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Research Article

Properties of a Generalized Class of Weights Satisfying Reverse Hölder's Inequality

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In this paper, we will prove some fundamental properties of the discrete power mean operator $\mathscr{M}_p u(n) = (1/n\sum_{k=1}^{n} u^p(k))^{1/p}$, for $n \in \mathbb{I} \subseteq \mathbb{Z}_+$, of order p, where u is a nonnegative discrete weight defined on $\mathbb{I} \subseteq \mathbb{Z}_+$ the set of the nonnegative integers. We also establish some lower and upper bounds of the composition of different operators with different powers. Next, we will study the structure of the generalized discrete class $\mathscr{B}_p^q(B)$ of weights that satisfy the reverse Hölder inequality $\mathscr{M}_q u \le B\mathscr{M}_p u$, for positive real numbers p, q, and B such that $0 < p < q$ and $B > 1$. For applications, we will prove some self-improving properties of weights from $\mathscr{B}_p^q(B)$ and derive the self improving properties of the discrete Gehring weights as a special case. The paper ends by a conjecture with an illustrative sharp example.

1. Introduction

In [1], Muckenhoupt introduced a full characterization of the A^p – class of weights in connection with the boundedness of the Hardy-Littlewood maximal operator in the space $L_w^p(\mathbb{R}_+)$ with a weight w. Another important class of weights, the Gehring class G^q, for $1 < q < \infty$, was introduced by Gehring [2, 3] in connection with local integrability properties of the gradient of quasiconformal mappings. Due to the importance of these two classes in mathematical and harmonic analysis, the structure of them has been studied by several authors, and various results regarding the relation between them and their applications have been established. We refer the reader to the papers [1–23] and the references cited therein.

In recent years, the study of the discrete analogues in harmonic analysis becomes an active field of research. For example, the study of regularity and boundedness of discrete operator on l^p analogues for L^p – regularity, higher summability, and structure of discrete Muckenhoupt and Gehring weights has been considered by some authors, and we refer the reader to the papers [24–34] and the references they are cited.

We confine ourselves, in this paper in proving some new fundamental properties of a generalized discrete space of weights that satisfy reverse Hölder's inequality and prove some self-improving properties. As special cases, we will derive the self-improving properties of the discrete Gehring weights.

In the following, for the sake of completeness, we present the background and the basic definitions that will be used in this paper. Throughout this paper, $\mathbb{Z}_+$ stands for the set of nonnegative integers, i.e., $\mathbb{Z}_+ = \{1, 2, \cdots\}$. By an interval J, we mean a finite subset of $\mathbb{Z}_+$ consisting of consecutive integers, i.e., $J = \{a, a+1, \cdots, a+n\}$, $a,n \in \mathbb{Z}_+$, and $|J|$ stands for its cardinality. We assume that $1 < p < \infty$ and fix an interval $\mathbb{I} \subset \mathbb{Z}_+$ of the form $\mathbb{I} = \{1, 2, \cdots, N\}$, where N a nonnegative integer (or $[1, N] \subset \mathbb{Z}_+$). A discrete weight v defined on $\mathbb{I}$ is a sequence of nonnegative real numbers.

A discrete weight u defined on $\mathbb{I} \subseteq \mathbb{Z}_+$ belongs to the discrete Muckenhoupt class $\mathscr{A}^2(A)$ for $p > 1$ and $A > 1$, if the inequality

$$\frac{1}{|J|}\sum_{k\in J} u(k) \le Au(k), \text{for all } k \in J \quad (1)$$

holds for every subinterval $J \subset \mathbb{I}$, where $|J|$ is the cardinality of the set J. A discrete weight u defined on $\mathbb{I} \subset \mathbb{Z}_+$ is said to belong to the discrete Muckenhoupt class $\mathscr{A}^p(A)$ for $p > 1$ and $A > 1$ if the inequality

$$\left(\frac{1}{|J|}\sum_{k\in J} u(k)\right)\left(\frac{1}{|J|}\sum_{k\in J} u^{\frac{-1}{p-1}}(k)\right)^{p-1} \le A \quad (2)$$

holds for every subinterval interval $J \subset \mathbb{I}$. For a given exponent $p > 1$, we define the $\mathscr{A}^p$ – norm by the following quantity

$$[\mathscr{A}^p(u)] \coloneqq \sup_{J\subset \mathbb{I}}\left(\frac{1}{|J|}\sum_{k\in J} u(k)\right)\left(\frac{1}{|I|}\sum_{k\in J} u^{\frac{-1}{p-1}}(k)\right)^{p-1}, \quad (3)$$

where the supremum is taken over all intervals $J \subset \mathbb{I}$. Note that by Hölder's inequality $[\mathscr{A}^p(u)] \ge 1$ for all $1 < p < \infty$ and the following inclusions are true:

$$\text{if } 1 < p \le q < \infty, \text{then } \mathscr{A}^1 \subset \mathscr{A}^p \subset \mathscr{A}^q \text{ and } [\mathscr{A}^q(u)] \le [\mathscr{A}^p(u)]. \quad (4)$$

For a given exponent $q > 1$ and a constant $\mathscr{K} > 1$, a discrete nonnegative weight u defined on $\mathbb{I}$ belongs to the discrete Gehring class $\mathscr{G}^q(\mathscr{K})$ (or satisfies the reverse Hölder inequality) if for every subinterval $J \subseteq \mathbb{I}$, we have

$$\left(\frac{1}{|J|_{k\in J}} u^q(k)\right)^{1/q} \le \mathscr{K}\left(\frac{1}{|J|}\sum_{k\in J} u(k)\right). \quad (5)$$

For a given exponent $q > 1$, we define the $\mathscr{G}^q$ – norm by

$$[\mathscr{G}^q(u)] \coloneqq \sup_{J\subset \mathbb{I}}\left[\left(\frac{1}{|J|}\sum_{k\in J} u(k)\right)^{-1}\left(\frac{1}{|J|}\sum_{k\in J} u^q(k)\right)^{\frac{1}{q}}\right]^{\frac{q}{q-1}}, \quad (6)$$

where the supremum is taken over all intervals $J \subseteq \mathbb{I}$ and represents the best constant for which the $\mathscr{G}^q$ – condition holds true independently on the interval $J \subseteq \mathbb{I}$. Note that by Hölder's inequality $[\mathscr{G}^q(u)] \ge 1$ for all $1 < q < \infty$ and the following inclusion is true:

$$\text{if } 1 < p \le q < \infty, \text{then } \mathscr{G}^q \subset \mathscr{G}^p \text{ and } 1 \le [\mathscr{G}^p(u)] \le [\mathscr{G}^q(u)]. \quad (7)$$

By the generalized power mean operator $\mathscr{M}_q u$ of order $q \ne 0$ and nonnegative weight u defined on $\mathbb{I}$, we mean an operator of the form

$$\mathscr{M}_q u \coloneqq \left(\frac{1}{|J|}\sum_{k\in J} u^q(k)\right)^{1/q}, \quad \text{for} J \subset \mathbb{I}. \quad (8)$$

In [27], Böttcher and Seybold considered the operator (8) and proved that if

$$\mathscr{M}_p u \mathscr{M}_{-q} u \le \mathscr{C}, \quad (9)$$

where $p > 1$ and $1/p + 1/q = 1$, then there exists a constant $\delta > 0$ and $\mathscr{C}_1 < \infty$ depending only on p and u such that

$$\mathscr{M}_{p(1+\varepsilon)} u \le \mathscr{C}_1 \mathscr{M}_p u, \quad (10)$$

for all $\varepsilon \in 0, \delta]$ and all J of the form $|J| = 2^r$ with $r \in \mathbb{N}$ (the set of natural numbers). In [34], the authors proved that $\mathscr{M}_{-1}u \le \mathscr{M}_1 u, \mathscr{M}_q u \ge \mathscr{M}_1 u$, for all $q \ge 1, \mathscr{M}_q u \le \mathscr{M}_1 u$, for all $q < 1$ and if $p \le q$, then

$$\mathscr{M}_p u \le \mathscr{M}_q u, \quad (11)$$

for any nonnegative weight u defined on $\mathbb{I}$. In the present paper, we consider the class $\mathscr{B}_p^q(B)$ of all nonnegative weights u that satisfy the reverse Hölder inequality

$$\mathscr{M}_q u \le B \mathscr{M}_p u, \text{for} J \subset \mathbb{I}, \quad (12)$$

where the constant $B > 1$ is independent of p,q, and J and $q > p$. The smallest constant B independent on the interval J and satisfies the inequality (12) is called the $\mathscr{B}_p^q$ – norm which is given by

$$\left[\mathscr{B}_p^q(u)\right] \coloneqq \sup_{J\subset \mathbb{I}}\left(\mathscr{M}_p u\right)^{-\frac{1}{p}}\left(\mathscr{M}_q u\right)^{\frac{1}{q}}, \text{for} J \subset \mathbb{I}. \quad (13)$$

We say that u is a $\mathscr{B}_p^q$ – weight if its $\mathscr{B}_p^q$ – norm is finite, i.e.,

$$v \in \mathscr{B}_p^q \Leftarrow \left[\mathscr{B}_p^q(u)\right] < +\infty. \quad (14)$$

When we fix a constant $\mathscr{C} > 1$, the triple of real numbers $(p, q, \mathscr{C})$ defines the $\mathscr{B}_p^q$ – discrete class:

$$u \in \mathscr{B}_p^q(\mathscr{C}) \Leftarrow \left[\mathscr{B}_p^q(u)\right] \le \mathscr{C}, \quad (15)$$

and we will refer to $\mathscr{C}$ as the $\mathscr{B}_p^q$ – constant of the class. It is immediate to observe that the classes $\mathscr{A}^p$ and $\mathscr{G}^q$ are special cases of the discrete class $\mathscr{B}_p^q$ of weights as follows:

$$\mathscr{A}^p \coloneqq \mathscr{B}_{1/p-1}^1, \text{ and } \mathscr{G}^q \coloneqq \mathscr{B}_1^q. \quad (16)$$

In this paper, we aim to study the structure of the general class $\mathscr{B}_p^q$ and use the new properties to prove some self-improving properties. The paper is organized as follows: In Section 2, we state and prove some basic lemmas concerning the bounds of the generalized power mean operator $\mathscr{M}_p$. In Section 3, we will establish some lower and upper bounds of the composition of operators by using two special functions ρ_p and ρ_q (will be defined later) and prove some inclusion properties. For example, we prove that if $u \in \mathscr{B}_p^q(B)$, then $\mathscr{M}_q u \in \mathscr{B}_p^\delta(B_1)$ with exact values of δ and B_1. In Section 4, we present some applications of the main results and prove the self-improving property of a monotone weights from $\mathscr{B}_p^q$, i.e., we will prove that if $u \in \mathscr{B}_p^q(B)$, then $u \in \mathscr{B}_p^\lambda(B_1)$ with exact values of λ and B_1. For illustrations, we will derive the self improving property of the discrete Gehring weights as special cases. The paper ends by a conjecture with the self-improving of the Muckenhoupt weights with an illustrative example.

