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Research Article

Existence Results for Fractional Semilinear Integrodifferential Equations of Mixed Type with Delay

Xue Wang [1,2] and Bo Zhu [2]

[1] *School of Mathematics, Shandong University, Jinan 250100, China*
[2] *School of Mathematics and Quantitative Economics, Shandong University of Finance and Economics, Jinan 250014, China*

Correspondence should be addressed to Bo Zhu; zhubo207@163.com

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In this paper, we discuss a class of fractional semilinear integrodifferential equations of mixed type with delay. Based on the theories of resolvent operators, the measure of noncompactness, and the fixed point theorems, we establish the existence and uniqueness of global mild solutions for the equations. An example is provided to illustrate the application of our main results.

1. Introduction

Fractional calculus can be used to describe some nonclassical phenomena in natural science and engineering applications. Fractional differential equations have been applied in different fields ranging from engineering, finance, and physics in the past few decades. Researchers have conducted extensive explorations on this subject and have achieved fruitful results for the fractional differential equations [1–13]. Zhu and Han [10] and Chadha and Pandey [11] studied the fractional integrodifferential equations and discussed the existence of mild solutions. Based on the theory of the resolvent family and fixed point theorems, Chen et al. [14–17] analyzed nonautonomous evolution equations in a Banach space. Moreover, some researchers considered sufficient conditions on the existence of mild solutions for fractional differential equations by the measure of noncompactness [4, 18, 19]. The initial boundary value problem for the fractional integrodifferential equations with delay has been investigated by using fixed point theorems [4, 5, 18, 20]. In [3, 21–24], differential equations of mixed type have been studied and some results have been concluded.

Chen [22] studied the fractional nonautonomous evolution equations of mixed type:

$$\begin{cases} {}^cD_t^{\beta} u(t) + A(t)u(t) = f(t, u(t), Tu(t), Su(t)), & t \in (0, a], \\ u(0) = A^{-1}(0)u_0, \end{cases} \tag{1}$$

where

$$\begin{aligned} Tu(t) &= \int_0^t K(t,s)u(s)ds, \\ Su(t) &= \int_0^a H(t,s)u(s)ds, \end{aligned} \tag{2}$$

where the kernels K and H are linear functions. The operator T is an integral with a variable upper limit, and the operator S is an ordinary definite integral; accordingly, problem (1) is called fractional semilinear integrodifferential equations of mixed type.

Li and Jia [25] investigated the existence of mild solutions for abstract delay fractional differential equations:

$$\begin{cases} {}^cD_t^{\beta} u(t) = Au(t) + J_t^{1-\beta} f(t, u_t), & t \in [0, T], \\ u(t) = \varphi(t), & t \in [-r, 0], \end{cases} \tag{3}$$

where $\beta \in (0,1)$, $J_t^{1-\beta}$ is the Riemann-Liouville fractional integral, the linear operator A is independent on t, and the Lipschitz coefficient of f is constant.

To the best of our knowledge, there are no results on the fractional integrodifferential equations of mixed type with delay. Motivated by this idea, we consider the following problem:

$$\begin{cases} {}^cD_t^\beta x(t) = A(t)x(t) + J_t^{1-\beta} f(t, x_t, \mathscr{K}x_t, \mathscr{H}x_t), & t \in [0, T_0], \\ x(t) = \phi(t), & t \in [-r, 0], \end{cases} \tag{4}$$

where $\beta \in (0,1]$, ${}^cD_t^\beta$ is the Caputo fractional derivative of order β, $A(t)$ is a closed and linear operator with domain $D(A)$ defined on a Banach space E, $J_t^{1-\beta}$ is the Riemann-Liouville fractional integral of order $1-\beta$, $\mathscr{K}$ and $\mathscr{H}$ are defined by

$$\mathscr{K}x_t = \int_0^t K(t, s, x_s)ds, \qquad \mathscr{H}x_t = \int_0^{T_0} H(t, s, x_s)ds, \tag{5}$$

where $K : D \times C([-r, 0]; E) \longrightarrow E$ and $H : D_0 \times C([-r, 0]; E) \longrightarrow E$ are continuous and nonlinear functions, $D = \{(t, s) \in R^2 : 0 \le s \le t \le T_0\}$, $J = [0, T_0]$, $D_0 = \{(t, s) \in R^2 : 0 \le t, s \le T_0\}$, $\phi \in C[-r, 0]$, f is to be specified later, and x_t means the element of $C([-r, 0]; E)$ defined by $x_t(\theta) = x(t+\theta)$, $-r \le \theta \le 0$, for $x \in C([-r, T_0]; E)$, $t \in J$.

