and (26), it follows that

$$\frac{d}{dt}\int_0^1 \omega(x,t)dx + \frac{k_3}{\alpha}\int_0^1 \omega(x,t)dx = 0. \tag{28}$$

So, by solving (28) and using the initial data of u, we get

$$\int_0^1 \omega(x,t)dx = \left(\int_0^1 \omega_0(x)dx\right) e^{-(t/\alpha)k_3}. \tag{29}$$

Consequently, if we let

$$\bar{\omega}(x,t) = \omega(x,t) - \left(\int_0^1 \omega_0(x)dx\right) e^{-(t/\alpha)k_3}, \tag{30}$$

we get

$$\int_0^1 \bar{\omega}(x,t)dx = 0, \quad \forall t \geq 0. \tag{31}$$

Therefore, the use of Poincare's inequality for $\bar{\omega}$ is justified. In addition, simple substitution shows that $(u, \phi, \theta, \bar{\omega}, \mathscr{Y})$ satisfies system (23). Henceforth, we work with $\bar{\omega}$ instead of ω but write ω for simplicity of notation.

In this paper, we consider $(u, \phi, \theta, w, \mathscr{Y})$ to be a solution of system (23)–(26) with the regularity needed to justify the calculations. In Section 2, the well posedness is established, and in Section 3, the exponential stability is proved. In all of the following, we mention that $c > 0$.

Remark 1. The coupling that we have proposed in this work with the presence of microtemperatures and distributed delay in problems of swelling in porous elasticity we believe constitutes a new contribution and differs from the previous studies.

2. Well Posedness

In this section, we established the well posedness of the system (23)–(26).

First, introducing the vector function

$$U = (u, u_t, \phi, \phi_t, \theta, w, \mathscr{Y})^T, \tag{32}$$

and the variables $v = u_t, \varphi = \phi_t$, then the system (23) writes as follows:

$$\begin{cases} U_t = \mathscr{A}U, \\ U(0) = U_0 = (u_0, u_1, \phi_0, \phi_1, \theta_0, w_0, f_0)^T, \end{cases} \tag{33}$$

where $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subset \mathscr{H} : \longrightarrow \mathscr{H}$ is the linear operator given by

$$\mathscr{A}U = \begin{pmatrix} v \\ -\frac{1}{\rho_u}[a_1 u_{xx} + a_2\phi_{xx} - \gamma\theta_x] \\ \varphi \\ \frac{1}{\rho_\phi}\left[a_3\phi_{xx} + a_2 u_{xx} - dw_x + m\theta - \mu_1\varphi - \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| \mathscr{Y}(x,1,\sigma,t)d\sigma\right] \\ -\frac{1}{c_0}[\gamma v_x + m\varphi + k_1 w_x] \\ \frac{1}{\alpha}[k_2 w_{xx} - k_3 w - k_1\theta_x - d\varphi_x] \\ -\frac{1}{\sigma}\mathscr{Y}_\rho \end{pmatrix}, \tag{34}$$

and $\mathscr{H}$ is the energy space given by

$$\mathscr{H} = H_0^1(0,1)\times L^2(0,1)\times H_0^1(0,1)\times L^2(0,1) \times L^2(0,1)\times L_*^2(0,1)\times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)), \tag{35}$$

where

$$\begin{aligned} L_*^2(0,1) &= \left\{\psi \in \frac{L^2(0,1)}{\int_0^1 \psi(x)dx} = 0\right\}, \\ H_*^1(0,1) &= H^1(0,1)\cap L_*^2(0,1), \\ H_*^2(0,1) &= \left\{\psi \in \frac{H^2(0,1)}{\psi_x(1)} = \psi_x(0) = 0\right\}. \end{aligned} \tag{36}$$

For any,

$$\begin{aligned} U &= (u, v, \phi, \varphi, \theta, w, \mathscr{Y})^T \in \mathscr{H}, \\ \widehat{U} &= (u\wedge, v\wedge, \phi\wedge, \varphi\wedge, \theta\wedge, w\wedge, \mathscr{Y}\wedge)^T \in \mathscr{H}, \end{aligned} \tag{37}$$

we equip $\mathscr{H}$ with the inner product defined by

$$
\begin{aligned}
<U,\widehat{U}>_{\mathscr{H}} = {} & \rho_u\int_0^1 v\widehat{v}dx + a_1\int_0^1 u_x\widehat{u}_x dx + \rho_\phi\int_0^1 \varphi\widehat{\varphi}dx \\
& + a_3\int_0^1 \phi_x\widehat{\phi}_x dx + c_0\int_0^1 \theta\widehat{\theta}dx \\
& + \alpha\int_0^1 w\widehat{w}dx + a_2\int_0^1 \left(u_x\widehat{\phi} + \widehat{u}_x\phi\right)dx \\
& + \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma|\mu_2(\sigma)|\mathscr{Y}\widehat{\mathscr{Y}}d\sigma d\rho dx.
\end{aligned} \tag{38}
$$

The domain of $\mathscr{A}$ is given by

$$
\mathscr{D}(\mathscr{A}) = \left\{ \begin{array}{c} U\in\mathscr{H}/u,\phi\in H^2(0,1)\cap H_0^1(0,1), v,\varphi,\theta\in H_0^1(0,1), \\ w\in H_*^2(0,1)\cap H_*^1(0,1), \\ \mathscr{Y},\mathscr{Y}_\rho\in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)), \mathscr{Y}(x,0,\sigma,t)=\varphi \end{array} \right\}. \tag{39}
$$

Clearly, $\mathscr{D}(\mathscr{A})$ is dense in $\mathscr{H}$.