2. Basic Lemmas

In this section, we state and prove the basic lemmas and establish some properties of the power mean operators that will be used to prove the main results later. We will assume that $\mathbb{I} \ll \{1, 2, \cdots, N\}$ is a fixed finite subset of $\mathbb{Z}_+$, and we recall the power mean operator $\mathscr{M}_q u$ that we will consider in this paper is given by

$$\mathscr{M}_q u(n) = \left(\frac{1}{n}\sum_{k=1}^{n} u^q(k)\right)^{1/q}, \text{for all } n \in \mathbb{I}, \tag{17}$$

for any nonnegative weight $u : \mathbb{I} \longrightarrow \mathbb{R}^+$ and $q \in \mathbb{R} \setminus \{0\}$ and by $\mathscr{M}_q^p u(k)$, we mean that $(\mathscr{M}_q u(k))^p$. For the sake of conventions, we assume that $0 \cdot \infty = 0$ and $0/0 = 0$ and $\sum_{k=a}^{b} y(k) = 0$, whenever $a > b$, and

$$\Delta\left(\sum_{k=1}^{n-1} y(k)\right) = y(n), \quad \sum_{k=1}^{n-1} \Delta y(k) = y(n) - y(1). \tag{18}$$

The product rule in the discrete form is given by

$$\Delta(u(k)v(k)) = u(k)\Delta v(k) + v(k+1)\Delta u(k), \tag{19}$$

where $\Delta u(k) = u(k+1) - u(k)$. The summation by parts now is given by

$$\sum_{k=1}^{n} \Delta u(k) v(k+1) = u(k)v(k)\big|_{k=1}^{n+1} - \sum_{k=1}^{n} u(k)\Delta v(k). \tag{20}$$

Lemma 1. *Let $p < q$ and $p.q \neq 0$, and $u : \mathbb{I} \longrightarrow \mathbb{R}^+$ is a nonnegative weight. Then, the following*

$$\sum_{k=1}^{n} \Delta\left[(k-1)\mathscr{M}_q^p u(k-1)\right] = (n)\mathscr{M}_q^p u(n), \tag{21}$$

$$\sum_{k=1}^{n} \Delta\left[(k-1)\mathscr{M}_p^q u(k-1)\right] = (n)\mathscr{M}_p^q u(n) \tag{22}$$

hold for all $n \in \mathbb{I}$.

Proof. By applying the second relation in (18) with $u(k) = (k-1)\mathscr{M}_q^p u(k-1)$, we have

$$\sum_{k=1}^{n} \Delta\left[(k-1)\mathscr{M}_q^p u(k-1)\right] = \sum_{k=1}^{n} \left[(k)\mathscr{M}_q^p u(k) - (k-1)\mathscr{M}_q^p u(k-1)\right] = (n)\mathscr{M}_q^p u(n), \tag{23}$$

which is the desired equations (21). Similarly, by applying the second relation in (18) with $u(k) = (k-1)\mathscr{M}_p^q u(k-1)$, we have

$$\sum_{k=1}^{n} \Delta\left[(k-1)\mathscr{M}_p^q u(k-1)\right] = \sum_{k=1}^{n} \left[(k)\mathscr{M}_p^q u(k) - (k-1)\mathscr{M}_p^q u(k-1)\right] = (n)\mathscr{M}_p^q u(n), \tag{24}$$

which is the equality (22). The proof is complete.

Lemma 2. *Assume that $u : \mathbb{I} \rightarrow \mathbb{R}^+$ be any nonnegative weight and $q \in \mathbb{R} \setminus \{0\}$. Then, following properties hold:*

(1) If u is nonincreasing, then $\mathscr{M}_q u$ is nonincreasing and $\mathscr{M}_q u(n) \geq u(n)$, for all $n \in \mathbb{I}$

(2) If u is nondecreasing, then $\mathscr{M}_q u$ is nondecreasing and $\mathscr{M}_q u(n) \leq u(n)$, for all $n \in \mathbb{I}$

Proof. (1)). From the definition of $\mathscr{M}_q u$ and the fact that u is nonincreasing, we get for $q = 1$ that

$$\mathscr{M}_1 u(n) = \left(\frac{1}{n}\sum_{k=1}^{n} u(k)\right) \geq \left(\frac{1}{n}\sum_{k=1}^{n} u(n)\right) = u(n). \tag{25}$$

For the general case when $q \neq 1$, we have also for all $n \in \mathbb{I}$ that

$$\mathscr{M}_q u(n) = \left(\frac{1}{n}\sum_{k=1}^{n} u^q(k)\right)^{1/q} \geq \left(\frac{1}{n}\sum_{k=1}^{n} u^q(n)\right)^{1/q} = u(n). \tag{26}$$

From this inequality, we get that

$$n u^q(n) \leq \sum_{k=1}^{n} u^q(k), \text{for all } n \in \mathbb{I}. \tag{27}$$

Now, by using (27) and the fact that u is nonincreasing, we obtain that

$$\begin{aligned}
&\Delta(\mathscr{M}_q u(n)) \\
&= \frac{1}{(n+1)^{1/q}}\left(\sum_{k=1}^{n+1} u^q(k)\right)^{1/q} - \frac{1}{n^{1/q}}\left(\sum_{k=1}^{n} u^q(k)\right)^{1/q} \\
&= \frac{\left[n\sum_{k=1}^{n+1} u^q(k)\right]^{1/q} - \left[(n+1)\sum_{k=1}^{n} u^q(k)\right]^{1/q}}{[n(n+1)]^{1/q}} \\
&= \frac{\left[nu^q(n+1) + n\sum_{k=1}^{n} u^q(k)\right]^{1/q} - \left[n\sum_{k=1}^{n} u^q(k) + \sum_{k=1}^{n} u^q(k)\right]^{1/q}}{[n(n+1)]^{1/q}} \\
&\leq \frac{\left[nu^q(n) + n\sum_{k=1}^{n} u^q(k)\right]^{1/q} - \left[n\sum_{k=1}^{n} u^q(k) + \sum_{k=1}^{n} u^q(k)\right]^{1/q}}{[n(n+1)]^{1/q}} \\
&\leq \frac{\left[\sum_{k=1}^{n} u^q(k) + n\sum_{k=1}^{n} u^q(k)\right]^{1/q} - \left[n\sum_{k=1}^{n} u^q(k) + \sum_{k=1}^{n} u^q(k)\right]^{1/q}}{[n(n+1)]^{1/q}} \\
&= 0,
\end{aligned} \tag{28}$$

and thus $\mathscr{M}_q u(n)$ is nonincreasing.

2). From the definition of $\mathscr{M}_q u(n)$ and the fact that $u(n)$ is nondecreasing, we have for $q = 1$ that

$$\mathscr{M}_1 u(n) = \frac{1}{n}\sum_{k=1}^{n} u(k) \leq \frac{1}{n}\sum_{k=1}^{n} u(n) = u(n). \tag{29}$$

For the general case when $q \neq 1$, we have also for all $n \in \mathbb{I}$ that

$$\mathscr{M}_q u(n) = \left(\frac{1}{n}\sum_{k=1}^{n} u^q(k)\right)^{1/q} \leq \left(\frac{1}{n}\sum_{k=1}^{n} u^q(n)\right)^{1/q} = u(n). \tag{30}$$

From this inequality, we see that

$$nu^q(n) \geq \sum_{k=1}^{n} u^q(k), \text{for all } n \in \mathbb{I}. \tag{31}$$

Then, by using inequality (31) and the fact that u is nondecreasing and proceeding as in the first case, we obtain that

$$\begin{aligned}
\Delta(\mathscr{M}_q u(n)) &= \left(\frac{1}{n+1}\sum_{k=1}^{n+1} u^q(k)\right)^{1/q} - \left(\frac{1}{n}\sum_{k=1}^{n} u^q(k)\right)^{1/q} \\
&= \frac{\left[n\sum_{k=1}^{n+1} u^q(k)\right]^{1/q} - \left[(n+1)\sum_{k=1}^{n} u^q(k)\right]^{1/q}}{[n(n+1)]^{1/q}}.
\end{aligned} \tag{32}$$

We proceed as in the proof of the nonincreasing case to get that $\Delta(\mathscr{M}_q u(n)) \geq 0$, and thus $\mathscr{M}_q u(n)$ is nondecreasing. The proof is complete.

The following lemma will play an important rule in proving the main results.

Lemma 3. *Let α and β be positive numbers and $g\colon \mathbb{I} \to \mathbb{R}^+$ be any nonnegative weight such that*

$$\alpha\left(\frac{1}{k}\sum_{\tau=1}^{k} g(\tau)\right) \leq g(k) \leq \beta\left(\frac{1}{k}\sum_{\tau=1}^{k} g(\tau)\right), \text{for all } k \in \mathbb{I}. \tag{33}$$

Then, for every $r, s \in 0, N]$ such that $r < s$, we have that

$$\left(\frac{s}{r}\right)^{\alpha-1}\left(\frac{1}{r}\sum_{k=1}^{r} g(k)\right) \leq \frac{1}{s}\sum_{k=1}^{s} g(k) \leq \left(\frac{s}{r}\right)^{\beta-1}\left(\frac{1}{r}\sum_{k=1}^{r} g(k)\right). \tag{34}$$

Proof. The left-side of inequality (33) writes

$$\frac{1}{k}\sum_{\tau=1}^{k} g(\tau) \leq \frac{1}{\alpha} g(k), \tag{35}$$

and by multiplying both sides by $k^{-\alpha}$and summating from $k = m > 1$ to s, we have

$$\sum_{k=m}^{s} k^{-\alpha-1}\sum_{\tau=1}^{k} g(\tau) \leq \sum_{k=m}^{s} \frac{1}{\alpha} k^{-\alpha} g(k). \tag{36}$$

The left-side of inequality (36) can be written in the form

$$\begin{aligned}
\sum_{k=m}^{s} k^{-\alpha-1}\sum_{\tau=1}^{k} g(\tau) &= \sum_{k=1}^{s}\left(k^{-\alpha-1}\sum_{\tau=1}^{k} g(\tau)\right) - \sum_{k=1}^{m-1}\left(k^{-\alpha-1}\sum_{\tau=1}^{k} g(\tau)\right), \\
&\geq \sum_{k=1}^{s}\left(k^{-\alpha-1}\sum_{\tau=1}^{k} g(\tau)\right) - \sum_{k=1}^{m-1}\left(k^{-\alpha-1}\sum_{\tau=1}^{k+1} g(\tau)\right).
\end{aligned} \tag{37}$$

By applying Fubini's Theorem on the right-hand side, we have that

$$\sum_{k=m}^{s} k^{-\alpha-1}\sum_{\tau=1}^{k} g(\tau) \geq \sum_{k=1}^{s} g(k)\sum_{\tau=k}^{s} \tau^{-\alpha-1} - \sum_{k=1}^{m-1} g(k)\sum_{\tau=k+1}^{m-1} \tau^{-\alpha-1}. \tag{38}$$