We demonstrate the existence and uniqueness of global mild solutions for problem (4) under the conditions of the compact resolvent operator and noncompact resolvent operator, respectively. The kernels K and H of the operators $\mathscr{K}$ and $\mathscr{H}$ are nonlinear functions. In addition, the operator $A(t)$ is dependent on t. The rest of this paper is organized as follows. Basic definitions and auxiliary results are presented in Section 2. In Section 3, we prove the existence and uniqueness of mild solutions via various fixed point theorems, the measure of noncompactness, and the Banach contraction mapping principle. An example is provided to illustrate the main theorems in Section 4. Finally, Section 5 is the summary of our results.

2. Preliminaries

Definition 1 [6, 26]. The Riemann-Liouville fractional integral J_t^β and derivative D_t^β of a function $f : (0,\infty) \longrightarrow R$ of order $\beta > 0$ are defined by

$$J_t^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)ds,$$
$$D_t^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\beta-1} f(s)ds, \quad n-1 < \beta \le n, \tag{6}$$

where $f(t) \in L^1((0, T_0); E)$, $\Gamma(\cdot)$ denotes the gamma function, and $n \in N$.

Remark 2 [25]. $D_t^\beta f(t) = D_t^m J_t^{m-\beta} f(t)$, where $D_t^m = d^m/dt^m$ and $J_t^{m-\beta} f(t) \in W^{m,1}((0, T_0); E)$.

Definition 3 [26, 27]. The Caputo fractional derivative of order $\beta > 0$ of a function $f : (0,\infty) \longrightarrow R$ is given by

$${}^cD_t^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} f^{(n)}(s)ds, \quad n-1 < \beta < n. \tag{7}$$

Remark 4 [25]. For the Riemann-Liouville fractional integral operator and the Caputo fractional derivative operator, the following conclusions are obtained:

$${}^cD_t^\beta f(t) = D_t^\beta \left(f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} f^{(k)}(0) \right),$$
$${}^cD_t^\beta \left(J_t^\beta f(t) \right) = f(t),$$
$$J_t^\beta \left({}^cD_t^\beta (f(t)) \right) = f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} f^{(k)}(0). \tag{8}$$

Definition 5 [28, 29]. Let $A(t)$ be a closed and linear operator with domain $D(A)$ defined on a Banach space E and $\beta > 0$. Let $\rho[A(t)]$ be the resolvent set of $A(t)$. $A(t)$ is called the generator of a β-resolvent family if there exist $\omega \ge 0$ and a strongly continuous function $U_\beta : R_+^2 \longrightarrow B(E)$ such that $\{\lambda^\beta : \mathrm{Re}\,\lambda > \omega\} \subset \rho(A)$ and

$$\left(\lambda^\beta I - A(s)\right)^{-1} x = \int_0^\infty e^{-\lambda(t-s)} U_\beta(t, s)x dt, \quad \mathrm{Re}\,(\lambda) > \omega, x \in E. \tag{9}$$

In this case, $U_\beta(t, s)$ is called the β-resolvent family generated by $A(t)$.

Remark 6 [29, 30]. $U_\beta(t, s)$ satisfies the following properties:

(1) $U_\beta(s, s) = I$ and $U_\beta(t, s) = U_\beta(t, r)U_\beta(r, s)$, for $0 \le s \le r \le t \le a$

(2) $(t, s) \longrightarrow U_\beta(t, s)$ is strongly continuous for $0 \le s \le t \le a$

(3) If $U_\beta(t, s)$ is compact for $t, s > 0$, then $U_\beta(t, s)$ is continuous in the uniform operator topology

Lemma 7 [21]. *Let $B \subset C[J, E]$ be equicontinuous and bounded; then, $\bar{C}oB \subset C[J, E]$ is also equicontinuous and bounded.*

Lemma 8 [24]. *Let $B \subset C[J, E]$ be equicontinuous and bounded; then, $\alpha(B(t))$ is continuous on J and*

$$\alpha\left(\int_J B(s)ds\right) \le \int_J \alpha(B(s)ds), \quad \alpha(B) = \max_{t\in J}\alpha(B(t)), \quad (10)$$

where α denotes the measure of noncompactness.