Theorem 2. *Let $U_0\in\mathscr{H}$ and assume that (27) holds. Then, there exists a unique solution $U\in\mathscr{C}(\mathbb{R}_+,\mathscr{H})$ of problem (33).*

Moreover, if $U_0\in\mathscr{D}(\mathscr{A})$, then

$$
U\in\mathscr{C}(\mathbb{R}_+,\mathscr{D}(\mathscr{A}))\cap\mathscr{C}^1(\mathbb{R}_+,\mathscr{H}). \tag{40}
$$

Proof. First, we prove that the operator $\mathscr{A}$ is dissipative. For any $U_0\in\mathscr{D}(\mathscr{A})$ and by using (38), we have

$$
\begin{aligned}
<\mathscr{A}U,U>_{\mathscr{H}} = {} & -\mu_1\int_0^1\varphi^2dx - \int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\varphi\mathscr{Y}(x,1,\sigma,t)d\sigma dx \\
& - \int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\mathscr{Y}_\rho\mathscr{Y}d\sigma d\rho dx \\
& - k_2\int_0^1 w_x^2dx - k_3\int_0^1 w^2dx.
\end{aligned} \tag{41}
$$

For the third term of the RHS of (41), we have

$$
\begin{aligned}
& -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\mathscr{Y}_\rho\mathscr{Y}d\sigma d\rho dx \\
& = -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\int_0^1|\mu_2(\sigma)|\frac{d}{d\rho}\mathscr{Y}^2 d\rho d\sigma dx \\
& = -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\mathscr{Y}^2(x,1,\sigma,t)d\sigma dx \\
& \quad + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\mathscr{Y}^2(x,0,\sigma,t)d\sigma dx.
\end{aligned} \tag{42}
$$

By using Young's inequality, we get

$$
\begin{aligned}
& -\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\varphi\mathscr{Y}(x,1,\sigma,t)d\sigma dx \\
& \le \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|d\sigma\right)\int_0^1\varphi^2dx \\
& \quad + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\mathscr{Y}^2(x,1,\sigma,t)d\sigma dx.
\end{aligned} \tag{43}
$$

Substituting (42), (43) into (41), using $\mathscr{Y}(x,0,\sigma,t)=\varphi(x,t)$ and (27), we find

$$
<\mathscr{A}U,U>_{\mathscr{H}} \le -\eta_0\int_0^1\varphi^2dx - k_2\int_0^1 w_x^2dx - k_3\int_0^1 w^2dx \le 0, \tag{44}
$$

where $\eta_0 = (\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|d\sigma) > 0$. Hence, $\mathscr{A}$ is dissipative operator.

Next, we prove $\mathscr{A}$ is maximal operator. It is sufficient to show that $(\lambda I - \mathscr{A})$ is surjective operator.

Indeed, for any $F=(f_1,f_2,f_3,f_4,f_5,f_6,f_7)^T\in\mathscr{H}$, we prove that there exists a unique $U=(u,v,\phi,\varphi,\theta,w,z)\in\mathscr{D}(\mathscr{A})$ such that

$$
(\lambda I - \mathscr{A})U = F. \tag{45}
$$

That is,

$$
\begin{cases}
\lambda u - v = f_1 \in H_0^1(0,1), \\
\rho_u\lambda v - a_1u_{xx} - a_2\phi_{xx} + \gamma\theta_x = \rho_u f_2 \in L^2(0,1), \\
\lambda\phi - \varphi = f_3 \in H_0^1(0,1), \\
\rho_\phi\lambda\varphi - a_3\phi_{xx} - a_2u_{xx} + dw_x - m\theta + \mu_1\varphi + \int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\mathscr{Y}(x,1,\sigma,t)d\sigma = \rho_\phi f_4 \in L^2, \\
c_0\lambda\theta + \gamma v_x + m\varphi + k_1w_x = c_0f_5 \in L^2(0,1), \\
\alpha\lambda w - k_2w_{xx} + k_3w + k_1\theta_x + d\varphi_x = \alpha f_6 \in H_*^1(0,1), \\
\sigma\lambda\mathscr{Y}_t(x,\rho,\sigma,t) + \mathscr{Y}_\rho(x,\rho,\sigma,t) = \sigma f_7 \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)).
\end{cases} \tag{46}
$$

We note that Equation $(46)_7$ with $\mathscr{Y}(x,0,\sigma,t)=\varphi(x,t)$ has a unique solution defined by

$$
\mathscr{Y}(x,\rho,\sigma,t) = e^{-\lambda\rho\sigma}\varphi + \sigma e^{\sigma\rho\lambda}\int_0^\rho e^{\lambda\sigma\rho}f_7(x,\rho,\sigma,t)d\rho, \tag{47}
$$

then

$$
\mathscr{Y}(x,1,\sigma,t) = e^{-\lambda\sigma}\varphi + \sigma e^{\lambda\sigma}\int_0^1 e^{\lambda\sigma\rho}f_7(x,\rho,\sigma,t)d\rho, \tag{48}
$$

and we have

$$
v = \lambda u - f_1, \quad \varphi = \lambda\phi - f_3. \tag{49}
$$

Inserting (48) and (49) in $(46)_2$, $(46)_4$, $(46)_5$, and $(46)_6$, we get

$$\begin{cases} \rho_u\lambda^2 u - a_1 u_{xx} - a_2\phi_{xx} + \gamma\theta_x = h_1, \\ \mu_3\phi - a_3\phi_{xx} - a_2 u_{xx} + d w_x - m\theta = h_2, \\ c_0\theta + \gamma u_x + m\phi + \dfrac{k_1}{\lambda} w_x = h_3, \\ \dfrac{\alpha\lambda + k_3}{\lambda} w - \dfrac{k_2}{\lambda} w_{xx} + \dfrac{k_1}{\lambda}\theta_x + d\phi_x = h_4, \end{cases} \quad (50)$$