By using the inequality,

$$\gamma x^{\gamma-1}(x-y) \geq x^\gamma - y^\gamma \geq \gamma y^{\gamma-1}(x-y), \text{for} x \geq y > 0, \gamma > 1 \text{or} \gamma < 0, \tag{39}$$

with $\gamma = -\alpha < 0$, and we have

$$\begin{aligned}
&\sum_{\tau=k}^{s} \tau^{-\alpha-1} \geq \sum_{\tau=k}^{s} \frac{-1}{\alpha}\Delta\tau^{-\alpha} = \frac{k^{-\alpha}}{\alpha} - \frac{(s+1)^{-\alpha}}{\alpha} \geq \frac{k^{-\alpha}}{\alpha} - \frac{s^{-\alpha}}{\alpha}, \\
&\sum_{\tau=k+1}^{m-1} \tau^{-\alpha-1} \leq \sum_{\tau=k+1}^{m-1} \frac{-1}{\alpha}\Delta(\tau-1)^{-\alpha} = \frac{k^{-\alpha}}{\alpha} - \frac{(m-1)^{-\alpha}}{\alpha}.
\end{aligned} \tag{40}$$

Then, (38) becomes

$$\sum_{k=m}^{s} k^{-\alpha-1} \sum_{\tau=1}^{k} g(\tau) \geq \sum_{k=1}^{s} \left[\frac{k^{-\alpha}}{\alpha} - \frac{s^{-\alpha}}{\alpha}\right] g(k) - \sum_{k=1}^{m-1} \left[\frac{k^{-\alpha}}{\alpha} - \frac{(m-1)^{-\alpha}}{\alpha}\right] g(k) = \sum_{k=m}^{s} \frac{k^{-\alpha}}{\alpha} g(k) - \frac{s^{-\alpha}}{\alpha} \sum_{k=1}^{s} g(k) + \frac{(m-1)^{-\alpha}}{\alpha} \sum_{k=1}^{m-1} g(k). \tag{41}$$

By substituting (41) into (36), we have

$$\sum_{k=m}^{s} \frac{1}{\alpha} k^{-\alpha} g(k) \geq \sum_{k=m}^{s} k^{-\alpha-1} \sum_{\tau=1}^{k} g(\tau) \geq \sum_{k=m}^{s} \frac{k^{-\alpha}}{\alpha} g(k) - \frac{s^{-\alpha}}{\alpha} \sum_{k=1}^{s} g(k) + \frac{(m-1)^{-\alpha}}{\alpha} \sum_{k=1}^{m-1} g(k), \tag{42}$$

which implies that

$$-\frac{s^{-\alpha}}{\alpha} \sum_{k=1}^{s} g(k) + \frac{(m-1)^{-\alpha}}{\alpha} \sum_{k=1}^{m-1} g(k) \leq 0, \tag{43}$$

which is equivalent to

$$\frac{1/r \sum_{k=1}^{r} g(k)}{1/s \sum_{k=1}^{s} g(k)} \leq \frac{s}{r} \left(\frac{r}{s}\right)^{\alpha} = \left(\frac{r}{s}\right)^{\alpha-1}, \tag{44}$$

where $r = m - 1 > 0$. This proves the left-side of inequality (34). Now, the right-side of inequality (33) writes

$$\frac{1}{k} \sum_{\tau=1}^{k} g(\tau) \geq \frac{1}{\beta} g(k), \tag{45}$$

and by multiplying both sides by $k^{-\beta}$, and summating from $k = m > 1$ to s, we have

$$\sum_{k=m}^{s} k^{-\beta-1} \sum_{\tau=1}^{k} g(\tau) \geq \sum_{k=m}^{s} \frac{1}{\beta} k^{-\beta} g(k). \tag{46}$$

The left-side of inequality (46) can be written in the form

$$\sum_{k=m}^{s} k^{-\beta-1} \sum_{\tau=1}^{k} g(\tau) = \sum_{k=1}^{s} \left(k^{-\beta-1} \sum_{\tau=1}^{k} g(\tau)\right) - \sum_{k=1}^{m-1} \left(k^{-\beta-1} \sum_{\tau=1}^{k} g(\tau)\right) \leq \sum_{k=1}^{s} \left(k^{-\beta-1} \sum_{\tau=1}^{k+1} g(\tau)\right) - \sum_{k=1}^{m-1} \left(k^{-\beta-1} \sum_{\tau=1}^{k} g(\tau)\right). \tag{47}$$

This implies by applying Fubini's Theorem on the left-hand side that

$$\sum_{k=m}^{s} k^{-\beta-1} \sum_{\tau=1}^{k} g(\tau) \leq \sum_{k=1}^{s} g(k) \sum_{\tau=k+1}^{s} \tau^{-\beta-1} - \sum_{k=1}^{m-1} g(k) \sum_{\tau=k}^{m-1} \tau^{-\beta-1}. \tag{48}$$

By using the inequality,

$$\gamma x^{\gamma-1}(x-y) \geq x^{\gamma} - y^{\gamma} \geq \gamma y^{\gamma-1}(x-y), \text{for} x \geq y > 0, \gamma > 1 \text{or} \gamma < 0, \tag{49}$$

with $\gamma = -\beta < 0$, and we have

$$\sum_{\tau=k+1}^{s} \tau^{-\beta-1} \leq \sum_{\tau=k+1}^{s} \frac{-1}{\beta} \Delta(\tau-1)^{-\beta} = \frac{k^{-\beta}}{\beta} - \frac{s^{-\beta}}{\beta},$$
$$\sum_{\tau=k}^{m-1} \tau^{-\beta-1} \geq \sum_{\tau=k}^{m-1} \frac{-1}{\beta} \Delta \tau^{-\beta} = \frac{k^{-\beta}}{\beta} - \frac{m^{-\beta}}{\beta} \geq \frac{k^{-\beta}}{\beta} - \frac{(m-1)^{-\beta}}{\beta}. \tag{50}$$

Then, (48) becomes

$$\sum_{k=m}^{s} k^{-\beta-1} \sum_{\tau=1}^{k} g(\tau) \leq \sum_{k=1}^{s} \left[\frac{k^{-\beta}}{\beta} - \frac{s^{-\beta}}{\beta}\right] g(k) - \sum_{k=1}^{m-1} \left[\frac{k^{-\beta}}{\beta} - \frac{(m-1)^{-\beta}}{\beta}\right] g(k) = \sum_{k=m}^{s} \frac{k^{-\beta}}{\beta} g(k) - \frac{s^{-\beta}}{\beta} \sum_{k=1}^{s} g(k) + \frac{(m-1)^{-\beta}}{\beta} \sum_{k=1}^{m-1} g(k). \tag{51}$$

By substituting (51) into (46), we have

$$\sum_{k=m}^{s} \frac{1}{\beta} k^{-\beta} g(k) \leq \sum_{k=m}^{s} k^{-\beta-1} \sum_{\tau=1}^{k} g(\tau) \leq \sum_{k=m}^{s} \frac{k^{-\beta}}{\beta} g(k) - \frac{s^{-\beta}}{\beta} \sum_{k=1}^{s} g(k) + \frac{(m-1)^{-\beta}}{\beta} \sum_{k=1}^{m-1} g(k), \tag{52}$$

which implies that

$$-\frac{s^{-\beta}}{\beta} \sum_{k=1}^{s} g(k) + \frac{(m-1)^{-\beta}}{\beta} \sum_{k=1}^{m-1} g(k) \geq 0, \tag{53}$$

which is equivalent to

$$\frac{1/r \sum_{k=1}^{r} g(k)}{1/s \sum_{k=1}^{s} g(k)} \geq \frac{s}{r} \left(\frac{r}{s}\right)^{\beta} = \left(\frac{r}{s}\right)^{\beta-1}, \tag{54}$$

where $r = m - 1 > 0$. This proves the right-side of inequality (34). The proof is complete.

3. Fundamental Properties of Power Mean Operators

In this section, we will prove some fundamental properties of the generalized power mean operator $\mathscr{M}_p u(n)$ which is given by

$$\mathscr{M}_p u(n) = \left(\frac{1}{n}\sum_{k=1}^{n} u^p(k)\right)^{1/p}, \text{for all } n \in \mathbb{I}, \tag{55}$$

where p is assumed to be positive for the rest of the paper. In order to prove the main results, we will use the properties of the function

$$\rho_p(\lambda) = \left(1 - \frac{p}{\lambda}\right)^{1/p}, p > 0 \tag{56}$$

of the variable $\lambda \in (-\infty,0) \cup p, \infty)$. It is clear that the function $\rho_p(\lambda)$ is continuous and increases from 1 to $+\infty$ on $(-\infty, 0)$ and from 0 to 1 on $[p, \infty)$ and for $\lambda \neq p$, we have that

$$\begin{aligned}\rho_p(\lambda)\rho_p(p-\lambda) &= \left(1 - \frac{p}{\lambda}\right)^{1/p}\left(1 - \frac{p}{p-\lambda}\right)^{1/p} \\ &= \left(\frac{\lambda - p}{\lambda}\right)^{1/p}\left(\frac{\lambda}{\lambda - p}\right)^{1/p} = 1.\end{aligned} \tag{57}$$

To understand the importance of the function $\rho_p(\lambda)$, we consider the sequence $u_0(n) = n^{-1/\lambda}$. Then, we have

$$\mathscr{M}_p u_0(n) = \left(\frac{1}{n}\sum_{k=1}^{n} k^{-p/\lambda}\right)^{1/p}. \tag{58}$$

We consider the different cases of the power $-p/\lambda + 1$. First, assume that $0 < -p/\lambda + 1 < 1$, and by employing the inequality

$$\gamma x^{\gamma-1}(x-y) \le x^\gamma - y^\gamma \le \gamma y^{\gamma-1}(x-y), \text{for} x \ge y > 0, 0 \le \gamma \le 1, \tag{59}$$

with $\gamma = -p/\lambda + 1$, we have

$$\Delta(k-1)^{1-p/\lambda} = k^{1-p/\lambda} - (k-1)^{1-p/\lambda} \ge (1 - p/\lambda)k^{-p/\lambda}. \tag{60}$$

Then, we have

$$\begin{aligned}\mathscr{M}_p u_0(n) &= \left(\frac{1}{n}\sum_{k=1}^{n} k^{-p/\lambda}\right)^{1/p} \le \left(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{1-p/\lambda}\Delta(k-1)^{1-p/\lambda}\right)^{1/p} \\ &= \left(\frac{1}{n}\frac{1}{1-p/\lambda}n^{1-p/\lambda}\right)^{1/p} = \frac{n^{-1/\lambda}}{(1-p/\lambda)^{1/p}} \\ &= \frac{u_0(n)}{\rho_p(\lambda)}, \text{for all } n \in \mathbb{I}.\end{aligned} \tag{61}$$

Next, we consider the case when $-p/\lambda + 1 > 1$, and by employing inequality (39) with $\gamma = 1 - p/\lambda > 1$, we have

$$\Delta(k-1)^{1-p/\lambda} = k^{1-p/\lambda} - (k-1)^{1-\frac{p}{\lambda}} \le \left(1 - \frac{p}{\lambda}\right)k^{-\frac{p}{\lambda}}. \tag{62}$$

Then, we have that

$$\begin{aligned}\mathscr{M}_p u_0(n) &= \left(\frac{1}{n}\sum_{k=1}^{n} k^{-p/\lambda}\right)^{1/p} \ge \left(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{1-p/\lambda}\Delta(k-1)^{1-p/\lambda}\right)^{1/p} \\ &= \frac{1}{(1-p/\lambda)^{1/p}}\left(\frac{n^{1-p/\lambda}}{n}\right)^{1/p} = \frac{u_0(n)}{\rho_p(\lambda)}, \text{for all } n \in \mathbb{I}.\end{aligned} \tag{63}$$

