Hindawi
Journal of Function Spaces
Volume 2021, Article ID 5524494, 10 pages
https://doi.org/10.1155/2021/5524494

Research Article

Common Best Proximity Point Theorems in JS-Metric Spaces Endowed with Graphs

Chaiporn Thangthong,[1] **Phakdi Charoensawan**,[1] **Supreedee Dangskul**,[1] **and Narawadee Phudolsitthiphat**[2,3]

[1]Advanced Research Center for Computational Simulation, Chiang Mai University, Chiang Mai 50200, Thailand
[2]Data Science Research Center, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand
[3]Research Center in Mathematics and Applied Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand

Correspondence should be addressed to Narawadee Phudolsitthiphat; narawadee_n@hotmail.co.th

Received 10 February 2021; Revised 18 March 2021; Accepted 3 May 2021; Published 11 May 2021

Academic Editor: Nawab Hussain

In this paper, we introduce a notion of G-proximal edge preserving and dominating G-proximal Geraghty for a pair of mappings, which will be used to present some existence and uniqueness results for common best proximity points. Here, the mappings are defined on subsets of a JS-metric space endowed with a directed graph. An example is also provided to support the results. Moreover, we apply our result to a similar setting, where the JS-metric space is endowed with a binary relation.

1. Introduction

Problems concerning objects that remain unchanged have been of great interest in sciences. Geneticists, for instance, have discovered that gene mutations may be delayed by raising the number of DNAs kept unaltered under exposure and hence cancer prevention. Mathematicians may interpret such objects in their own environment as points kept fixed under self-mapping, known as fixed points.

Theory of fixed points and their related notions have been expansively explored, not only in pure mathematics itself but also in real-world problems, where optimal solutions are sought. There have been a large number of publications that contribute to the subject in various approaches (see [1, 2] for some fixed point theorems and see [3–9] for results regarding best proximity points and pairs, to mention but a few). The reader may also be referred to [10] for some applications of the theory in economics.

Best proximity and common best proximity points, in particular, have become one of the most studied topics in the field of fixed point theory. These notions generalize fixed points and allow us to deal with nonself-mappings. Several settings and techniques have been used in order to determine which circumstances a best (common best) proximity point can be guaranteed. Hussain and his coauthors are amongst those who have actively contributed results to this research area (see [11, 12], where different kinds of contractions are employed; see [13, 14], where generalized notions of metric spaces are considered; and see [15] for best proximity results in nonlinear dynamical systems). There are many more results on common best proximity points in the literature (see [16–20] for some of the key works).

One of the most popular research approaches in the theory of fixed points is to appropriately adjust mappings that control the distance between two points. A well-known result by S. Banach, known as Banach contraction principle, gives rise to a variety of modifications. Instead of contractions, in [21], Ayari considered a new class of mappings containing all $\theta : [0,+\infty) \to [0,1]$ with property that

$$\lim_{n\to+\infty} \theta(t_n) = 1 \text{ implies } \lim_{n\to+\infty} t_n = 0, \tag{1}$$

which generalizes contractions and also Geraghty's work [22]. With this generalization, existence and uniqueness results for best proximity points in a closed subset of a complete metric space can be established. A recent work by Khemphet et al. [20] also benefits from this class of mappings—a notion of dominating proximal generalized Geraghty property of a pair of mappings is presented for some existence and uniqueness results of common best proximity coincidence points in complete metric spaces, improving Chen's work [19].

One may try to control the distance between two points in a metric space using directed graphs. This idea was first introduced by Jachymski [23]. Given a directed graph $G = (V, E)$, the set of edges E is contained in the Cartesian product $V \times V$. If V is also a metric space, one could impose some conditions on the distance between points x and y, provided that $(x, y) \in E$. There have been a number of articles employing this graph-like theoretic approach (see [24, 25] for some fixed point theorems, see [26–28] for some results on Hilbert spaces, and see [29–31] for some assertions regarding common fixed points, coincidence points, and best proximity points).