where

$$\begin{cases} h_1 = \rho_u(\lambda f_1 + f_2), \\ h_2 = \left(\rho_\phi\lambda + \mu_1 + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| e^{-\sigma\lambda} d\sigma\right) f_3 \\ \quad - \displaystyle\int_{\tau_1}^{\tau_2} \sigma|\mu_2(\sigma)| e^{\sigma\lambda} \int_0^1 e^{\lambda\sigma\rho} f_7(x,\rho,\sigma,t)\, d\rho\, d\sigma, \\ h_3 = \dfrac{1}{\lambda}(\gamma f_{1x} + m f_3 + c_0 f_5), \\ h_4 = \dfrac{1}{\lambda}(\alpha f_6 + d\lambda f_3), \\ \mu_3 = \rho_\phi\lambda^2 + \mu_1\lambda + \lambda\displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| e^{-\lambda\sigma} d\sigma. \end{cases} \quad (51)$$

We multiply (50) by $\widehat{u}, \widehat{\phi}, \widehat{\theta}, \widehat{w}$, respectively, and integrate their sum over $(0,1)$ to find the following variational formulation:

$$B\left((u,\phi,\theta,w), \left(\widehat{u}, \widehat{\phi}, \widehat{\theta}, \widehat{w}\right)\right) = \Gamma\left(\widehat{u}, \widehat{\phi}, \widehat{\theta}, \widehat{w}\right), \quad (52)$$

where

$$B : \left(H_0^1(0,1) \times H_0^1(0,1) \times L^2(0,1) \times H_*^1(0,1)\right)^2 \longrightarrow \mathbb{R}, \quad (53)$$

is the bilinear form given by

$$\begin{aligned} B\left((u,\phi,\theta,w), \left(\widehat{u}, \widehat{\phi}, \widehat{\theta}, \widehat{w}\right)\right) &= \rho_u\lambda^2 \int_0^1 u\widehat{u}\,dx + a_1\int_0^1 u_x\widehat{u}_x\,dx \\ &+ a_2\int_0^1 \phi_x\widehat{u}_x\,dx + \gamma\int_0^1 \theta_x\widehat{u}\,dx + \mu_3\int_0^1 \phi\widehat{\phi}\,dx \\ &+ a_3\int_0^1 \phi_x\widehat{\phi}_x\,dx + a_2\int_0^1 u_x\widehat{\phi}_x\,dx + d\int_0^1 w_x\widehat{\phi}\,dx - m\int_0^1 \theta\widehat{\phi}\,dx \\ &+ c_0\int_0^1 \theta\widehat{\theta}\,dx + \gamma\int_0^1 u_x\widehat{\theta}\,dx + m\int_0^1 \phi\widehat{\theta}\,dx + \frac{k_1}{\lambda}\int_0^1 w_x\widehat{\theta}\,dx \\ &+ \frac{\alpha\lambda + k_3}{\lambda}\int_0^1 w\widehat{w}\,dx + \frac{k_2}{\lambda}\int_0^1 w_x\widehat{w}_x\,dx + \frac{k_1}{\lambda}\int_0^1 \theta_x\widehat{w}\,dx + d\int_0^1 \phi_x\widehat{w}\,dx, \end{aligned} \quad (54)$$

and

$$\Gamma : \left(H_0^1(0,1) \times H_0^1(0,1) \times L^2(0,1) \times H_*^1(0,1)\right) \longrightarrow \mathbb{R}, \quad (55)$$

is the linear functional defined by

$$\Gamma\left(\widehat{u}, \widehat{\phi}, \widehat{\theta}, \widehat{w}\right) = \int_0^1 h_1\widehat{u}\,dx + \int_0^1 h_2\widehat{\phi}\,dx + \int_0^1 h_3\widehat{\theta}\,dx + \int_0^1 h_4\widehat{w}\,dx. \quad (56)$$

Now, for $V = H_0^1(0,1) \times H_0^1(0,1) \times L^2(0,1) \times H_*^1(0,1)$, equipped with the norm,

$$\begin{aligned} \|u,\phi,\theta,w\|_V^2 &= \|u\|_2^2 + \|u_x\|_2^2 + \|\phi\|_2^2 + \|\phi_x\|_2^2 \\ &+ \|\theta\|_2^2 + \|w_x\|_2^2 + \|w\|_2^2, \end{aligned} \quad (57)$$

then we have

$$\begin{aligned} B((u,\phi,\theta,w),(u,\phi,\theta,w)) &= \rho_u\lambda^2\int_0^1 u^2\,dx + a_1\int_0^1 u_x^2\,dx \\ &+ \mu_3\int_0^1 \phi^2\,dx + a_3\int_0^1 \phi_x^2\,dx + 2a_2\int_0^1 u_x\phi_x\,dx \\ &+ c_0\int_0^1 \theta^2\,dx + \frac{\alpha\lambda + k_3}{\lambda}\int_0^1 w^2\,dx + \frac{k_2}{\lambda}\int_0^1 w_x^2\,dx. \end{aligned} \quad (58)$$

On the other hand, we can write

$$\begin{aligned} a_1 u_x^2 + 2a_2 u_x\phi_x + a_3\phi_x^2 &= \frac{1}{2}\left[a_1\left(u_x + \frac{a_2}{a_3}\phi_x\right)^2 + a_3\left(\phi_x + \frac{a_2}{a_1}u_x\right)^2\right. \\ &\left. + u_x^2\left(a_1 + \frac{a_2^2}{a_3}\right) + \phi_x^2\left(a_3 + \frac{a_2^2}{a_1}\right)\right]. \end{aligned} \quad (59)$$