Lemma 9 [21]. *Let E be a Banach space and $D \subset E$ be bounded; then, there exists a countable set $D_0 \subset D$ such that $\alpha(D) \le 2\alpha(D_0)$.*

Lemma 10 [31]. *Let E be a Banach space and $D \subset E$ be a bounded closed and convex set. Assume that $Q : D \longrightarrow D$ is a strict set contraction mapping; then, Q has at least one fixed point in D.*

Definition 11. A function $x \in C([-r, T_0]; E)$ is a mild solution of problem (4), if x satisfies the following equations:

$$x(t) = \begin{cases} U_\beta(t,0)\phi(0) + \int_0^t U_\beta(t,s) f(s, x_s, \mathscr{K}x_s, \mathscr{H}x_s)ds, & t \in [0, T_0], \\ \phi(t), & t \in [-r, 0]. \end{cases} \quad (11)$$

3. Main Results

Let us introduce the operator $\Psi : C([-r, T_0]; E) \longrightarrow C([-r, T_0]; E)$ by

$$\Psi x(t) = \begin{cases} U_\beta(t,0)\phi(0) + \int_0^t U_\beta(t,s) f(s, x_s, \mathscr{K}x_s, \mathscr{H}x_s)ds, & t \in [0, T_0], \\ \phi(t), & t \in [-r, 0]. \end{cases} \quad (12)$$

Theorem 12. *Assume that the following conditions hold:*

(H_1). The resolvent operator $U_\beta(t, s)$ is compact for all $t, s > 0$, $M^\star = \max \|U_\beta(t, s)\| < +\infty, 0 \le s \le t \le T_0$.

(H_2). $K : D \times C([-r, 0]; E) \longrightarrow E$ and $H : D_0 \times C([-r, 0]; E) \longrightarrow E$ are continuous; there exist nonnegative Lebesgue integrable functions $p_i \in L(J, R_+) (i = 1, 2)$ such that $\|K(t, s, x)\| \le p_1(t)\|x\|_{C([-r,0];E)}$ and $\|H(t, s, x)\| \le p_2(t)\|x\|_{C([-r,0];E)}$, for all $(t, s) \in D$, $(t, s) \in D_0$, $x \in C([-r, 0]; E)$.

(H_3). $f : J \times C([-r, 0]; E) \times C([-r, 0]; E) \times C([-r, 0]; E) \longrightarrow E$ is continuous; there exist nonnegative Lebesgue integrable functions $a, L_i \in L(J, R_+) (i = 1, 2, 3)$ such that $\|f(t, x_1, x_2, x_3)\| \le a(t) + \sum_{i=1}^3 L_i(t)\|x_i\|_{C([-r,0];E)}$, for all $t \in J$, $x_i \in C([-r, 0]; E)$.

Then, problem (4) has at least one mild solution $x \in C([-r, T_0]; E)$.

Proof. Let us set the notation $R_1 > 0$ such that

$$R_1 \ge \frac{M^\star \phi_0 + M^\star \int_0^{T_0} a(s)ds}{1 - M^\star \left(\int_0^{T_0} L_1(s)ds + \int_0^{T_0} L_2(s) \int_0^{T_0} p_1(\nu)d\nu ds + \int_0^{T_0} L_3(s) \int_0^{T_0} p_2(\nu)d\nu ds \right)}, \quad (13)$$

where $\phi_0 = \|\phi(0)\|$ and $(\int_0^{T_0} L_1(s)ds + \int_0^{T_0} L_2(s)\int_0^{T_0} p_1(\nu)d\nu ds + \int_0^{T_0} L_3(s)\int_0^{T_0} p_2(\nu)d\nu ds)^{-1} > M^\star$.

First of all, we consider the set $B_{R_1} = \{x \in C([-r, T_0]; E) : \|x\|_{C([-r,T_0];E)} \le R_1\}$ and show that $\Psi B_{R_1} \subset B_{R_1}$. By using conditions (H_2) and (H_3), for all $x \in B_{R_1}$, we have