This paper is aimed at establishing some common best proximity point theorems (Theorems 12 and 13) on a more general setting, the so-called JS-metric spaces, introduced in [32]. Our space will also be endowed with a directed graph G. More specifically, the main theorems rely on two key assumptions that a pair of mappings is G-proximal edge preserving and dominating G-proximal Geraghty. To the best of our knowledge, this study approach has not yet been investigated.

The paper is organized as follows. Section 2 collects basic definitions and facts regarding JS-metric spaces, common proximity points, and spaces endowed with directed graphs. Section 3 presents our main results and some example. Section 4 provides some consequences of our main theorems by concretely selecting Geraghty-like functions. Last but not least, Section 5 is devoted for an application of our results for a pair of mappings between subsets of a JS-metric space endowed with a binary relation.

2. Preliminaries

2.1. JS-Metric Spaces. In [32], a weaker notion of metrics was introduced by Jleli and Samet, known as JS-metrics. These generalized metrics lack the triangle inequality, which somewhat ruins the intuition of distance. It turns out, however, that various types of topological spaces are JS-metric spaces.

Let X be a nonempty set, and let $D : X \times X \to [0,+\infty]$ be a function. For each $x \in X$, let us define

$$C(D, X, x) = \left\{ \{x_n\} \subseteq X : \lim_{n \to +\infty} D(x_n, x) = 0 \right\} \tag{2}$$

to be the set of sequences in X that converges to x, with respect to D.

Definition 1 (see [32]). Let X be a nonempty set. A function $D : X \times X \to [0,+\infty]$ is called *JS-metric* on a set X if it satisfies the following conditions:

(JS_1) For any $x, y \in X$, $D(x, y) = 0$ implies $x = y$

(JS_2) For any $x, y \in X$, $D(x, y) = D(y, x)$

(JS_3) There is a constant $C_X > 0$ such that

$$D(x, y) \le C_X \limsup_{n \to +\infty} D(x_n, y), \tag{3}$$

whenever $x, y \in X$ and $\{x_n\} \in C(D, X, x)$.

The pair (X, D) is called a *JS-metric space*.

Conditions (JS_1) and (JS_3) imply the following fact.

Proposition 2 (see [33]). *Let (X, D) be a JS-metric space and $x \in X$. If $C(D, X, x) \neq \emptyset$, then $D(x, x) = 0$.*

As in a metric space, convergence and completeness for a JS-metric space can be defined in a similar way.

Definition 3 (see [32]). *Let (X, D) be a JS-metric space and $\{x_n\}$ a sequence in X.*

(i) $\{x_n\}$ *D-converges* to x if $\{x_n\} \in C(D, X, x)$

(ii) $\{x_n\}$ is called a *D-Cauchy* sequence if $\lim_{m,n \to +\infty} D(x_m, x_n) = 0$

(iii) The space (X, D) is said to be *complete* if every D-Cauchy sequence is D-convergent

In a metric space, triangle inequality forces any convergent sequence to have a unique limit. Here, the condition (JS_3) plays that role.

Proposition 4 (see [32]). *Let (X, D) be a JS-metric space, and let $\{x_n\}$ be a sequence in X. For any $x, y \in X$, if $\{x_n\} \in C(D, X, x) \cap C(D, X, y)$, then $x = y$.*

Pointwise continuity can then be defined, using convergence of sequences.

Definition 5. Let (X, D) be a JS-metric space. A mapping $P : X \to X$ is said to be continuous at a point $x_0 \in X$ if $\{x_n\} \in C(D, X, x_0)$ implies $\{Px_n\} \in C(D, X, Px_0)$. In addition, P is said to be continuous if it is continuous at each x in X.