Since (14), we deduce

$$a_1 u_x^2 + 2a_2 u_x\phi_x + a_3\phi_x^2 > \frac{1}{2}\left[u_x^2\left(a_1 + \frac{a_2^2}{a_3}\right) + \phi_x^2\left(a_3 + \frac{a_2^2}{a_1}\right)\right], \quad (60)$$

then, for some $M_0 > 0$

$$B((u,\phi,\theta,w),(u,\phi,\theta,w)) \geq M_0\|(u,\phi,\theta,w)\|_V^2. \quad (61)$$

Thus, B is coercive. Hence, we use the Lax-Milgram theorem to conclude that (52) has a unique solution:

$$\begin{aligned} &u, \phi \in H_0^1(0,1), \\ &w \in H_*^1(0,1), \\ &\theta \in L^2(0,1). \end{aligned} \quad (62)$$

Substituting u, ϕ, θ, and w into $(46)_{1,3}$, we have

$$v, \varphi \in H_0^1(0,1). \tag{63}$$

Similarly, the compensation of v in (47) with $(46)_7$, gives

$$\mathscr{Y}, \mathscr{Y}_\rho \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)). \tag{64}$$

Moreover, if we take $\hat{u} = \hat{\theta} = \hat{w} = 0$ in (54), we get

$$\begin{aligned} a_3\int_0^1 \phi_x\hat{\phi}_x dx + \mu_3\int_0^1 \phi\hat{\phi} dx + a_2\int_0^1 u_x\hat{\phi}_x dx \\ + d\int_0^1 w_x\hat{\phi} dx - m\int_0^1 \theta\hat{\phi} dx \\ = \int_0^1 h_2\hat{\phi} dx, \quad \forall\hat{\phi} \in H_0^1(0,1), \end{aligned} \tag{65}$$

which implies

$$\begin{aligned} a_3\int_0^1 \phi_x\hat{\phi}_x dx = \int_0^1 (h_2 - \mu_3\phi + a_2u_{xx} - dw_x \\ + m\theta)\hat{\phi} dx, \quad \forall\hat{\phi} \in H_0^1(0,1), \end{aligned} \tag{66}$$

that is

$$a_3\phi_{xx} = \mu_3\phi - a_2u_{xx} + dw_x - m\theta - h_2 \in L^2(0,1). \tag{67}$$

Consequently

$$\phi \in H^2(0,1) \cap H_0^1(0,1). \tag{68}$$

Similarly, we get

$$\begin{aligned} u \in H^2(0,1) \cap H_0^1(0,1), \\ \theta \in H_0^1(0,1), \end{aligned} \tag{69}$$

and, if we let $\hat{u} = \hat{\theta} = \hat{\phi} = 0$ in (54), we get

$$\begin{aligned} \frac{\alpha\lambda + k_3}{\lambda}\int_0^1 w\hat{w} dx + \frac{k_2}{\lambda}\int_0^1 w_x\hat{w}_x dx + \frac{k_1}{\lambda}\int_0^1 \theta_x\hat{w} dx \\ + d\int_0^1 \phi_x\hat{w} dx - m\int_0^1 \theta\hat{\phi} dx \\ = \int_0^1 h_4\hat{w} dx, \quad \forall\hat{w} \in H_*^1(0,1), \end{aligned} \tag{70}$$

which implies

$$\begin{aligned} \frac{k_2}{\lambda}\int_0^1 w_x\psi_x dx = \int_0^1 \left(-h_4 + \frac{\alpha\lambda + k_3}{\lambda}w + \frac{k_1}{\lambda}\theta_x + d\phi_x\right)\psi dx, \\ \forall\psi \in C^1(0,1) \subset H_*^1(0,1), \end{aligned} \tag{71}$$

Thus, using integration by parts, we get

$$w_x(1)\psi(1) - w_x(0)\psi(0) = 0, \quad \forall\psi \in C^1(0,1). \tag{72}$$

Therefore,

$$w_x(1) = w_x(0) = 0. \tag{73}$$

Consequently,

$$w \in H_*^2(0,1) \cap H_*^1(0,1). \tag{74}$$

Finally, the application of regularity theory for the linear elliptic equations guarantees the existence of unique $U \in \mathscr{D}(\mathscr{A})$ such that (45) is satisfied.

Consequently, we conclude that $\mathscr{A}$ is a maximal dissipative operator. Hence, by Lumer-Philips theorem (see [29]), we have the well-posedness result. This completes the proof.□

3. Exponential Decay

In this section, we prove our stability result of the system (23)–(26).

For this, we have the following lemmas.

Lemma 3. *The energy functional E, defined by*

$$\begin{aligned} E(t) = \frac{1}{2}\int_0^1 \left[\rho_u u_t^2 + a_1u_x^2 + \rho_\phi\phi_t^2 + a_3\phi_x^2 + 2a_2u_x\phi_x + c_0\theta^2 + \alpha w^2\right] dx \\ + \frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma|\mu_2(\sigma)|\mathscr{Y}^2(x,\rho,\sigma,t) d\sigma d\rho dx, \end{aligned} \tag{75}$$

satisfies

$$E'(t) \leq -k_2\int_0^1 w_x^2 dx - k_3\int_0^1 w^2 dx - \eta_0\int_0^1 \phi_t^2 dx \leq 0, \tag{76}$$

where $\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| d\sigma > 0$.