The meaning of $\rho_p(\lambda)$ now arises from the fact that the sequence $u_0(n) = n^{-1/\lambda}$ satisfies the equivalence between $\mathscr{M}_p u_0(n)$ and the fraction $u_0(n)/\rho_p(\lambda)$, for all $n \in \mathbb{I}$. Let $0 < p < q$ and define the function $S_{p,q}(\lambda)$ by

$$S_{p,q}(\lambda) := \frac{\rho_p(\lambda)}{\rho_q(\lambda)}, \text{ for } \lambda \in (-\infty,0) \cup (q,\infty). \tag{64}$$

The function $C_{p,q}(\lambda)$ is continuous and increases on the interval $(-\infty, 0)$ and decreases on the interval (q, ∞) with $C_{p,q}((\beta,\infty)) = (1,\infty)$. Therefore, for any $B > 1$, the equation

$$S_{p,q}(\lambda) = \frac{(1-p/\lambda)^{1/p}}{(1-q/\lambda)^{1/q}} = B \tag{65}$$

has exactly two roots: a positive root $\lambda^+ \in (q,\infty)$ and a negative root $\lambda^- \in (-\infty,0)$. The nonnegative weight $v : \mathbb{I} \longrightarrow \mathbb{R}^+$ is said to be belong to $\mathscr{B}_p^q(B)$ if v satisfies the reverse Hölder inequality

$$\mathscr{M}_q v(n) \le B\mathscr{M}_p v(n), \quad \text{for all } n \in \mathbb{I}; \tag{66}$$

that is,

$$\left(\frac{1}{n}\sum_{k=1}^{n} v^q(k)\right)^{1/q} \le B\left(\frac{1}{n}\sum_{k=1}^{n} v^p(k)\right)^{1/p}, \tag{67}$$

for all $n \in \mathbb{I}$, where the constant $B > 1$ is independent of p,q, and $q > p$. Now, we are ready to state and prove the main properties of the operator (55) and the composition of different operators with different powers.

Theorem 4. *Let $0 < p < q$, and $u : \mathbb{I} \longrightarrow \mathbb{R}^+$ be any nonnegative, monotone weight. If $u \in \mathscr{B}_p^q(B)$ for $B > 1$, then*

$$\rho_p(\lambda^+) \le \frac{\mathscr{M}_q u(n)}{\mathscr{M}_p(\mathscr{M}_q u)(n)} \le \rho_p(\lambda^-), \textit{ for all } n \in \mathbb{I}, \tag{68}$$

where λ^+ and λ^- are the roots of (65).

Proof. By applying the product rule (19) on the term $\Delta[(k-1)\mathscr{M}_q^p u(k-1)]$ with $u = k-1$ and $v = \mathscr{M}_q^p u(k-1)$, we obtain that

$$\begin{aligned}\Delta\left[(k-1)\mathscr{M}_q^p u(k-1)\right] &= \Delta\left[(k-1)\left(\frac{1}{k-1}\sum_{s=1}^{k-1} u^q(s)\right)^{p/q}\right] \\ &= \left(\frac{1}{k}\sum_{s=1}^{k} u^q(s)\right)^{p/q} + (k-1)\Delta\left(\frac{1}{k-1}\sum_{s=1}^{k-1} u^q(s)\right)^{p/q}.\end{aligned} \tag{69}$$

Now, we find the estimate of the second term in (69) and consider two cases of the behavior of the monotone weight u. First, we assume that u is nondecreasing. Then, by Lemma 2, we have that

$$\mathscr{M}_q u(k) = \left(\frac{1}{k}\sum_{s=1}^{k} u^q(s)\right)^{1/q} \tag{70}$$

is also nondecreasing and by applying the elementary inequality (59), for $\gamma = p/q < 1$, we obtain

$$\begin{aligned}(k-1)\Delta\left(\frac{\sum_{s=1}^{k-1} u^q(s)}{k-1}\right)^{p/q} &= (k-1)\left[\left(\frac{\sum_{s=1}^{k} u^q(s)}{k}\right)^{p/q} - \left(\frac{\sum_{s=1}^{k-1} u^q(s)}{k-1}\right)^{p/q}\right] \\ &\geq \frac{p}{q}(k-1)\left(\frac{1}{k}\sum_{s=1}^{k} u^q(s)\right)^{p/q-1}\left(\frac{(k-1)\sum_{s=1}^{k} u^q(s) - k\sum_{s=1}^{k-1} u^q(s)}{k(k-1)}\right) \\ &= \frac{p}{q}(k-1)\left(\frac{1}{k}\sum_{s=1}^{k} u^q(s)\right)^{p/q-1}\left(\frac{ku^q(k) - \sum_{s=1}^{k} u^q(s)}{k(k-1)}\right) \\ &= \frac{p}{q}\left(\frac{1}{k}\sum_{s=1}^{k} u^q(s)\right)^{p/q-1} u^q(k) - \frac{p}{q}\left(\frac{1}{k}\sum_{s=1}^{k} u^q(s)\right)^{p/q}.\end{aligned} \tag{71}$$

By combining (71) and (69), we obtain

$$\begin{aligned}\Delta\left[(k-1)\mathscr{M}_q^p u(k-1)\right] \geq{}& \left(\frac{q-p}{q}\right)\left(\frac{\sum_{s=1}^{k} u^q(s)}{k}\right)^{p/q} \\ &+ \frac{p}{q}\left(\frac{\sum_{s=1}^{k} u^q(s)}{k}\right)^{p/q-1} u^q(k).\end{aligned} \tag{72}$$

Next, we assume that u is nonincreasing. Then, by Lemma 2, we see that $\mathscr{M}_q u(k)$ is nonincreasing and by employing the inequality (59) again, we have that

$$\begin{aligned}(k-1)\Delta\left(\frac{\sum_{s=1}^{k-1} u^q(s)}{k-1}\right)^{p/q} &= -(k-1)\left[\left(\frac{\sum_{s=1}^{k-1} u^q(s)}{k-1}\right)^{p/q} - \left(\frac{\sum_{s=1}^{k} u^q(s)}{k}\right)^{p/q}\right] \\ &\geq -\frac{p}{q}(k-1)\left(\frac{\sum_{s=1}^{k} u^q(s)}{k}\right)^{p/q-1}\left(\frac{k\sum_{s=1}^{k-1} u^q(s) - (k-1)\sum_{s=1}^{k} u^q(s)}{k(k-1)}\right) \\ &= -\frac{p}{q}(k-1)\left(\frac{1}{k}\sum_{s=1}^{k} u^q(s)\right)^{p/q-1}\left[\frac{\sum_{s=1}^{k} u^q(s)}{k(k-1)} - \frac{u^q(k)}{k-1}\right] \\ &= -\frac{p}{q}\left(\frac{1}{k}\sum_{s=1}^{k} u^q(s)\right)^{p/q} + \frac{p}{q}\left(\frac{1}{k}\sum_{s=1}^{k} u^q(s)\right)^{p/q-1} u^q(k).\end{aligned} \tag{73}$$

By combining (69) and (73), we again obtain the inequality (72). Now, by summing (72) from 1 to n, and applying (21), we obtain

$$\begin{aligned}\left[\mathscr{M}_q u(n)\right]^p \geq{}& \left(1-\frac{p}{q}\right)\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{s=1}^{k} u^q(s)\right)^{p/q} \\ &+ \frac{p}{q}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{s=1}^{k} u^q(s)\right)^{p/q-1} u^q(k).\end{aligned} \tag{74}$$

From the definition of $\mathscr{M}_q u$, we see that the first term in (74) is given by

$$\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{s=1}^{k} u^q(s)\right)^{p/q} = \left[\mathscr{M}_p\left(\mathscr{M}_q u\right)(n)\right]^p. \tag{75}$$

Now, we simplify the term

$$\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{s=1}^{k} u^q(s)\right)^{p/q-1} u^q(k). \tag{76}$$

By applying reverse Hölder's inequality for $p/q < 1$ and $p/(p-q)$, we obtain that

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{s=1}^{k} u^q(s)\right)^{p/q-1} u^q(k) &\geq \left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{s=1}^{k} u^q(s)\right)^{p/q}\right)^{(p-q)/p}\left(\frac{1}{n}\sum_{k=1}^{n} u^p(k)\right)^{q/p} \\ &= \left[\mathscr{M}_p\left(\mathscr{M}_q u\right)(n)\right]^{p-q}\left[\mathscr{M}_p u(n)\right]^q.\end{aligned} \tag{77}$$

By substituting (75) and (77) into (74), dividing by $p[\mathscr{M}_p(\mathscr{M}_q u)(n)]^p$ and then applying (66), we obtain

$$\begin{aligned}\frac{1}{p}\frac{[\mathscr{M}_q u(n)]^p}{[\mathscr{M}_p(\mathscr{M}_q u)(n)]^p} &\geq \left(\frac{1}{p}-\frac{1}{q}\right)+\frac{1}{q}\frac{[\mathscr{M}_p u(n)]^q}{[\mathscr{M}_p(\mathscr{M}_q u)(n)]^q}\\ &\geq \left(\frac{1}{p}-\frac{1}{q}\right)+B^{-q}\frac{1}{q}\frac{[\mathscr{M}_q u(n)]^q}{[\mathscr{M}_p(\mathscr{M}_q u)(n)]^q}.\end{aligned} \tag{78}$$

By setting

$$\lambda := p\frac{[\mathscr{M}_p(\mathscr{M}_q u)(n)]^p}{[\mathscr{M}_p(\mathscr{M}_q u)(n)]^p-[\mathscr{M}_q u(n)]^p}, \tag{79}$$

we see that the inequality (78) can be written in the form

$$\begin{aligned}&B^{-q}\left[1-\frac{[\mathscr{M}_p(\mathscr{M}_q u)(n)]^p-[\mathscr{M}_q u(n)]^p}{[\mathscr{M}_p(\mathscr{M}_q u)(n)]^p}\right]^{q/p}\\ &=B^{-q}\left[\frac{[\mathscr{M}_q u(n)]^p}{[\mathscr{M}_p(\mathscr{M}_q u)(n)]^p}\right]^{q/p}\leq 1-\frac{q}{p}+\frac{q}{p}\frac{[\mathscr{M}_q u(n)]^p}{[\mathscr{M}_p(\mathscr{M}_q u)(n)]^p}\\ &\leq 1-\frac{q}{p}\left(1-\frac{[\mathscr{M}_q u(n)]^p}{[\mathscr{M}_p(\mathscr{M}_q u)(n)]^p}\right)\\ &=1-\frac{q}{p}\left(\frac{[\mathscr{M}_p(\mathscr{M}_q u)(n)]^p-[\mathscr{M}_q u(n)]^p}{[\mathscr{M}_p(\mathscr{M}_q u)(n)]^p}\right).\end{aligned} \tag{80}$$

This inequality can be written now as

$$\left(1-\frac{p}{\lambda}\right)^{1/p}\leq B\left(1-\frac{q}{\lambda}\right)^{1/q}, \tag{81}$$

or equivalently

$$S_{p,q}(\lambda)=\frac{(1-p/\lambda)^{1/p}}{(1-q/\lambda)^{1/q}}\leq B, \text{for all } p<q. \tag{82}$$

This means that $\lambda\in(-\infty,\lambda^-]\cup\lambda^+,+\infty)$. The properties of ρ_p imply that

$$\rho_p(\lambda^+)\leq\rho_p(\lambda)\leq\rho_p(\lambda^-), \tag{83}$$

and since

$$\begin{aligned}\rho_p(\lambda)=\left(1-\frac{p}{\lambda}\right)^{1/p}&=\left(1-\frac{[\mathscr{M}_p(\mathscr{M}_q u)(n)]^p-[\mathscr{M}_q u(n)]^p}{[\mathscr{M}_p(\mathscr{M}_q u)(n)]^p}\right)^{1/p}\\ &=\left(1-1+\frac{[\mathscr{M}_q u(n)]^p}{[\mathscr{M}_p(\mathscr{M}_q u)(n)]^p}\right)^{1/p}\\ &=\frac{[\mathscr{M}_q u(n)]}{\mathscr{M}_p(\mathscr{M}_q u)(n)},\end{aligned} \tag{84}$$

we obtain that

$$\rho_p(\lambda^+)\leq\frac{[\mathscr{M}_q u(n)]}{\mathscr{M}_p(\mathscr{M}_q u)(n)}\leq\rho_p(\lambda^-), \tag{85}$$

which is the desired inequality (68). The proof is complete.