2.2. Common Proximity. Throughout the paper, for nonempty subsets A and B of a JS-metric space (X, D), the following notations will be used:

$$\begin{aligned} D(A, B) &:= \inf \{D(a, b) : a \in A, b \in B\}, \\ A_0 &:= \{a \in A : \text{there exists } b \in B \text{ such that } D(a, b) = D(A, B)\}, \\ B_0 &:= \{b \in B : \text{there exists } a \in A \text{ such that } D(a, b) = D(A, B)\}. \end{aligned} \tag{4}$$

Notice that A_0 and B_0 can be empty or even undefined. If $D(A, B) = +\infty$ and $D(a, b) = +\infty$, then it is unclear whether $a \in A_0$. Throughout the paper, the distance $D(A, B)$ will be always assumed to be finite.

For nonempty subsets A and B of a JS-metric space (X, D), let us recall that a *best proximity point* of a mapping $T : A \to B$ is a point $x^* \in A$ with $D(x^*, Tx^*) = D(A, B)$. If $D(A, B) = 0$, then x^* becomes a fixed point. The term "common" comes into play when we deal with a pair of mappings.

Definition 6 (see [18]). *Let $P, Q : A \to B$ be mappings. An element $x^* \in A$ is said to be a common best proximity point of the pair (P, Q) if*

$$D(x^*, Qx^*) = D(A, B) = D(x^*, Px^*). \tag{5}$$

The set of common best proximity points of P and Q is denoted by $\mathrm{CP}(P, Q)$.

Observe also that if $D(A, B) = 0$, then x^* becomes a common fixed point of the pair (P, Q).

Definition 7 (see [17]). Let $P, Q : A \to B$ be mappings. A pair (P, Q) is said to *commute proximally* if for each $x, u, v \in A$,

$$D(u, Qx) = D(A, B) = D(v, Px) \text{ implies } Pu = Qv. \tag{6}$$

2.3. Graph Endowment. Given a nonempty set X, a directed graph $G = (V_G, E_G)$ will be constructed as follows.

(i) The set of vertices V_G is the set X itself

(ii) The set of edges $E_G \subseteq X \times X$ contains all the loops; that is, $\{(x, x) \mid x \in X\} \subseteq E_G$

(iii) E_G contains no parallel edges

We say that X is said to be *endowed with a directed graph* $G = (V_G, E_G)$.

Let (X, D) be a JS-metric space endowed with a directed graph $G = (V_G, E_G)$, denoted by (X, D, G). We next introduce a notion of continuity with respect to the graph G.

Definition 8 (see [23]). *A mapping $P : X \to X$ is called G-continuous at $x_0 \in X$ if for any sequence $\{x_n\}$ in X with $(x_n, x_{n+1}) \in E_G$ and $\{x_n\} \in C(D, X, x_0)$, it follows that $\{Px_n\} \in C(D, X, Px_0)$.*

Notice that G-continuity is a weaker notion than usual continuity, with respect to the JS-metric. In other words, any continuous mapping on a space endowed with a directed graph G is G-continuous. A counterexample of the converse is easily found, when G has an isolated vertex, for example.

Let us now introduce some more terminology used in our main results.

Definition 9. Let A, B be nonempty subsets of (X, D, G) and $P, Q : A \to B$ be mappings. A pair (P, Q) is said to be G-proximal edge preserving if the following hold:

(i) If $(Px, Py) \in E_G$, $(Qx, Qy) \in E_G$

(ii) For any $x, u, v \in A$ with $(Px, Qx) \in E_G$ and

$$D(u, Qx) = D(A, B) = D(v, Px), \tag{7}$$

it follows that $(v, u) \in E_G$

Define the class Γ of functions to be

$$\Gamma = \left\{ \gamma : [0, +\infty] \to [0, 1] \mid \lim_{n \to +\infty} \gamma(t_n) = 1 \text{ implies } \lim_{n \to +\infty} t_n = 0 \right\}. \tag{8}$$