Proof. Multiplying Equation $(23)_{1,2,3,4}$ by u_t, ϕ_t, θ, and w, integrating by parts over $(0,1)$, and using (26), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_u u_t^2 + a_1u_x^2 + \rho_\phi\phi_t^2 + a_3\phi_x^2 + 2a_2u_x\phi_x + c_0\theta^2 + \alpha w^2\right] dx \\ + \mu_1\int_0^1 \phi_t^2 dx + \int_0^1 \phi_t\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)|\mathscr{Y}(x,1,\sigma,t) d\sigma dx \\ + k_3\int_0^1 w^2 dx + k_2\int_0^1 w_x^2 dx = 0. \end{aligned} \tag{77}$$

Now, multiplying Equation $(23)_5$ by $\mathscr{Y}|\mu_2(\sigma)|$, and integrating the result over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\sigma|\mu_2(\sigma)|\mathscr{Y}^2(x,\rho,\sigma,t)d\sigma d\rho dx\\
&\quad=-\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\mathscr{Y}\mathscr{Y}_\rho(x,\rho,\sigma,t)d\sigma d\rho dx\\
&\quad=-\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\frac{d}{d\rho}\mathscr{Y}^2(x,\rho,\sigma,t)d\sigma d\rho dx\\
&\quad=\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\left(\mathscr{Y}^2(x,0,\sigma,t)-\mathscr{Y}^2(x,1,\sigma,t)\right)d\sigma dx\\
&\quad=\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|d\sigma\int_0^1\phi_t^2dx\\
&\qquad-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\mathscr{Y}^2(x,1,\sigma,t)d\sigma dx.
\end{aligned}\tag{78}$$

Now, by substituting (78) into (77), and using Young's inequality, we have

$$E'(t)\le -k_3\int_0^1 w^2dx-k_2\int_0^1 w_x^2dx-\left(\mu_1-\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|d\sigma\right)\int_0^1\phi_t^2dx,\tag{79}$$

then, by (27), $\exists\eta_0>0$ so that

$$E'(t)\le -k_3\int_0^1 w^2dx-k_2\int_0^1 w_x^2dx-\eta_0\int_0^1\phi_t^2dx,\tag{80}$$

then we obtain (76) (E is a nonincreasing function).□

Remark 4. Using (14), we conclude that $E(t)$ satisfies

$$\begin{aligned}
E(t)>&\frac{1}{2}\int_0^1\left[\rho_u u_t^2+a_4u_x^2+\rho_\phi\phi_t^2+a_5\phi_x^2+c_0\theta^2+\alpha w^2\right]dx\\
&+\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\sigma|\mu_2(\sigma)|\mathscr{Y}^2(x,\rho,\sigma,t)d\sigma d\rho dx,
\end{aligned}\tag{81}$$

where

$$\begin{aligned}
a_4&=\frac{1}{2}\left(a_1-\frac{a_2^2}{a_3}\right)>0,\\
a_5&=\frac{1}{2}\left(a_3-\frac{a_2^2}{a_1}\right)>0.
\end{aligned}\tag{82}$$

Then, the function $E(t)$ is nonnegative.

Lemma 5. *The functional*

$$D_1(t):=\rho_\phi\int_0^1\phi_t\phi dx-\frac{a_2}{a_1}\rho_u\int_0^1\phi u_t dx+\frac{\mu_1}{2}\int_0^1\phi^2dx,\tag{83}$$

satisfies, for any $\varepsilon_1>0$

$$\begin{aligned}
D_1'(t)\le&-\frac{a}{2}\int_0^1\phi_x^2dx+\varepsilon_1\int_0^1 u_t^2dx+c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1\phi_t^2dx\\
&+c\int_0^1 w^2dx+c\int_0^1\theta^2dx+c\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\mathscr{Y}^2(x,1,\sigma,t)d\sigma dx.
\end{aligned}\tag{84}$$

Proof. Direct computation using integration by parts and Young's inequality yields

$$\begin{aligned}
D_1'(t)=&-a_3\int_0^1\phi_x^2dx+\rho_\phi\int_0^1\phi_t^2dx+\frac{a_2^2}{a_1}\int_0^1\phi_x^2dx\\
&+\frac{a_2}{a_1}\rho_u\int_0^1\phi_t u_t dx-d\int_0^1 w_x\phi dx+m\int_0^1\theta\phi dx\\
&+\frac{a_2\gamma}{a_1}\int_0^1\theta_x\phi dx-\int_0^1\phi\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\mathscr{Y}(x,1,\sigma,t)d\sigma dx\\
\le&-\left(a_3-\frac{a_2^2}{a_1}\right)\int_0^1\phi_x^2dx+\rho_\phi\int_0^1\phi_t^2dx\\
&+\frac{a_2}{a_1}\rho_u\int_0^1\phi_t u_t dx-d\int_0^1 w_x\phi dx+m\int_0^1\theta\phi dx\\
&+\frac{a_2\gamma}{a_1}\int_0^1\theta_x\phi dx-\int_0^1\phi\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\mathscr{Y}(x,1,\sigma,t)d\sigma dx.
\end{aligned}\tag{85}$$

We use Cauchy-Schwartz, Young's, and Poincare's inequalities; for $\delta_1,\varepsilon_1>0$, we obtain

$$\begin{aligned}
D_1'(t)\le&-\left(a_3-\frac{a_2^2}{a_1}-\mu_1c\delta_1\right)\int_0^1\phi_x^2dx+\varepsilon_1\int_0^1 u_t^2dx\\
&+c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1\phi_t^2dx+d\int_0^1 w\phi_x dx+m\int_0^1\theta\phi dx\\
&-\frac{a_2\gamma}{a_1}\int_0^1\theta\phi_x dx+\frac{1}{4\delta_1}\int_0^t\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\mathscr{Y}^2(x,1,\sigma,t)d\sigma dx.
\end{aligned}\tag{86}$$