Theorem 5. *Let $0<p<q$, and $u:\mathbb{I}\to\mathbb{R}^+$ be any nonnegative, monotone weight. If $u\in\mathscr{B}_p^q(B)$ for $B>1$, then*

$$\rho_q(\lambda^+)\leq\frac{\mathscr{M}_p u(n)}{\mathscr{M}_q(\mathscr{M}_p u)(n)}\leq\rho_q(\lambda^-), \textit{for all } n\in\mathbb{I}, \tag{86}$$

where λ^+ and λ^- are the roots of (65).

Proof. By applying the product rule (19) on the term $\Delta[(k-1)\mathscr{M}_p^q u(k-1)]$, with $u=k-1$ and $v=\mathscr{M}_p^q u(k-1)$, we obtain that

$$\begin{aligned}&\Delta[(k-1)\mathscr{M}_p^q u(k-1)]\\ &=\Delta\left[(k-1)\left(\frac{1}{k-1}\sum_{s=1}^{k-1}u^p(s)\right)^{q/p}\right]\\ &=\left(\frac{1}{k}\sum_{s=1}^{k}u^p(s)\right)^{q/p}+(k-1)\Delta\left(\frac{1}{k-1}\sum_{s=1}^{k-1}u^p(s)\right)^{q/p}.\end{aligned} \tag{87}$$

First, we assume that u is nonincreasing. Then, by Lemma 2, we have then $\mathscr{M}_p u(k)$ that is nonincreasing. By employing inequality (39) with $\gamma=q/p>1$, we obtain

$$\begin{aligned}&(k-1)\Delta\left(\frac{\sum_{s=1}^{k-1}u^p(s)}{k-1}\right)^{q/p}\\ &=-(k-1)\left[\left(\frac{\sum_{s=1}^{k-1}u^p(s)}{k-1}\right)^{q/p}-\left(\frac{\sum_{s=1}^{k}u^p(s)}{k}\right)^{q/p}\right]\\ &\leq-\frac{q}{p}\left(\frac{1}{k}\sum_{s=1}^{k}u^p(s)\right)^{q/p-1}\left[\frac{\sum_{s=1}^{k}u^p(s)-ku^p(s)}{k}\right]\\ &=-\frac{q}{p}\left(\frac{1}{k}\sum_{s=1}^{k}u^p(s)\right)^{q/p}+\frac{q}{p}\left(\frac{1}{k}\sum_{s=1}^{k}u^p(s)\right)^{q/p-1}u^p(s).\end{aligned} \tag{88}$$

By substituting (88) into (87), we obtain

$$\begin{aligned}\Delta\left[(k-1)\mathscr{M}_p^q u(k-1)\right] \\ \leq \left(\frac{p-q}{p}\right)\left(\frac{\sum_{s=1}^{k} u^p(s)}{k}\right)^{q/p} + \frac{q}{p}\left(\frac{\sum_{s=1}^{k} u^p(s)}{k}\right)^{q/p-1} u^p(k).\end{aligned} \tag{89}$$

Next, we assume that u is nondecreasing. Then, by Lemma 2, we have that $\mathscr{M}_p u(k)$ is nondecreasing and by applying the inequality (39) with $\gamma = q/p > 1$, we get

$$\begin{aligned}&(k-1)\Delta\left(\frac{\sum_{s=1}^{k-1} u^p(s)}{k-1}\right)^{q/p} \\ &= (k-1)\left[\left(\frac{\sum_{s=1}^{k} u^p(s)}{k}\right)^{q/p} - \left(\frac{\sum_{s=1}^{k-1} u^p(s)}{k-1}\right)^{q/p}\right] \\ &\leq \frac{q}{p}(k-1)\left(\frac{1}{k}\sum_{s=1}^{k} u^p(s)\right)^{q/p-1}\left[\frac{-\sum_{s=1}^{k} u^p(s) + k u^p(s)}{k(k-1)}\right] \\ &= -\frac{q}{p}\left(\frac{1}{k}\sum_{s=1}^{k} u^p(s)\right)^{q/p} + \frac{q}{p}\left(\frac{1}{k}\sum_{s=1}^{k} u^p(s)\right)^{q/p-1} u^p(k).\end{aligned} \tag{90}$$

Now, by combining (87) and (90), we obtain again (89). Now, by summing (89) from 1 to n and applying (22), we obtain that

$$\begin{aligned}\left[\mathscr{M}_p u(n)\right]^q \leq \left(1-\frac{q}{p}\right)\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{s=1}^{k} u^p(s)\right)^{q/p} \\ + \frac{q}{p}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{s=1}^{k} u^p(s)\right)^{q/p-1} u^p(k).\end{aligned} \tag{91}$$

The first term in (91) is given by

$$\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{s=1}^{k} u^p(s)\right)^{q/p} = \left[\mathscr{M}_q\left(\mathscr{M}_p u\right)(n)\right]^q. \tag{92}$$

Now, we simplify the second term

$$\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{s=1}^{k} u^p(s)\right)^{q/p-1} u^p(k), \tag{93}$$

in (91), by applying Hölder's inequality for $q/p > 1$ and $q/(q-p)$, to obtain

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{s=1}^{k} u^p(s)\right)^{q/p-1} u^p(k) \\ &\leq \left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{s=1}^{k} u^p(s)\right)^{q/p}\right)^{(q-p)/q}\left(\frac{1}{n}\sum_{k=1}^{n} u^q(k)\right)^{p/q} \\ &= \left[\mathscr{M}_q\left(\mathscr{M}_p u\right)(n)\right]^{q-p}\left[\mathscr{M}_q u(n)\right]^p.\end{aligned} \tag{94}$$

By substituting (92) and (94) into (91), dividing by $q\left[\mathscr{M}_q(\mathscr{M}_p u)(n)\right]^q$ and applying (66), we obtain

$$\begin{aligned}\frac{\left[\mathscr{M}_p u(n)\right]^q}{q\left[\mathscr{M}_q\left(\mathscr{M}_p u\right)(n)\right]^q} &\leq \left(\frac{1}{q}-\frac{1}{p}\right) + \frac{1}{p}\frac{\left[\mathscr{M}_p u(n)\right]^p}{\left[\mathscr{M}_q\left(\mathscr{M}_p u\right)(n)\right]^p} \\ &\leq \left(\frac{1}{q}-\frac{1}{p}\right) + \frac{B^p}{p}\frac{\left[\mathscr{M}_p u(n)\right]^p}{\left[\mathscr{M}_q\left(\mathscr{M}_p u\right)(n)\right]^p}.\end{aligned} \tag{95}$$

Inequality (95) now takes the form

$$\begin{aligned}&\left[B\left(1-\frac{\left[\mathscr{M}_q\left(\mathscr{M}_p u\right)(n)\right]^q - \left[\mathscr{M}_p u(n)\right]^q}{\left[\mathscr{M}_q\left(\mathscr{M}_p u\right)(n)\right]^q}\right)^{1/q}\right]^p \\ &= \left[B\left(\frac{\left[\mathscr{M}_p u(n)\right]^q}{\left[\mathscr{M}_q\left(\mathscr{M}_p u\right)(n)\right]^q}\right)^{1/q}\right]^p = B^p\frac{\left[\mathscr{M}_p u(n)\right]^p}{\left[\mathscr{M}_q\left(\mathscr{M}_p u\right)(n)\right]^p} \\ &\geq 1-\frac{p}{q}+\frac{p}{q}\frac{\left[\mathscr{M}_p u(n)\right]^q}{\left[\mathscr{M}_q\left(\mathscr{M}_p u\right)(n)\right]^q} = 1-\frac{p}{q}\left(1-\frac{\left[\mathscr{M}_p u(n)\right]^q}{\left[\mathscr{M}_q\left(\mathscr{M}_p u\right)(n)\right]^q}\right) \\ &= 1-\frac{p}{q}\frac{\left[\mathscr{M}_q\left(\mathscr{M}_p u\right)(n)\right]^q - \left[\mathscr{M}_p u(n)\right]^q}{\left[\mathscr{M}_q\left(\mathscr{M}_p u\right)(n)\right]^q}.\end{aligned} \tag{96}$$

By setting

$$\lambda := q\frac{\left[\mathscr{M}_q\left(\mathscr{M}_p u\right)(n)\right]^q}{\left[\mathscr{M}_q\left(\mathscr{M}_p u\right)(n)\right]^q - \left[\mathscr{M}_p u(n)\right]^q}, \tag{97}$$

we see that inequality (96) becomes

$$\left(1-\frac{p}{\lambda}\right)^{1/p} \leq B\left(1-\frac{q}{\lambda}\right)^{1/q}, \tag{98}$$

or equivalently,

$$S_{p,q}(\lambda) = \frac{(1-p/\lambda)^{1/p}}{(1-q/\lambda)^{1/q}} \leq B. \tag{99}$$

This means that $\lambda \in (-\infty, \lambda^-] \cup \lambda^+, +\infty)$. The properties of ρ_q implies that

$$\rho_q\left(\lambda^+\right) \leq \rho_q(\lambda) \leq \rho_q\left(\lambda^-\right), \tag{100}$$

and since

$$\rho_q(\lambda) = \left(1 - \frac{q}{\lambda}\right)^{1/q} = \left(1 - \frac{\left[\mathcal{M}_q\left(\mathcal{M}_p u\right)(n)\right]^q - \left[\mathcal{M}_p u(n)\right]^q}{\left[\mathcal{M}_q\left(\mathcal{M}_p u\right)(n)\right]^q}\right)^{1/q} = \left(1 - 1 + \frac{\left[\mathcal{M}_p u(n)\right]^q}{\left[\mathcal{M}_q\left(\mathcal{M}_p u\right)(n)\right]^q}\right)^{1/q} = \frac{\left[\mathcal{M}_p u(n)\right]}{\left[\mathcal{M}_q\left(\mathcal{M}_p u\right)(n)\right]}, \tag{101}$$

we obtain that

$$\rho_q\left(\lambda^+\right) \le \frac{\mathcal{M}_p u(n)}{\mathcal{M}_q\left(\mathcal{M}_p u\right)(n)} \le \rho_q\left(\lambda^-\right), \tag{102}$$

which is the required inequality (86). The proof is complete.