$$\begin{aligned} \|(\Psi x)(t)\| &\le \|U_\beta(t,0)\phi(0)\| + \int_0^t \|U_\beta(t,s) f(s, x_s, \mathscr{K}x_s, \mathscr{H}x_s)\|ds \\ &\le M^\star \phi_0 + M^\star \int_0^t \|f(s, x_s, \mathscr{K}x_s, \mathscr{H}x_s)\|ds \le M^\star \phi_0 \\ &\quad + M^\star \int_0^t (a(s) + L_1(s)\|x_s\| + L_2(s)\|\mathscr{K}x_s\| + L_3(s)\|\mathscr{H}x_s\|)ds \\ &\le M^\star \phi_0 + M^\star \int_0^t a(s)ds + M^\star \left(\int_0^t L_1(s)ds + \int_0^t L_2(s) \right. \\ &\quad \left. \cdot \int_0^s p_1(\nu)d\nu ds + \int_0^t L_3(s) \int_0^{T_0} p_2(\nu)d\nu ds \right) \|x\|_{C([-r,0];E)} \\ &\le M^\star \phi_0 + M^\star \int_0^{T_0} a(s)ds + M^\star \left(\int_0^{T_0} L_1(s)ds + \int_0^{T_0} L_2(s) \right. \\ &\quad \left. \cdot \int_0^{T_0} p_1(\nu)d\nu ds + \int_0^{T_0} L_3(s) \int_0^{T_0} p_2(\nu)d\nu ds \right) \|x\|_{C([-r,T_0];E)} \le R_1. \end{aligned} \quad (14)$$

So, we conclude that Ψ maps B_{R_1} into itself.

Second, we prove that $\Psi : B_{R_1} \longrightarrow B_{R_1}$ is continuous.

Let $\{x_n\}_0^\infty \subset C([-r, T_0]; E)$, with $x_n \longrightarrow x (n \longrightarrow \infty)$, $x \in C([-r, T_0]; E)$. Using the fact that $K : D \times C([-r, 0]; E) \longrightarrow E$, $H : D_0 \times C([-r, 0]; E) \longrightarrow E$, and $f : J \times C([-r, 0]; E) \times C([-r, 0]; E) \times C([-r, 0]; E) \longrightarrow E$ are continuous, we obtain

$$f\left(t, (x_n)_t, \mathscr{K}(x_n)_t, \mathscr{H}(x_n)_t\right) \longrightarrow f(t, x_t, \mathscr{K}x_t, \mathscr{H}x_t)(n \longrightarrow \infty), \quad (15)$$

for any $t \in J$ uniformly. That is, for any $\varepsilon > 0$, there exists a natural number N_0, for $n > N_0$, $t \in J$, such that

$$\left\| f\left(t, (x_n)_t, \mathscr{K}(x_n)_t, \mathscr{H}(x_n)_t\right) - f(t, x_t, \mathscr{K}x_t, \mathscr{H}x_t) \right\| \le \frac{\varepsilon}{M^\star T_0}, \quad (16)$$

which implies that

$$\begin{aligned} \|(\Psi x_n)(t) - (\Psi x)(t)\| &= \left\| \int_0^t U_\beta(t,s) f\left(s, (x_n)_s, \mathscr{K}(x_n)_s, \mathscr{H}(x_n)_s\right)ds \right. \\ &\quad \left. - \int_0^t U_\beta(t,s) f(s, x_s, \mathscr{K}x_s, \mathscr{H}x_s)ds \right\| \\ &\le M^\star \int_0^t \left\| f\left(s, (x_n)_s, \mathscr{K}(x_n)_s, \mathscr{H}(x_n)_s\right) \right. \\ &\quad \left. - f(s, x_s, \mathscr{K}x_s, \mathscr{H}x_s) \right\| ds \le M^\star T_0 \frac{\varepsilon}{M^\star T_0} = \varepsilon. \end{aligned} \quad (17)$$

In consequence, $\Psi : B_{R_1} \longrightarrow B_{R_1}$ is continuous.

Furthermore, we prove that $\Psi(B_{R_1})$ is equicontinuous.

To do this, let $L(s)=L_1(s)+L_2(s)\int_0^{T_0}p_1(\nu)d\nu+L_3(s)\int_0^{T_0}p_2(\nu)d\nu$. Obviously, it is a nonnegative Lebesgue integrable function. For all $x\in B_{R_1}, t_1, t_2\in J(t_1<t_2)$, we have

$$\begin{aligned}
\|(\Psi x)(t_2)-(\Psi x)(t_1)\| &\le \|(U_\beta(t_2,0)-U_\beta(t_1,0))\phi(0)\| \\
&+\left\|\int_{t_1}^{t_2}U_\beta(t_2,s)f(s,x_s,\mathscr{K}x_s,\mathscr{H}x_s)ds\right\| \\
&+\left\|\int_0^{t_1}(U_\beta(t_2,s)-U_\beta(t_1,s))f(s,x_s,\mathscr{K}x_s,\mathscr{H}x_s)ds\right\| \\
&\le \phi_0\|U_\beta(t_2,0)-U_\beta(t_1,0)\|+M^\star\int_{t_1}^{t_2}\|f(s,x_s,\mathscr{K}x_s,\mathscr{H}x_s)\|ds \\
&+\sup_{s\in J}\|U_\beta(t_2,s)-U_\beta(t_1,s)\|\int_0^{t_1}\|f(s,x_s,\mathscr{K}x_s,\mathscr{H}x_s)\|ds \\
&\le \phi_0\|U_\beta(t_2,0)-U_\beta(t_1,0)\|+M^\star\int_{t_1}^{t_2}(a(s)+L(s)R_1)ds \\
&+\sup_{s\in J}\|U_\beta(t_2,s)-U_\beta(t_1,s)\|\int_0^{t_1}(a(s)+L(s)R_1)ds \\
&=: I_1+I_2+I_3.
\end{aligned} \tag{18}$$