Bearing in mind (14), and letting $\delta_1=a/2c$, we obtain the estimate (84).□

Lemma 6. *The functional*

$$D_2(t):=a_2\left(\int_0^1\phi_t u dx-\int_0^1\phi u_t dx\right),\tag{87}$$

satisfies,

$$\begin{aligned}
D_2'(t)\le&-\frac{a_2^2}{2\rho_\phi}\int_0^1 u_x^2dx+c\int_0^1\phi_x^2dx+c\int_0^1\phi_t^2dx+c\int_0^1 w^2dx\\
&+c\int_0^1\theta^2dx+c\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\mathscr{Y}^2(x,1,\sigma,t)d\sigma dx.
\end{aligned}\tag{88}$$

Proof. By differentiating D_2, then using (23), integration by parts, and (26), we obtain

$$D_2'(t) = -\frac{a_2^2}{\rho_\phi}\int_0^1 u_x^2 dx + \frac{a_2^2}{\rho_u}\int_0^1 \phi_x^2 dx - \left(\frac{a_2 a_3}{\rho_\phi} - \frac{a_1 a_2}{\rho_u}\right)\int_0^1 \phi_x u_x dx - \frac{a_2\mu_1}{\rho_\phi}\int_0^1 u\phi_t dx + \frac{a_2 d}{\rho_\phi}\int_0^1 w u_x dx + \frac{a_2 m}{\rho_\phi}\int_0^1 \theta u dx - \frac{a_2\gamma}{\rho_u}\int_0^1 \theta\phi_x dx - \frac{a_2}{\rho_\phi}\int_0^1 u\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\mathscr{Y}(x,1,\sigma,t)d\sigma dx. \tag{89}$$

Now, we estimate the last six terms in the RHS of (89), using Young's, Cauchy-Schwartz, and Poincare's inequalities. For $\delta_2, \delta_3, \delta_4, \delta_5, \delta_6 > 0$, we have

$$\begin{aligned}
-\left(\frac{a_2 a_3}{\rho_\phi} - \frac{a_1 a_2}{\rho_u}\right)\int_0^1 \phi_x u_x dx &\le \delta_2\int_0^1 u_x^2 dx + \left(\frac{a_2 a_3}{\rho_\phi} - \frac{a_1 a_2}{\rho_u}\right)^2 \frac{1}{4\delta_2}\int_0^1 \phi^2 dx. \\
\frac{a_2 d}{\rho_\phi}\int_0^1 w u_x dx &\le \delta_3\int_0^1 u_x^2 dx + \frac{c}{4\delta_3}\int_0^1 w^2 dx, \\
-\frac{a_2\mu_1}{\rho_\phi}\int_0^1 u\phi_t dx &\le c\delta_4\int_0^1 u_x^2 dx + \frac{c}{4\delta_4}\int_0^1 \phi_t^2 dx,
\end{aligned} \tag{90}$$

and

$$\begin{aligned}
\frac{a_2 m}{\rho_\phi}\int_0^1 \theta u dx &\le \delta_5 c\int_0^1 u_x^2 dx + \frac{c}{4\delta_5}\int_0^1 \theta^2 dx, \\
-\frac{a_2}{\rho_\phi}\int_0^1 u\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\mathscr{Y}(x,1,\sigma,t)d\sigma dx &\le c\delta_6\int_0^1 u_x^2 dx, \\
&\quad -\frac{c}{4\delta_6}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\mathscr{Y}^2(x,1,\sigma,t)d\sigma dx.
\end{aligned} \tag{91}$$

By letting $\delta_2 = \delta_3 = a_2/10\rho_\phi$, $\delta_4 = \delta_5 = \delta_6 = a_2/10c\rho_\phi$, and substituting into (89), we get (88).□

Lemma 7. *The functional*

$$D_3(t) := -\rho_u\int_0^1 u_t u dx, \tag{92}$$

satisfies

$$D_3'(t) \le -\rho_u\int_0^1 u_t^2 dx + 3a_1\int_0^1 u_x^2 dx + \frac{a_3}{4}\int_0^1 \phi_x^2 dx + \frac{\gamma^2}{4a_1}\int_0^1 \theta^2 dx. \tag{93}$$

Proof. Direct computations give

$$D_3'(t) = -\rho_u\int_0^1 u_t^2 dx + a_1\int_0^1 u_x^2 dx + a_2\int_0^1 u_x\phi_x dx - \gamma\int_0^1 u_x\theta dx. \tag{94}$$

Estimating (93) easily follows by using Young's inequality and (14).□

Lemma 8. *The functional*

$$D_4(t) := -c_0\alpha\int_0^1 \theta\left(\int_0^x w(y)dy\right)dx, \tag{95}$$

satisfies

$$D_4'(t) \le -\frac{c_0 k_1}{2}\int_0^1 \theta^2 dx + \varepsilon_2\int_0^1 u_t^2 dx + c\left(1 + \frac{1}{\varepsilon_2}\right)\int_0^1 w^2 dx + c\int_0^1 \phi_t^2 dx + c\int_0^1 w_x^2 dx. \tag{96}$$

Proof. Direct computations give

$$D_4'(t) = -c_0 k_1\int_0^1 \theta^2 dx + \alpha k_1\int_0^1 w^2 dx + \alpha\gamma\int_0^1 u_t w dx + c_0 d\int_0^1 \theta\phi_t dx - \alpha m\int_0^1 \phi_t\left(\int_0^x w(y)dy\right)dx + c_0 k_2\int_0^1 w_x\theta dx - c_0 k_3\int_0^1 \theta\left(\int_0^x w(y)dy\right)dx. \tag{97}$$

Estimate (96) easily follows by using Young's and Cauchy-Schwartz inequalities.□

Now, let us introduce the following functional used by.