This extends the class of functions in [19, 21]. Notice that any function in $\gamma \in \Gamma$ has a property that

$$\gamma(t) = 1 \text{ implies } t = 0. \tag{9}$$

Definition 10. Let A, B be nonempty subsets of (X, D, G) and $P, Q : A \to B$ be mappings. A pair (P, Q) is said to be dominating G-proximal Geraghty if there exists a function $\gamma \in \Gamma$ such that for each $x_1, x_2, u_1, u_2, v_1, v_2 \in A$ satisfying

$$\begin{gathered}(Px_1, Qx_1) \in E_G, \\ (Px_2, Qx_2) \in E_G, \\ D(u_1, Qx_1) = D(u_2, Qx_2) = D(A, B) = D(v_1, Px_1) = D(v_2, Px_2),\end{gathered} \tag{10}$$

it follows that

$$D(u_1, u_2) \leq \gamma(M(v_1, v_2, u_1, u_2)) M(v_1, v_2, u_1, u_2), \tag{11}$$

where $M(v_1, v_2, u_1, u_2) = \max\{D(v_1, v_2), D(v_1, u_1), D(v_2, u_2)\}$.

3. Existence Theorems

Throughout this section, let (X, D, G) be a complete JS-metric space endowed with a directed graph G, A, B be nonempty subsets of X, and $P, Q : A \to B$ be mappings. The following assumptions will be imposed.

(A0) $A_0, B_0 \neq \emptyset$

(A1) A_0 is bounded with respect to D and closed, in the sense that any convergent sequence in A_0 has its limit in A_0

(A2) $Q(A_0) \subseteq B_0$

(A3) For any $u, v \in A$, if there exist $x, y \in A$ such that

$$D(u, Qx) = D(A, B) = D(v, Py), \tag{12}$$

then $D(u, v) < +\infty$

(A4) The pair (P, Q) commutes proximally

(A5) The pair (P, Q) is dominating G-proximal Geraghty

Lemma 11. *If $Pu = Qu$ for some $u \in A_0$, then $CP(P, Q) \neq \emptyset$.*

Proof. Let $u \in A_0$ such that $Pu = Qu$. By (A2), there exists $x^* \in A_0$ such that

$$D(x^*, Qu) = D(A, B) = D(x^*, Pu). \tag{13}$$

Since the pair (P, Q) commutes proximally, we have $Px^* = Qx^*$. Again, the assumption (A2) implies that

$$D(z^*, Qx^*) = D(A, B) = D(z^*, Px^*), \tag{14}$$

for some $z^* \in A_0$. From (13) and (14), the assumption (A3) yields $D(x^*, x^*) < +\infty$, $D(z^*, z^*) < +\infty$, and $D(x^*, z^*) < +\infty$. Since $(Pu, Qu) = (Pu, Pu) \in E_G$, it follows from (13) and (A5) that

$$\begin{aligned} D(x^*, x^*) &\leq \gamma(M(x^*, x^*, x^*, x^*))M(x^*, x^*, x^*, x^*) \\ &= \gamma(D(x^*, x^*))D(x^*, x^*) \leq D(x^*, x^*). \end{aligned} \tag{15}$$

That is, $\gamma(D(x^*, x^*)) = 1$, and hence, $D(x^*, x^*) = 0$. Similarly, by (14) and $(Px^*, Qx^*) = (Px^*, Px^*) \in E_G$, we also get $D(z^*, z^*) = 0$.

Now, observe that (14) will prove the lemma if $x^* = z^*$. In fact, the equality can be achieved by a similar argument above. Since

$$\begin{aligned} M(x^*, z^*, x^*, z^*) &= \max\{D(x^*, z^*), D(x^*, x^*), D(z^*, z^*)\} \\ &= D(x^*, z^*), \\ D(x^*, z^*) &\leq \gamma(M(x^*, z^*, x^*, z^*))M(x^*, z^*, x^*, z^*) \\ &= \gamma(D(x^*, z^*))D(x^*, z^*) \leq D(x^*, z^*), \end{aligned} \tag{16}$$

it follows that $\gamma(D(x^*, z^*)) = 1$ yielding $D(x^*, z^*) = 0$. Thus, $x^* \in \mathrm{CP}(P, Q)$.