The assumptions and the conclusions of Theorems 4 and 5 will be used in proving the following theorems.

Theorem 6. *Assume that the conditions in Theorems 4 and 5 hold. Then, the compositions*

$$(n)^{1/\lambda^-}\mathcal{M}_q\left(\mathcal{M}_p u\right)(n), \text{and} (n)^{1/\lambda^-}\mathcal{M}_p\left(\mathcal{M}_q u\right)(n) \tag{103}$$

are nonincreasing for all $n \in \mathbb{I}$, and the compositions

$$(n)^{1/\lambda^+}\mathcal{M}_p\left(\mathcal{M}_q u\right)(n), \text{and} (n)^{1/\lambda^+}\mathcal{M}_q\left(\mathcal{M}_p u\right)(n) \tag{104}$$

are nondecreasing for all $n \in \mathbb{I}$.

Proof. Raise the inequality (68) to the power p, we get that

$$\left(1 - \frac{p}{\lambda^+}\right) \le \frac{\left(1/k\sum_{\tau=1}^{k} u^q(\tau)\right)^{p/q}}{1/s\sum_{k=1}^{s}\left(1/k\sum_{\tau=1}^{k} u^q(\tau)\right)^{p/q}} \le \left(1 - \frac{p}{\lambda^-}\right). \tag{105}$$

Setting

$$g(k) = \mathcal{M}_q u(k) = \left(\frac{1}{k}\sum_{s=1}^{k} u^q(s)\right)^{p/q}, \tag{106}$$

we see that $g(k)$ satisfies the inequality (34) in Lemma 3 with

$$\alpha = \left(1 - \frac{p}{\lambda^+}\right) \quad \text{and } \beta = \left(1 - \frac{p}{\lambda^-}\right). \tag{107}$$

Therefore, we see that

$$\left(\frac{s}{r}\right)^{\alpha-1} \le \frac{1/s\sum_{k=1}^{s}\left(1/k\sum_{\tau=1}^{k} u^q(\tau)\right)^{p/q}}{1/r\sum_{k=1}^{r}\left(1/k\sum_{s=1}^{k} u^q(\tau)\right)^{p/q}} \le \left(\frac{s}{r}\right)^{\beta-1}, \tag{108}$$

and so we have that

$$\left(\frac{s}{r}\right)^{-\frac{p}{\lambda^+}} \le \frac{1/s\sum_{k=1}^{s}\left(1/k\sum_{\tau=1}^{k} u^q(\tau)\right)^{p/q}}{1/r\sum_{k=1}^{r}\left(1/k\sum_{s=1}^{k} u^q(\tau)\right)^{p/q}} \le \left(\frac{s}{r}\right)^{-\frac{p}{\lambda^-}}. \tag{109}$$

If raised the last inequality to the power $-1/p$, we see that this inequality is equivalent to the monotonicity of the compositions

$$(n)^{1/\lambda^\pm}\mathcal{M}_p\left(\mathcal{M}_q u\right)(n). \tag{110}$$

Analogously, the inequality (86) and the same proof imply the monotonicity of the compositions

$$(n)^{1/\lambda^\pm}\mathcal{M}_q\left(\mathcal{M}_p u\right)(n). \tag{111}$$

The proof is complete.

Theorem 7. *Let $0 < p < q$ and u be any nonnegative weight, and λ^+ and λ^- are the roots of equation (65).*

(i) If $u \in \mathcal{B}_p^q(B)$ for $B > 1$ and $\lambda < \lambda^+$ and $\lambda \neq 0$, then

$$\mathcal{M}_p u \in \mathcal{B}_q^\lambda\left(\frac{\rho_q(\lambda^-)}{\rho_\lambda(\lambda^+)}\right), \text{ and } \mathcal{M}_q u \in \mathcal{B}_p^\lambda\left(\frac{\rho_p(\lambda^-)}{\rho_\lambda(\lambda^+)}\right). \tag{112}$$

(ii) If $u \in \mathcal{B}_p^q(B)$ for $B > 1$ and $\lambda > \lambda^-$, and $\lambda \neq 0$, then

$$\mathcal{M}_p u \in \mathcal{B}_\lambda^q\left(\frac{\rho_\lambda(\lambda^-)}{\rho_q(\lambda^+)}\right), \text{ and } \mathcal{M}_q u \in \mathcal{B}_\lambda^p\left(\frac{\rho_\lambda(\lambda^-)}{\rho_p(\lambda^+)}\right). \tag{113}$$

Proof. (i). Since λ is either positive or negative, we will discuss the two cases:

(1) Assume that $\lambda > 0$. By raising (86) to the power λ, we obtain for $m < n$ that

$$m^{-\lambda/\lambda^-}\rho_q^\lambda\left(\lambda^+\right)\left[m^{1/\lambda^-}\mathcal{M}_q\left(\mathcal{M}_p u\right)(m)\right]^\lambda \le \left(\mathcal{M}_p u(m)\right)^\lambda \le m^{-\lambda/\lambda^+}\rho_q^\lambda\left(\lambda^-\right)\left[m^{1/\lambda^+}\mathcal{M}_q\left(\mathcal{M}_p u\right)(m)\right]^\lambda. \tag{114}$$

By using the monotonicity of

$$n^{1/\lambda^-}\mathcal{M}_q\left(\mathcal{M}_p u\right)(n), \text{ and } n^{1/\lambda^+}\mathcal{M}_q\left(\mathcal{M}_p u\right)(n), \tag{115}$$

(see Theorem 6), we have that

$$m^{-\lambda/\lambda^-}\rho_q^\lambda(\lambda^+)\left[n^{1/\lambda^-}\mathscr{M}_q(\mathscr{M}_p u)(n)\right]^\lambda \le (\mathscr{M}_p u(m))^\lambda \le m^{-\lambda/\lambda^+}\rho_q^\lambda(\lambda^-)\left[n^{1/\lambda^+}\mathscr{M}_q(\mathscr{M}_p u)(n)\right]^\lambda. \quad (116)$$

Since $\lambda < \lambda^+$, by summing (116) from $m = 1$ to n, dividing by n, and raising it to the power $1/\lambda > 0$, we get

$$\left(\frac{\rho_q^\lambda(\lambda^+)\left[n^{1/\lambda^-}\mathscr{M}_q(\mathscr{M}_p u)(n)\right]^\lambda}{n}\sum_{m=1}^{n} m^{-\lambda/\lambda^-}\right)^{1/\lambda} \le \left(\frac{1}{n}\sum_{m=1}^{n}(\mathscr{M}_p u(m))^\lambda\right)^{1/\lambda} \le \left(\frac{\rho_q^\lambda(\lambda^-)\left[n^{1/\lambda^+}\mathscr{M}_q(\mathscr{M}_p u)(n)\right]^\lambda}{n}\sum_{m=1}^{n} m^{-\lambda/\lambda^+}\right)^{1/\lambda}. \quad (117)$$

Since $-1 < -\lambda/\lambda^+ < 0$, we have that $0 < -\lambda/\lambda^+ + 1 < 1$. By applying (59) with $\gamma = -\lambda/\lambda^+ + 1 < 1$, we have

$$\Delta(m-1)^{-\lambda/\lambda^+ + 1} = m^{-\lambda/\lambda^+ + 1} - (m-1)^{-\lambda/\lambda^+ + 1} \ge (-\lambda/\lambda^+ + 1)m^{-\lambda/\lambda^+}, \quad (118)$$

and then

$$\sum_{m=1}^{n} m^{-\lambda/\lambda^+} \le \frac{1}{-\lambda/\lambda^+ + 1}\sum_{m=1}^{n}\Delta(m-1)^{-\lambda/\lambda^+ + 1} = \frac{n^{-\lambda/\lambda^+ + 1}}{(-\lambda/\lambda^+) + 1}. \quad (119)$$

So, we have

$$\left(\frac{\rho_q^\lambda(\lambda^-)\left[n^{1/\lambda^+}\mathscr{M}_q(\mathscr{M}_p u)(n)\right]^\lambda}{n}\sum_{m=1}^{n} m^{-\lambda/\lambda^+}\right)^{1/\lambda} \le \left(\frac{\rho_q^\lambda(\lambda^-)\left[n^{1/\lambda^+}\mathscr{M}_q(\mathscr{M}_p u)(n)\right]^\lambda}{n}\frac{n^{-\lambda/\lambda^+ + 1}}{(-\lambda/\lambda^+) + 1}\right)^{1/\lambda} = \left(\rho_q^\lambda(\lambda^-)\left[\mathscr{M}_q(\mathscr{M}_p u)(n)\right]^\lambda\frac{1}{(1-\lambda/\lambda^+)}\right)^{1/\lambda} = \frac{\rho_q(\lambda^-)}{\rho_\lambda(\lambda^+)}\mathscr{M}_q(\mathscr{M}_p u)(n). \quad (120)$$

Similarly, since $\lambda^- < 0$, then $-\lambda/\lambda^- > 0$ and hence $-\lambda/\lambda^- + 1 > 1$. By applying (39) with $\gamma = -\lambda/\lambda^- + 1 > 1$, we have

$$\Delta(m-1)^{-\lambda/\lambda^- + 1} = m^{-\lambda/\lambda^- + 1} - (m-1)^{-\lambda/\lambda^- + 1} \le (-\lambda/\lambda^- + 1)m^{-\lambda/\lambda^-}, \quad (121)$$

and then

$$\sum_{m=1}^{n} m^{-\lambda/\lambda^-} \ge \frac{\sum_{m=1}^{n}\Delta(m-1)^{-\lambda/\lambda^- + 1}}{(-\lambda/\lambda^- + 1)} = \frac{n^{-\lambda/\lambda^- + 1}}{(-\lambda/\lambda^- + 1)}. \quad (122)$$

In this case, we have

$$\left(\frac{\rho_q^\lambda(\lambda^+)\left[n^{1/\lambda^-}\mathscr{M}_q(\mathscr{M}_p u)(n)\right]^\lambda}{n}\sum_{m=1}^{n} m^{-\lambda/\lambda^-}\right)^{1/\lambda} \ge \left(\frac{\rho_q^\lambda(\lambda^+)\left[n^{1/\lambda^-}\mathscr{M}_q(\mathscr{M}_p u)(n)\right]^\lambda}{n}\frac{n^{-\lambda/\lambda^- + 1}}{(-\lambda/\lambda^- + 1)}\right)^{1/\lambda} = \frac{\rho_q(\lambda^+)}{\rho_\lambda(\lambda^-)}\mathscr{M}_q(\mathscr{M}_p u)(n). \quad (123)$$

By substituting (120) and (123) into (117), we obtain that

$$\frac{\rho_q(\lambda^+)}{\rho_\lambda(\lambda^-)}\mathscr{M}_q(\mathscr{M}_p u)(n) \le \mathscr{M}_\lambda(\mathscr{M}_p u)(n) \le \frac{\rho_q(\lambda^-)}{\rho_\lambda(\lambda^+)}\mathscr{M}_q(\mathscr{M}_p u)(n), \quad (124)$$

which implies that

$$\mathscr{M}_p u \in \mathscr{B}_q^\lambda\left(\frac{\rho_q(\lambda^-)}{\rho_\lambda(\lambda^+)}\right), \quad (125)$$

that is the first relation in (112) holds for $\lambda > 0$.