In view of condition (H_1), compactness of the resolvent operator $U_\beta(t,s)(t,s)>0$ implies the continuity in the uniform operator topology. That is, for any $\varepsilon>0$, there exists $\delta_1>0$, for any $|t_2-t_1|<\delta_1, t_1, t_2\in J$, such that $I_3<\varepsilon/3$. Hence, for the above $\varepsilon>0$, by using properties of $U_\beta(t,s)$ and the above inequalities, there exists $\delta>0$ $(\delta<\delta_1)$ such that $\|(\Psi x)(t_2)-(\Psi x)(t_1)\|<\varepsilon$, for any $|t_2-t_1|<\delta, t_1, t_2\in J$. Consequently, $\Psi(B_{R_1})$ is equicontinuous.

In the end, we prove that $\Psi(B_{R_1})$ is precompact.

For any fixed $t(t\in[-r,T_0])$ and $0<\varepsilon<t$, the operator $(\Psi_\varepsilon x)(t)$ is defined by

$$(\Psi_\varepsilon x)(t)=\begin{cases} U_\beta(t,0)\phi(0)+\int_0^{t-\varepsilon}U_\beta(t,s)f(s,x_s,\mathscr{K}x_s,\mathscr{H}x_s)ds, & t\in[0,T_0], \\ \phi(t), & t\in[-r,0]. \end{cases} \tag{19}$$

Since $U_\beta(t,s)(t,s)>0$ is a compact resolvent operator, then the set $Y_\varepsilon(t)=\{(\Psi_\varepsilon x)(t): x\in B_{R_1}\}$ is relatively compact in E for any ε $(0<\varepsilon<t)$.

Moreover, for any $x\in B_{R_1}$, one can find that

$$\begin{aligned}
\|(\Psi x)(t)-(\Psi_\varepsilon x)(t)\| &= \left\|\int_{t-\varepsilon}^{t}U_\beta(t,s)f(s,x_s,\mathscr{K}x_s,\mathscr{H}x_s)ds\right\| \\
&\le M^\star\left\|\int_{t-\varepsilon}^{t}f(s,x_s,\mathscr{K}x_s,\mathscr{H}x_s)ds\right\| \\
&\le M^\star\int_{t-\varepsilon}^{t}(a(s)+L(s)R_1)ds \\
&\le M^\star(\|a(s)\|+\|L(s)\|R_1)\varepsilon.
\end{aligned} \tag{20}$$

Thus, $Y(t)=\{(\Psi x)(t): x\in B_{R_1}\}$ is totally bounded. Hence, $Y(t)$ is relatively compact in E, and so, based on the Arzelà-Ascoli theorem, $\Psi: B_{R_1}\longrightarrow B_{R_1}$ is completely continuous. As all the assumptions of the Schauder fixed point theorem are satisfied, the conclusion implies that the operator Ψ has a fixed point x in $C([-r;T_0],E)$, which is a global mild solution of problem (4). This completes the proof.

Next, we develop the existence of global mild solutions for problem (4) via the measure of noncompactness and fixed point theorem. Furthermore, we employ the notations: $T_R=\{x\in C([-r,T_0];E): \|x\|_{C([-r,T_0];E)}\le R\}$, $k_0=\sup\{\|K(t,s,x_s)\|: (t,s,x_s)\in D\times C([-r,0];E)\}$, $h_0=\sup\{\|H(t,s,x_s)\|: (t,s,x_s)\in D_0\times C([-r,0];E)\}$, and $R\ge\max\{T_0k_0, T_0h_0\}$.

Theorem 13. *Assume that (H_1) and the following conditions hold:*

(H_4). The function $f: J\times T_R\times T_R\times T_R\longrightarrow E$ is bounded and continuous, which satisfies

$$\lim_{R\to\infty}\sup\frac{M(R)}{R}<\frac{1}{T_0M^\star}, \tag{21}$$

where $M(R)=\sup\{\|f(t,x_1,x_2,x_3)\|: (t,x_1,x_2,x_3)\in J\times T_R\times T_R\times T_R\}$.