Lemma 9. *The functional*

$$D_5(t) := \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma e^{-\sigma\rho}|\mu_2(\sigma)|\mathscr{Y}^2(x,\rho,\sigma,t)d\sigma d\rho dx, \tag{98}$$

satisfies

$$D_5'(t) \le -\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma|\mu_2(\sigma)|\mathscr{Y}^2(x,\rho,\sigma,t)d\sigma d\rho dx + \mu_1\int_0^1 \phi_t^2 dx - \eta_1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\sigma)|\mathscr{Y}^2(x,1,\sigma,t)d\sigma dx, \tag{99}$$

where $\eta_1 > 0$.

Proof. By differentiating D_5, with respect to t and using the last equation in (23), we have

$$\begin{aligned} D_5'(t) &= -2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-\sigma\rho}|\mu_2(\sigma)|\mathcal{Y}\mathcal{Y}_\rho(x,\rho,\sigma,t)d\sigma d\rho dx \\ &= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma e^{-\sigma\rho}|\mu_2(\sigma)|\mathcal{Y}^2(x,\rho,\sigma,t)d\sigma d\rho dx \\ &\quad - \int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)|\left[e^{-\sigma}\mathcal{Y}^2(x,1,\sigma,t) - \mathcal{Y}^2(x,0,\sigma,t)\right]d\sigma dx. \end{aligned} \tag{100}$$

Using the fact that $\mathcal{Y}(x,0,\sigma,t) = \phi_t(x,t)$, and $e^{-\sigma} \le e^{-\sigma\rho} \le 1$, for all $0 < \rho < 1$, we obtain

$$\begin{aligned} D_5'(t) &= -\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma|\mu_2(\sigma)|\mathcal{Y}^2(x,\rho,\sigma,t)d\sigma d\rho dx \\ &\quad - \int_0^1\int_{\tau_1}^{\tau_2} e^{-\sigma}|\mu_2(\sigma)|\mathcal{Y}^2(x,1,\sigma,t)d\sigma dx \\ &\quad + \left(\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)|\, d\sigma\right)\int_0^1 \phi_t^2 dx. \end{aligned} \tag{101}$$

□

Because $-e^{-\sigma}$ is a increasing function, we have $-e^{-\sigma} \le -e^{-\tau_2}$, for all $\sigma \in [\tau_1,\tau_2]$.

Finally, setting $\eta_1 = e^{-\tau_2}$ and recalling (27), we find (99). We are now ready to prove the main result.

Theorem 10. *Assume (27) holds. Then, $\forall t_0 > 0$, there exist $\beta_1, \beta_2 > 0$ such that the energy functional given by (75) satisfies*

$$E(t) \le \beta_1 e^{-\beta_2 t}, \quad \forall t \ge 0. \tag{102}$$

Proof. We define the functional of Lyapunov

$$\begin{aligned} \mathcal{L}(t) &:= NE(t) + N_1D_1(t) + N_2D_2(t) \\ &\quad + D_3(t) + N_4D_4(t) + N_5D_5(t), \end{aligned} \tag{103}$$

where $N, N_1, N_2, N_4, N_5 > 0$ we will assign them later.

By differentiating (103) and using (75), (84), (88), (93), (96), and (99), we have

$$\begin{aligned} \mathcal{L}'(t) &\le -\left[\frac{aN_1}{2} - cN_2 - \frac{a_3}{4}\right]\int_0^1 \phi_x^2 dx - \left[\rho_u - N_1\varepsilon_1 - N_4\varepsilon_2\right]\int_0^1 u_t^2 dx \\ &\quad - \left[\frac{a_2^2N_2}{2\rho_\phi} - 3a_1\right]\int_0^1 u_x^2 dx - \left[k_2N - cN_4\right]\int_0^1 w_x^2 dx \\ &\quad - \left[\eta_0N - cN_1\left(1 + \frac{1}{\varepsilon_1}\right) - N_2c - N_4c - \mu_1N_5\right]\int_0^1 \phi_t^2 dx \\ &\quad - \left[k_3N - cN_1 - cN_2 - cN_4\left(1 + \frac{1}{\varepsilon_2}\right)\right]\int_0^1 w^2 dx \\ &\quad - \left[\frac{c_0k_1N_4}{2} - cN_1 - cN_2 - \frac{\gamma^2}{4a_1}\right]\int_0^1 \theta^2 dx \\ &\quad - \left[N_5\eta_1 - cN_1 - cN_2\right]\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)|\mathcal{Y}^2(x,1,\sigma,t)d\sigma dx \\ &\quad - N_5\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma|\mu_2(\sigma)|\mathcal{Y}^2(x,\rho,\sigma,t)d\sigma d\rho dx. \end{aligned} \tag{104}$$

By setting

$$\varepsilon_1 = \frac{\rho_u}{4N_1}, \quad \varepsilon_2 = \frac{\rho_u}{4N_4}, \tag{105}$$

we obtain

$$\begin{aligned} \mathcal{L}'(t) &\le -\left[\frac{aN_1}{2} - cN_2 - \frac{a_3}{4}\right]\int_0^1 \phi_x^2 dx - \left[\frac{\rho_u}{2}\right]\int_0^1 u_t^2 dx \\ &\quad - \left[\frac{a_2^2N_2}{2\rho_\phi} - 3a_1\right]\int_0^1 u_x^2 dx - \left[k_2N - cN_4\right]\int_0^1 w_x^2 dx \\ &\quad - \left[\eta_0N - cN_1(1 + N_1) - N_2c - N_4c - \mu_1N_5\right]\int_0^1 \phi_t^2 dx \\ &\quad - \left[k_3N - cN_1 - cN_2 - cN_4(1 + N_4)\right]\int_0^1 w^2 dx \\ &\quad - \left[\frac{c_0k_1N_4}{2} - cN_1 - cN_2 - \frac{\gamma^2}{4a_1}\right]\int_0^1 \theta^2 dx \\ &\quad - \left[N_5\eta_1 - cN_1 - cN_2\right]\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)|\mathcal{Y}^2(x,1,\sigma,t)d\sigma dx \\ &\quad - N_5\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma|\mu_2(\sigma)|\mathcal{Y}^2(x,\rho,\sigma,t)d\sigma d\rho dx. \end{aligned} \tag{106}$$

At this point, we choose our constants.