(2). Assume that $\lambda < 0$. By raising (86) to the power λ, we obtain for $m < n$ that

$$m^{-\lambda/\lambda^-}\rho_q^\lambda(\lambda^+)\left[m^{1/\lambda^-}\mathscr{M}_q(\mathscr{M}_p u)(m)\right]^\lambda \ge (\mathscr{M}_p u(m))^\lambda \ge m^{-\lambda/\lambda^+}\rho_q^\lambda(\lambda^-)\left[m^{1/\lambda^+}\mathscr{M}_q(\mathscr{M}_p u)(m)\right]^\lambda. \quad (126)$$

By using the monotonicity of

$$n^{1/\lambda^-}\mathscr{M}_q(\mathscr{M}_p u)(n), \text{and} n^{1/\lambda^+}\mathscr{M}_q(\mathscr{M}_p u)(n), \quad (127)$$

(see Theorem 6), we have that

$$m^{-\lambda/\lambda^-}\rho_q^\lambda(\lambda^+)\left[n^{1/\lambda^-}\mathscr{M}_q(\mathscr{M}_p u)(n)\right]^\lambda \ge (\mathscr{M}_p u(m))^\lambda \ge m^{-\lambda/\lambda^+}\rho_q^\lambda(\lambda^-)\left[n^{1/\lambda^+}\mathscr{M}_q(\mathscr{M}_p u)(n)\right]^\lambda. \quad (128)$$

Since $\lambda < \lambda^+$, by summing (128) from $m = 1$ to n, dividing by n, and raising it to the power $1/\lambda > 0$, we get

$$\left(\rho_q^\lambda(\lambda^+)\left[n^{1/\lambda^-}\mathcal{M}_q(\mathcal{M}_p u)(n)\right]^\lambda \frac{1}{n}\sum_{m=1}^{n} m^{-\lambda/\lambda^-}\right)^{1/\lambda} \le \left(\frac{1}{n}\sum_{m=1}^{n}\left(\mathcal{M}_p u(m)\right)^\lambda\right)^{\frac{1}{\lambda}} \le \left(\rho_q^\lambda(\lambda^-)\left[n^{\frac{1}{\lambda^+}}\mathcal{M}_q(\mathcal{M}_p u)(n)\right]^\lambda \frac{1}{n}\sum_{m=1}^{n} m^{-\frac{\lambda}{\lambda^+}}\right)^{\frac{1}{\lambda}}. \tag{129}$$

Since $0 < -\lambda/\lambda^+$, then $-\lambda/\lambda^+ + 1 > 1$ and by applying (39) with $\gamma = -\lambda/\lambda^+ + 1$, we have

$$\Delta(m-1)^{-\lambda/\lambda^+ + 1} = m^{-\lambda/\lambda^+ + 1} - (m-1)^{-\lambda/\lambda^+ + 1} \le \left(-\lambda/\lambda^+ + 1\right) m^{-\lambda/\lambda^+}, \tag{130}$$

and then

$$\sum_{m=1}^{n} m^{-\lambda/\lambda^+} \ge \frac{1}{-\lambda/\lambda^+ + 1}\sum_{m=1}^{n}\Delta(m-1)^{-\lambda/\lambda^+ + 1} = \frac{n^{-\lambda/\lambda^+ + 1}}{-\lambda/\lambda^+ + 1}. \tag{131}$$

Since $\lambda < 0$, we have that

$$\left(\frac{\rho_q^\lambda(\lambda^-)\left[n^{1/\lambda^+}\mathcal{M}_q(\mathcal{M}_p u)(n)\right]^\lambda}{n}\sum_{m=1}^{n} m^{-\lambda/\lambda^+}\right)^{1/\lambda} \le \left(\frac{\rho_q^\lambda(\lambda^-)\left[n^{1/\lambda^+}\mathcal{M}_q(\mathcal{M}_p u)(n)\right]^\lambda}{n}\frac{n^{-\lambda/\lambda^+ + 1}}{-\lambda/\lambda^+ + 1}\right)^{1/\lambda} = \frac{\rho_q(\lambda^-)}{\rho_\lambda(\lambda^+)}\mathcal{M}_q(\mathcal{M}_p u)(n)\left[\frac{n^{-\lambda/\lambda^+ + 1}}{n^{-\lambda/\lambda^+ + 1}}\right]^{1/\lambda} = \frac{\rho_q(\lambda^-)}{\rho_\lambda(\lambda^+)}\mathcal{M}_q(\mathcal{M}_p u)(n). \tag{132}$$

Similarly, since $\lambda^- < 0$ and $\lambda < 0$, then $-\lambda/\lambda^- < 0$ and $-\lambda/\lambda^- + 1 < 1$. If $0 < -\lambda/\lambda^- + 1 < 1$, we apply inequality (59) to obtain

$$\Delta(m-1)^{-\lambda/\lambda^- + 1} = m^{-\lambda/\lambda^- + 1} - (m-1)^{-\lambda/\lambda^- + 1} \ge (-\lambda/\lambda^- + 1) m^{-\lambda/\lambda^-}, \tag{133}$$

and then

$$\sum_{m=1}^{n} m^{-\lambda/\lambda^-} \le \frac{1}{-\lambda/\lambda^- + 1}\sum_{m=1}^{n}\Delta(m-1)^{-\lambda/\lambda^- + 1} = \frac{n^{-\lambda/\lambda^- + 1}}{-\lambda/\lambda^- + 1}. \tag{134}$$

Since $\lambda < 0$, we have that

$$\left(\frac{\rho_q^\lambda(\lambda^+)\left[n^{1/\lambda^-}\mathcal{M}_q(\mathcal{M}_p u)(n)\right]^\lambda}{n}\sum_{m=1}^{n} m^{-\lambda/\lambda^-}\right)^{1/\lambda} \ge \left(\frac{\rho_q^\lambda(\lambda^+)\left[n^{1/\lambda^-}\mathcal{M}_q(\mathcal{M}_p u)(n)\right]^\lambda}{n}\frac{\left[n^{-\lambda/\lambda^- + 1}\right]}{-\lambda/\lambda^- + 1}\right)^{1/\lambda} = \frac{\rho_q(\lambda^+)}{\rho_\lambda(\lambda^-)}\mathcal{M}_q(\mathcal{M}_p u)(n)\left[\frac{n^{-\lambda/\lambda^- + 1}}{n^{-\lambda/\lambda^- + 1}}\right]^{1/\lambda} = \frac{\rho_q(\lambda^+)}{\rho_\lambda(\lambda^-)}\mathcal{M}_q(\mathcal{M}_p u)(n). \tag{135}$$

If $-\lambda/\lambda^- + 1 < 0$, we apply (39) to obtain

$$\Delta(m-1)^{-\lambda/\lambda^- + 1} = m^{-\lambda/\lambda^- + 1} - (m-1)^{-\lambda/\lambda^- + 1} \le (-\lambda/\lambda^- + 1) m^{-\lambda/\lambda^-}, \tag{136}$$

and then

$$\sum_{m=1}^{n} m^{-\lambda/\lambda^-} \le \frac{1}{-\lambda/\lambda^- + 1}\sum_{m=1}^{n}\Delta(m-1)^{-\lambda/\lambda^- + 1} \ge \frac{n^{-\lambda/\lambda^- + 1}}{-\lambda/\lambda^- + 1}. \tag{137}$$

In this case, we have

$$\left(\frac{\rho_q^\lambda(\lambda^+)\left[n^{1/\lambda^-}\mathcal{M}_q(\mathcal{M}_p u)(n)\right]^\lambda}{n}\sum_{m=1}^{n} m^{-\lambda/\lambda^-}\right)^{1/\lambda} \ge \left(\frac{\rho_q^\lambda(\lambda^+)\left[n^{1/\lambda^-}\mathcal{M}_q(\mathcal{M}_p u)(n)\right]^\lambda}{n}\frac{n^{-\lambda/\lambda^- + 1}}{-\lambda/\lambda^- + 1}\right)^{1/\lambda} = \frac{\rho_q(\lambda^+)}{\rho_\lambda(\lambda^-)}\mathcal{M}_q(\mathcal{M}_p u)(n). \tag{138}$$

From the two cases, we conclude that for all λ^-, we have

$$\left(\frac{\rho_q^\lambda(\lambda^+)\left[n^{1/\lambda^-}\mathcal{M}_q(\mathcal{M}_p u)(n)\right]^\lambda}{n}\sum_{m=1}^{n} m^{-\lambda/\lambda^-}\right)^{1/\lambda} \ge \frac{\rho_q(\lambda^+)}{\rho_\lambda(\lambda^-)}\mathcal{M}_q(\mathcal{M}_p u)(n). \tag{139}$$

By substituting (132) and (139) into (117), we have

$$\frac{\rho_q(\lambda^+)}{\rho_\lambda(\lambda^-)}\mathcal{M}_q(\mathcal{M}_p u)(n) \le \mathcal{M}_\lambda(\mathcal{M}_p u)(n) \le \frac{\rho_q(\lambda^-)}{\rho_\lambda(\lambda^+)}\mathcal{M}_q(\mathcal{M}_p u)(n). \tag{140}$$

This gives us that

$$\mathcal{M}_p u \in \mathcal{B}_q^\lambda\left(\frac{\rho_q(\lambda^-)}{\rho_\lambda(\lambda^+)}\right), \tag{141}$$

which is the first relation in (112) in the case $\lambda < 0$. Similarly, we can prove the first relation in (112) by using relation (68). Analogously, we prove the two relations in (113) by using the same technique and inequalities (68) and (86). The proof is complete.

4. Self-Improving Properties

In Theorem 7, we proved that the power mean operators $\mathcal{M}_p u$ and $\mathcal{M}_q u$ of the weight $u \in \mathcal{B}_p^q$ satisfy the reverse Hölder inequality with some better exponents. However, the fact that the mean $\mathcal{M}_p u$ or $\mathcal{M}_q u$ belongs to some class $\mathcal{B}_{p'}^{q'}$ does not imply that the weight u itself belongs to $\mathcal{B}_{p'}^{q'}$. Thus, Theorem 7 does not guarantee the self-improvement of the summability exponents of the weight $u \in \mathcal{B}_p^q$. But if we additionally assume the condition of the monotonicity of the weight u, then we can obtain the following results for self-improving of exponents.