(H_5). For any R, there exist nonnegative Lebesgue integrable functions $q_i\in L(J,R_+)$, $(i=1,2,3,4,5)$ such that for any equicontinuous and countable set $D_i\subset T_R(i=1,2,3)$, $\alpha(f(t,D_1,D_2,D_3))\le\sum_{i=1}^{3}q_i(t)\alpha(D_i)$, $\alpha(K(t,s,D_2))\le q_4(t)\alpha(D_2)$, and $\alpha(H(t,s,D_3))\le q_5(t)\alpha(D_3)$.

(H_6). $2M^\star\int_0^{T_0}(q_1(s)+q_2(s)\int_0^{T_0}q_4(\nu)d\nu+q_3(s)\int_0^{T_0}q_5(\nu)d\nu)ds<1$.

Then, problem (4) has at least one mild solution.

Proof. By (H_4), there exists $0<\mu<1/T_0M^\star$ and $R_0>0$, for any $R\ge R_0$, such that

$$M(R)<\mu R. \tag{22}$$

Let $R^\star=\max\{R_0, M^\star\phi_0(1-M^\star T_0\mu)^{-1}\}$; we first consider the set $B_{R^\star}=\{x\in C([-r,T_0];E): \|x\|_{C([-r,T_0];E)}\le R^\star\}$ and show that $\Psi B_{R^\star}\subset B_{R^\star}$. From the above inequality, for all $x\in B_{R^\star}$, we have

$$\begin{aligned}
\|\Psi x\|_{C([-r,T_0];E)} &\le \|U_\beta(t,0)\phi(0)\|+\int_0^t\|U_\beta(t,s)\|\|f(s,x_s,\mathscr{K}x_s,\mathscr{H}x_s)\|ds \\
&\le M^\star\phi_0+M^\star T_0M(R^\star)\le M^\star\phi_0+M^\star T_0\mu R^\star\le R^\star.
\end{aligned} \tag{23}$$

Meanwhile, applying the arguments employed in the proof of Theorem 12, we conclude that Ψ is a continuous and bounded operator on $B_{R^\star}$.

Then, we prove that $\Psi(B_{R^\star})$ is equicontinuous. For any $x \in B_{R^\star}, t_1, t_2 \in J (t_1 < t_2)$, we have

$$\begin{aligned}
\|(\Psi x)(t_2) - (\Psi x)(t_1)\| &\le \|(U_\beta(t_2,0) - U_\beta(t_1,0))\phi(0)\| \\
&\quad + \int_{t_1}^{t_2} \|U_\beta(t_2,s) f(s, x_s, \mathscr{K}x_s, \mathscr{H}x_s)\| ds \\
&\quad + \int_0^{t_1} \|(U_\beta(t_2,s) - U_\beta(t_1,s)) f(s, x_s, \mathscr{K}x_s, \mathscr{H}x_s)\| ds \\
&\le \phi_0 \|U_\beta(t_2,0) - U_\beta(t_1,0)\| + M^\star(t_2 - t_1) M(R^\star) \\
&\quad + \sup_{s\in J} \|U_\beta(t_2,s) - U_\beta(t_1,s)\| M(R^\star) t_1.
\end{aligned} \tag{24}$$

By (H_1), the compactness of $U_\beta(t,s)$, for $(t,s) > 0$, implies the continuity in the uniform operator topology. Namely, for any $\varepsilon > 0$, there exists $\delta_1 > 0$, for any $|t_2 - t_1| < \delta_1, t_1, t_2 \in J$, such that

$$\sup_{s\in J} \|U_\beta(t_2,s) - U_\beta(t_1,s)\| M(R^\star) t_1 < \frac{\varepsilon}{3}. \tag{25}$$

Therefore, for the above $\varepsilon > 0$, there exists $\delta > 0$ $(\delta < \delta_1)$ such that $\|(\Psi x)(t_2) - (\Psi x)(t_1)\| < \varepsilon$, for all $x \in B_{R^\star}, |t_2 - t_1| < \delta, t_1, t_2 \in J$, which shows that $\Psi(B_{R^\star})$ is equicontinuous. In view of Lemma 7, $\bar{C}o\Psi(B_{R^\star}) \subset B_{R^\star}$ is bounded and equicontinuous.