We choose N_2 large enough so that

$$\alpha_1 = \frac{a_2^2N_2}{2\rho_\phi} - 3a_1 > 0, \tag{107}$$

then we pick N_1 large enough such that

$$\alpha_2 = \frac{aN_1}{2} - cN_2 - \frac{a_3}{4} > 0, \tag{108}$$

then we select N_4 and N_5 large enough such that

$$\begin{aligned} \alpha_3 &= \frac{c_0k_1N_4}{2} - cN_1 - cN_2 - \frac{\gamma^2}{4a_1} > 0, \\ \alpha_4 &= N_5\eta_1 - cN_1 - cN_2 > 0. \end{aligned} \tag{109}$$

Thus, we arrive at

$$\begin{aligned}\mathscr{L}'(t) \le& -\alpha_2\int_0^1 \phi_x^2 dx - \frac{\rho_u}{2}\int_0^1 u_t^2 dx - \alpha_1\int_0^1 u_x^2 dx \\ &- [\eta_0 N - c]\int_0^1 \phi_t^2 dx - [k_3 N - c]\int_0^1 w^2 dx \\ &- [k_2 N - c]\int_0^1 w_x^2 dx - \alpha_3\int_0^1 \theta^2 dx \\ &- \alpha_4\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)|\mathscr{Y}^2(x,1,\sigma,t)\, d\sigma dx \\ &- \alpha_5\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma|\mu_2(\sigma)|\mathscr{Y}^2(x,\rho,\sigma,t)\, d\sigma d\rho dx.\end{aligned} \tag{110}$$

where $\alpha_5 = \eta_1 N_5$.

On the other hand, if we let

$$\mathfrak{L}(t) = N_1 D_1(t) + N_2 D_2(t) + D_3(t) + N_4 D_4(t) + N_5 D_5(t), \tag{111}$$

then

$$\begin{aligned}|\mathfrak{L}(t)| \le& N_1\rho_\phi\int_0^1 |\phi\phi_t| dx + N_1\frac{a_2}{a_1}\rho_u\int_0^1 |\phi u_t| dx + N_1\frac{\mu_1}{2}\int_0^1 \phi^2 dx \\ &+ N_2 a_2\int_0^1 |\phi u_t - u\phi_t| dx + \rho_u\int_0^1 |u_t u| dx \\ &+ N_4 c_0\alpha\int_0^1 \left|\theta\left(\int_0^x w(y)dy\right)\right| dx \\ &+ N_5\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma e^{-\sigma\rho}|\mu_2(\sigma)|\mathscr{Y}^2(x,\rho,\sigma,t)\, d\sigma d\rho dx.\end{aligned} \tag{112}$$

According Young's, Cauchy-Schwartz, and Poincaré inequalities, we find

$$\begin{aligned}|\mathfrak{L}(t)| \le& c\int_0^1 \left(u_t^2 + \phi_t^2 + \phi_x^2 + u_x^2 + \theta^2 + w^2\right) dx \\ &+ c\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma|\mu_2(\sigma)|\mathscr{Y}^2(x,\rho,\sigma,t)\, d\sigma d\rho.\end{aligned} \tag{113}$$

On the other hand, we can write

$$\begin{aligned}a_1 u_x^2 + 2a_2\phi_x u_x + a_4\phi_x^2 =& \frac{1}{2}\left[a_1\left(u_x + \frac{a_2}{a_1}\phi_x\right)^2 + a_4\left(\phi_x + \frac{a_2}{a_4}u_x\right)^2\right. \\ &\left. + \left(a_1 - \frac{a_2^2}{a_4}\right)u_x^2 + \left(a_4 - \frac{a_2^2}{a_1}\right)\phi_x^2\right].\end{aligned} \tag{114}$$

Since $a_1 a_3 > a_2^2$ and (27), we deduce that

$$a_1 u_x^2 + 2a_2\phi_x u_x + a_4\phi_x^2 > \frac{1}{2}\left[\left(a_1 - \frac{a_2^2}{a_4}\right)u_x^2 + \left(a_4 - \frac{a_2^2}{a_1}\right)\phi_x^2\right]. \tag{115}$$

Hence, we get

$$|\mathfrak{L}(t)| = |\mathscr{L}(t) - NE(t)| \le cE(t), \tag{116}$$

that is,

$$(N - c)E(t) \le \mathscr{L}(t) \le (N + c)E(t). \tag{117}$$

At this point, we choose N large enough such that

$$N - c > 0, N\eta_0 - c > 0, Nk_3 - c > 0, Nk_2 - c > 0, \tag{118}$$

and exploiting (75), the estimates (110) and (117), respectively, gives

$$c_2 E(t) \le \mathscr{L}(t) \le c_3 E(t), \quad \forall t \ge 0, \tag{119}$$

$$\mathscr{L}'(t) \le -d_1 E(t), \quad \forall t \ge 0, \tag{120}$$

for some $d_1, c_2, c_3 > 0$.

Consequently, for some $\beta_2 > 0$, we find

$$\mathscr{L}'(t) \le -\beta_2\mathscr{L}(t), \quad \forall t \ge 0. \tag{121}$$

Integration of (120) over $(0, t)$ gives

$$\mathscr{L}(t) \le \mathscr{L}(0)e^{-\beta_2 t}, \quad \forall t \ge 0. \tag{122}$$

Consequently, (102) is established by virtue of (117) and (121).□

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

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