Theorem 8. *Let $0 < p < q$ and u be any nonnegative weight belongs to $\mathcal{B}_p^q(B)$ for $B > 1$, and λ^+ and λ^- are the roots of equation (65).*

(1) If u is nonincreasing, and $\lambda < \lambda^+$, then

$$u \in \mathcal{B}_q^\lambda\left(\frac{\rho_p(\lambda^-)}{\rho_\lambda(\lambda^+)\rho_p(\lambda^+)}\right), \quad u \in \mathcal{B}_p^\lambda\left(\frac{\rho_p(\lambda^-)}{\rho_p(\lambda^+)\theta_\lambda(\lambda^+)}\right). \tag{142}$$

(2) If u is nondecreasing, and $\lambda > \lambda^-$, then

$$u \in \mathcal{B}_\lambda^q\left(\frac{\rho_q(\lambda^-)\rho_\lambda(\lambda^-)}{\rho_q(\lambda^+)}\right), u \in \mathcal{B}_\lambda^p\left(\frac{\rho_p(\lambda^-)\rho_\lambda(\lambda^-)}{\rho_p(\lambda^+)}\right). \tag{143}$$

Proof. (1). Since u is nonincreasing, then Lemma 2 implies that $\mathcal{M}_q u(n)$ is also nonincreasing and $\mathcal{M}_q u(n) \geq u(n)$, and hence

$$\mathcal{M}_\lambda(\mathcal{M}_q u)(n) \geq \mathcal{M}_\lambda u(n). \tag{144}$$

By applying the second relation in (112), we obtain that

$$\mathcal{M}_\lambda(\mathcal{M}_q u)(n) \leq \frac{\rho_p(\lambda^-)}{\rho_\lambda(\lambda^+)} \mathcal{M}_p(\mathcal{M}_q u)(n). \tag{145}$$

By applying the left-inequality in (68) and since $\mathcal{M}_q u(n)$ is nonincreasing (see Lemma 2), we from (144) and (145) that

$$\begin{aligned} \mathcal{M}_\lambda u(n) \leq \mathcal{M}_\lambda(\mathcal{M}_q u)(n) &\leq \frac{\rho_p(\lambda^-)}{\rho_\lambda(\lambda^+)} \mathcal{M}_p(\mathcal{M}_q u)(n) \\ &\leq \frac{\rho_p(\lambda^-)}{\rho_\lambda(\lambda^+)\rho_p(\lambda^+)} \mathcal{M}_q u(n). \end{aligned} \tag{146}$$

That is,

$$u \in \mathcal{B}_q^\lambda\left(\frac{\rho_p(\lambda^-)}{\rho_\lambda(\lambda^+)\rho_p(\lambda^+)}\right), \tag{147}$$

which is the first relation in (142). Similarly, since u is nonincreasing, we have $\mathcal{M}_p u(n) \geq u(n)$, and so

$$\mathcal{M}_\lambda(\mathcal{M}_p u)(n) \geq \mathcal{M}_\lambda u(n). \tag{148}$$

By applying the first relation in (112), we obtain that

$$\mathcal{M}_\lambda(\mathcal{M}_p u(n)) \leq \frac{\rho_q(\lambda^-)}{\rho_\lambda(\lambda^+)} \mathcal{M}_q(\mathcal{M}_p u(n)). \tag{149}$$

By applying the left-inequality in (86), and using (148), (149), and the fact that $\mathcal{M}_p u(n)$ is nonincreasing (see Lemma 2), we have that

$$\begin{aligned} \mathcal{M}_\lambda(u(n)) \leq \mathcal{M}_\lambda(\mathcal{M}_p u(n)) &\leq \frac{\rho_q(\lambda^-)}{\rho_\lambda(\lambda^+)} \mathcal{M}_q(\mathcal{M}_p u(n)) \\ &\leq \frac{\rho_q(\lambda^-)}{\rho_\lambda(\lambda^+)\rho_q(\lambda^+)} \mathcal{M}_p u(n). \end{aligned} \tag{150}$$

That is,

$$u \in \mathcal{B}_p^\lambda\left(\frac{\rho_q(\lambda^-)}{\rho_\lambda(\lambda^+)\rho_q(\lambda^+)}\right), \tag{151}$$

which is the second relation in (142).

(2). Since u is nondecreasing, then Lemma 2 implies that $\mathcal{M}_p u(n)$ is nondecreasing and $\mathcal{M}_p u(n) \leq u(n)$, and we have that

$$\mathcal{M}_\lambda(\mathcal{M}_p u)(n) \leq \mathcal{M}_\lambda u(n). \tag{152}$$

By applying the first relation in (113), we obtain that

$$\mathcal{M}_\lambda(\mathcal{M}_p u)(n) \geq \frac{\rho_q(\lambda^+)}{\rho_\lambda(\lambda^-)} \mathcal{M}_q(\mathcal{M}_p u)(n). \tag{153}$$

By applying the right-inequality in (86), we have that

$$\mathcal{M}_\lambda(\mathcal{M}_p u)(n) \geq \frac{\rho_q(\lambda^+)}{\rho_\lambda(\lambda^-)} \mathcal{M}_q(\mathcal{M}_p u)(n) \geq \frac{\rho_q(\lambda^+)\mathcal{M}_p u(n)}{\rho_q(\lambda^-)\rho_\lambda(\lambda^-)}. \tag{154}$$

By combining (152) and (154), we have that

$$\mathscr{M}_p u(n) \leq \frac{\rho_q(\lambda^-)\rho_\lambda(\lambda^-)}{\rho_q(\lambda^+)}\mathscr{M}_\lambda u(n); \tag{155}$$

that is,

$$u \in \mathscr{B}_\lambda^p\left(\frac{\rho_q(\lambda^-)\rho_\lambda(\lambda^-)}{\rho_q(\lambda^+)}\right), \tag{156}$$

which is the second relation in (143). Again, since u is nondecreasing, then Lemma 2 implies that $\mathscr{M}_q u(n)$ is nondecreasing and $\mathscr{M}_q u(n) \leq u(n)$, and so

$$\mathscr{M}_\lambda\left(\mathscr{M}_q u\right)(n) \leq \mathscr{M}_\lambda u(n). \tag{157}$$

By applying the second relation in (113), we obtain that

$$\mathscr{M}_\lambda\left(\mathscr{M}_q u\right)(n) \geq \frac{\rho_p(\lambda^+)}{\rho_\lambda(\lambda^-)}\mathscr{M}_p\left(\mathscr{M}_q u\right)(n). \tag{158}$$

By applying the right-inequality in (68), we have that

$$\begin{aligned}\mathscr{M}_\lambda(u)(n) \geq \mathscr{M}_\lambda\left(\mathscr{M}_q u\right)(n) &\geq \frac{\rho_p(\lambda^+)}{\rho_\lambda(\lambda^-)}\mathscr{M}_p\left(\mathscr{M}_q u\right)(n) \\ &\geq \frac{\rho_p(\lambda^+)}{\rho_p(\lambda^-)\rho_\lambda(\lambda^-)}\mathscr{M}_q u(n).\end{aligned} \tag{159}$$

By combining (157) and (159), we have that

$$\mathscr{M}_q u(n) \leq \frac{\rho_p(\lambda^-)\rho_\lambda(\lambda^-)}{\rho_p(\lambda^+)}\mathscr{M}_\lambda u(n). \tag{160}$$

That is,

$$u \in \mathscr{B}_\lambda^q\left(\frac{\rho_p(\lambda^-)\rho_\lambda(\lambda^-)}{\rho_p(\lambda^+)}\right), \tag{161}$$

which is the first relation in (143). The proof is complete.

Now, we derive the self-improving properties of the class $\mathscr{G}^q := \mathscr{B}_1^q$.

Theorem 9. *Let $q > 1$ and u be any nondecreasing weight, add weight after nondecreasing $\mathscr{G}^q(B)$ for $B > 1$. Then, $u \in \mathscr{G}^\lambda(B_1'')$ for $\lambda \in q,\lambda^+)$, where λ^+ is the root of the equation*

$$\left(\frac{x-1}{x}\right)\left(\frac{x}{x-q}\right)^{\frac{1}{q}} = B. \tag{162}$$

Proof. Since $\mathscr{G}^q := \mathscr{B}_1^q$, then equation (65) becomes $C_{1,q}(x) = B_1'$ which writes

$$\left(\frac{x}{x-1}\right)^{-1}\left(\frac{x}{x-q}\right)^{\frac{1}{q}} = B_1', \tag{163}$$

which is the desired equation (162), and the constant B_1'' is obtained from (142) and given by

$$B_1'' = \frac{\rho_1(\lambda^-)\rho_\lambda(\lambda^-)}{\rho_1(\lambda^+)}. \tag{164}$$

The proof is complete.

Remark 10. Our technique is only applicable in the case when $0 < p < q$, and it remains an open problem to prove the main results for all p and q such that $p < q$ and $p.q \neq 0$ to be able to get the main results of Muckenhoupt weights similar to the Gehring weights. This leads to the following conjecture.

Theorem 11. *Let $p < q$ such that $p.q \neq 0$ and u be any nonnegative weight that belongs to $\mathscr{B}_p^q(B)$ for $B > 1$, and λ^+ and λ^- are the roots of equation (65).*

(1) If u is nonincreasing, and $\lambda < \lambda^+$, then

$$u \in \mathscr{B}_q^\lambda\left(\frac{\rho_p(\lambda^-)}{\rho_\lambda(\lambda^+)\rho_p(\lambda^+)}\right), \quad u \in \mathscr{B}_p^\lambda\left(\frac{\rho_p(\lambda^-)}{\rho_p(\lambda^+)\theta_\lambda(\lambda^+)}\right). \tag{165}$$

(2) If u is nondecreasing, and $\lambda > \lambda^-$ then

$$u \in \mathscr{B}_\lambda^q\left(\frac{\rho_q(\lambda^-)\rho_\lambda(\lambda^-)}{\rho_q(\lambda^+)}\right), u \in \mathscr{B}_\lambda^p\left(\frac{\rho_p(\lambda^-)\rho_\lambda(\lambda^-)}{\rho_p(\lambda^+)}\right). \tag{166}$$

From this theorem, we can obtain the following sharp result.

Theorem 12. *Let $p > 1$ and ube any nonnegative and nondecreasing belong to$\mathscr{A}^p(B)$ for $B > 1$. Then, $u \in \mathscr{A}^\lambda(B_2'')$ for $\lambda \in (x^-,p]$, where x^- is the root of the equation*

$$\frac{p-x}{p-1}(Bx)^{\frac{1}{p-1}} = 1. \tag{167}$$

Proof. Since $\mathscr{A}^p := \mathscr{B}_{1/1-p}^1$, then equation (65) becomes $C_{p,q}(x) = B_2'$ which is written by

$$\left(\frac{x}{x-1}\right)\left(\frac{(p-1)x}{(p-1)x+1}\right)^{p-1} = B. \tag{168}$$

By applying the transform $\lambda \to 1/(1-x)$, we see that λ^- is determined from the equation

$$\frac{p-\lambda}{p-1}(B\lambda)^{\frac{1}{p-1}}=1, \tag{169}$$

which is the desired equation (167), and the constant B_2'' is obtained from (143) and given by

$$B_2''=\frac{\rho_1(\lambda^-)\rho_\lambda(\lambda^-)}{\rho_1(\lambda^+)}. \tag{170}$$

The proof is complete.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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