Finally, we prove that $\Psi : \bar{C}o\Psi(B_{R^\star}) \longrightarrow \bar{C}o\Psi(B_{R^\star})$ is a condensing operator. By Lemma 9, for any $D \subset \bar{C}o\Psi(B_{R^\star})$, there exists a countable set $D_0 = \{x_n\} \subset D$ such that

$$\alpha(\Psi(D)) \le 2\alpha(\Psi(D_0)). \tag{26}$$

By using condition (H_5) and Lemma 8, we obtain

$$\begin{aligned}
\alpha(\Psi(D_0)(t)) &= \alpha\left(\int_0^t U_\beta(t,s) f(s, (D_0)_s, \mathscr{K}(D_0)_s, \mathscr{H}(D_0)_s) ds\right) \\
&\le M^\star \int_0^t \alpha(f(s, (D_0)_s, \mathscr{K}(D_0)_s, \mathscr{H}(D_0)_s)) ds \\
&\le M^\star \int_0^t (q_1(s)\alpha((D_0)_s) + q_2(s)\alpha(\mathscr{K}(D_0)_s) \\
&\quad + q_3(s)\alpha(\mathscr{H}(D_0)_s)) ds \\
&\le M^\star \int_0^t \left(q_1(s) + q_2(s)\int_0^s q_4(\nu) d(\nu)\right. \\
&\quad \left. + q_3(s) \int_0^{T_0} q_5(\nu) d(\nu)\right) ds\alpha(D).
\end{aligned} \tag{27}$$

In addition, using Lemma 8, we have

$$\alpha(\Psi(D_0)) = \max_{t\in J} \alpha(\Psi(D_0)(t)). \tag{28}$$

Consequently,

$$\begin{aligned}
\alpha(\Psi(D)) \le 2M^\star \int_0^{T_0} &\left(q_1(s) + q_2(s)\int_0^{T_0} q_4(\nu) d(\nu)\right. \\
&\left. + q_3(s) \int_0^{T_0} q_5(\nu) d(\nu)\right) ds\alpha(D).
\end{aligned} \tag{29}$$

By (H_6), we obtain that Ψ is a condensing operator on $\bar{C}o\Psi(B_{R^\star})$. By Lemma 10, there exists at least one fixed point $x \in \bar{C}o\Psi(B_{R^\star}) \subset C([-r, T_0]; E)$ for Ψ. In conclusion, problem (4) has at least one global mild solution. This completes the proof.

Remark 14. Theorems 12 and 13 above are concluded under the conditions that $U_\beta(t,s)$ is compact for $t, s > 0$ and the functions f, K, and H satisfy corresponding conditions; in contrast, when the resolvent operator $U_\beta(t,s)$ is noncompact, we could obtain Theorem 15 if f, K, and H meet the Lipschitz conditions.

Theorem 15. *Assume that the following conditions hold:*

(H_7). $f : J \times C([-r,0]; E) \times C([-r,0]; E) \times C([-r,0]; E) \longrightarrow E$ is continuous; there exist nonnegative Lebesgue integrable functions $g_i \in L(J, R_+) (i = 1, 2, 3)$, for all $t \in J, u_i, v_i \in E$, such that

$$\|f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3)\| \le \sum_{i=1}^{3} g_i(t) \|u_i - v_i\|_{C([-r,0];E)}. \tag{30}$$

(H_8). $K : D \times C([-r,0]; E) \longrightarrow E$ and $H : D_0 \times C([-r,0]; E) \longrightarrow E$; there exist nonnegative Lebesgue integrable functions $g_4, g_5 \in L(J, R_+)$, for all $u, v \in E, (t,s) \in D, (t,s) \in D_0$ such that

$$\begin{aligned}
\|K(t,s,u) - K(t,s,v)\| &\le g_4(t) \|u - v\|_{C([-r,0];E)}, \\
\|H(t,s,u) - H(t,s,v)\| &\le g_5(t) \|u - v\|_{C([-r,0];E)}.
\end{aligned} \tag{31}$$

(H_9). $M^\star \int_0^{T_0} (g_1(s) + g_2(s) \int_0^{T_0} g_4(\nu) d\nu + g_3(s) \int_0^{T_0} g_5(\nu) d\nu) ds < 1$.

Then, problem (4) has a unique mild solution.

Proof. For any $u, v \in C([-r, T_0]; E)$,

$$\begin{aligned}
\|(\Psi u)(t) - (\Psi v)(t)\| &\le M^\star \int_0^t \|f(s, u_s, \mathscr{K}u_s, \mathscr{H}u_s) \\
&\quad - f(s, v_s, \mathscr{K}v_s, \mathscr{H}v_s)\| ds \le M^\star \int_0^t (g_1(s)\|u_s - v_s\| \\
&\quad + g_2(s)\|\mathscr{K}u_s - \mathscr{K}v_s\| + g_3(s)\|\mathscr{H}u_s - \mathscr{H}v_s\|) ds \le M^\star \int_0^{T_0} \\
&\quad \cdot \left(g_1(s) + g_2(s)\int_0^{T_0} g_4(\nu) d\nu + g_3(s)\int_0^{T_0} g_5(\nu) d\nu\right) ds \|u - v\|_{C([-r,T_0];E)}.
\end{aligned} \tag{32}$$

By (H_9), we have $\|\Psi u - \Psi v\|_{C([-r,T_0];E)} < \|u - v\|_{C([-r,T_0];E)}$.

These arguments enable us to conclude that the operator Ψ is a contraction mapping. Hence, the operator Ψ has a unique fixed point $x^* \in C([-r, T_0]; E)$, which implies that problem (4) has a unique global mild solution. This completes the proof.

Remark 16. In Theorem 15, we develop the uniqueness of the mild solution for problem (4) via the Banach contraction mapping principle. In conditions (H_7) and (H_8), $g_i \in L(J, R_+)(i = 1, 2, 3, 4, 5)$ turn out to be nonnegative Lebesgue integrable functions instead of constants.

4. An Application

In order to show the application of the main results, we consider the following problem:

$$\begin{cases} {}^cD_t^\beta x(z,t) = t^2 \dfrac{\partial^2}{\partial z^2} x(z,t) + J_t^{1-\beta}\left(\dfrac{t}{1+t^2} x(z,t+\theta) + \dfrac{1}{1+t^2}\displaystyle\int_0^t a(s)x(z,s+\theta)ds + \dfrac{1}{1+e^t}\displaystyle\int_0^1 b(s)x(z,s+\theta)ds\right), \\ 0 < t \le 1, \quad z \in \Omega, \theta \in [-r, 0], \\ x(z,\theta) = \varphi(z,\theta), \quad z \in \Omega, \theta \in [-r, 0], \end{cases} \tag{33}$$

where $0 < \beta < 1$, ${}^cD_t^\beta$ is the Caputo fractional derivative of order β, $J_t^{1-\beta}$ is the Riemann-Liouville fractional integral of order $1-\beta$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with regular boundary $\partial\Omega$, and $\varphi \in C([-r, 0]; E), E = C(\bar{\Omega}; \mathbb{R}), \bar{\Omega} = \Omega \bigcup \partial\Omega$.

By setting $x(t) = x(\cdot, t)$, problem (33) can be rewritten as the following abstract form:

$$\begin{cases} {}^cD_t^\beta x(t) = A(t)x(t) + J_t^{1-\beta} f(t, x_t, \mathscr{K}x_t, \mathscr{H}x_t), \quad t \in [0, 1], \\ x(t) = \varphi(t), \quad t \in [-r, 0], \end{cases} \tag{34}$$

where $x_t = x(t+\theta)$, $f(t, x_t, \mathscr{K}x_t, \mathscr{H}x_t) = ((t/(1+t^2))x_t + (1/(1+t^2))\int_0^t a(s)x_s ds + (1/(1+e^t))\int_0^1 b(s)x_s ds)$, and

$$\begin{cases} D(A) = \left\{x \in C(\bar{\Omega}, R) : x'' \in C(\bar{\Omega}, R)\right\}, \\ A(t)x = x'', \quad t \in [-r, 0]. \end{cases} \tag{35}$$

It is well known that the operator $A(t)$ generates a β-resolvent family $U_\beta(t, s)$ [23, 25]. Let equation (34) satisfy the conditions of Theorems 12–15; then, problem (34) has a global mild solution, which means that problem (33) has a mild solution.

5. Conclusion

In this paper, we study the existence and uniqueness of the global mild solutions for the fractional integrodifferential equations of mixed type with delay. Under the condition of the compact resolvent operator, we obtain Theorems 12 and 13, respectively, via various fixed point theorems and the measure of noncompactness. Theorem 15 is established by using the Banach contraction mapping principle under the condition of the noncompact resolvent operator. Furthermore, an example is provided to illustrate the main theorems. The kernels K and H of the operators $\mathscr{K}$ and $\mathscr{H}$ are nonlinear functions; meanwhile, the operator $A(t)$ is dependent on t. As a consequence, our main theorems improve and generalize many corresponding results by using different methods.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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