Theorem 12. *Assume (A0)–(A5) as before. In addition, if the following are satisfied*

(i) $Q(A_0) \subseteq P(A_0)$

(ii) there exists $x_0 \in A_0$ *such that* $(Px_0, Qx_0) \in E_G$

(iii) (P, Q) *is G-proximal edge preserving*

(iv) P and Q are G-continuous,

then $CP(P, Q) \neq \varnothing$. *Moreover, if* $(Px, Qx) \in E_G$, *for all* $x \in CP(P, Q)$, *then the pair* (P, Q) *has a unique common best proximity point.*

Proof. Let $x_0 \in A_0$ be such that $(Px_0, Qx_0) \in E_G$. From the assumption $Q(A_0) \subseteq P(A_0)$ and the G-proximal edge preserving property of (P, Q), we can construct a sequence $\{x_n\}$ in A_0 satisfying

$$\begin{gathered} Qx_n = Px_{n+1}, \\ (Px_n, Qx_n) \in E_G, \end{gathered} \tag{17}$$

for all integers $n \geq 0$. For each n, since $Q(A_0) \subseteq B_0$, there exists an element $u_n \in A_0$ such that

$$D(u_n, Qx_n) = D(A, B) = D(u_n, Px_{n+1}). \tag{18}$$

If $u_{n_0} = u_{n_0+1}$ for some n_0, we then have

$$D\left(u_{n_0+1}, Qx_{n_0+1}\right) = D(A, B) = D\left(u_{n_0}, Px_{n_0+1}\right). \tag{19}$$

It follows from (A4) that $Q(u_{n_0}) = P(u_{n_0+1}) = P(u_{n_0})$. Applying Lemma 11 yields $\mathrm{CP}(P, Q) \neq \varnothing$.

Let us assume $u_n \neq u_{n+1}$ for all integers $n \geq 0$. From (18), we get

$$\begin{aligned} D(u_n, Qx_n) &= D(u_{n+1}, Qx_{n+1}) = D(A, B) = D(u_{n-1}, Px_n) \\ &= D(u_n, Px_{n+1}), \end{aligned} \tag{20}$$

and (A3) gives $D(u_n, u_{n+1}) < +\infty$ for all n. Since (P, Q) is dominating G-proximal Geraghty, we have that

$$\begin{aligned} D(u_n, u_{n+1}) &\leq \gamma(M(u_{n-1}, u_n, u_n, u_{n+1}))M(u_{n-1}, u_n, u_n, u_{n+1}) \\ &\leq M(u_{n-1}, u_n, u_n, u_{n+1}) \\ &= \max\{D(u_{n-1}, u_n), D(u_n, u_{n+1})\}, \end{aligned} \tag{21}$$

implying that $D(u_n, u_{n+1}) \leq D(u_{n-1}, u_n)$ for all integers $n \geq 1$. Let $\{(D(u_n, u_{n+1})\}$ converge to a real number $r \geq 0$. So does $\{M(u_{n-1}, u_n, u_n, u_{n+1})\}$.

If r were positive, we would obtain

$$1 = \lim_{n\to+\infty} \frac{D(u_n, u_{n+1})}{M(u_{n-1}, u_n, u_n, u_{n+1})} \leq \lim_{n\to+\infty} \gamma(M(u_{n-1}, u_n, u_n, u_{n+1})) \leq 1, \tag{22}$$

which leads to a contradiction as follows:

$$\lim_{n\to+\infty} \gamma(M(u_{n-1}, u_n, u_n, u_{n+1})) = 1 \text{ implies } \lim_{n\to+\infty} M(u_{n-1}, u_n, u_n, u_{n+1}) = 0. \tag{23}$$

Therefore, $\lim_{n\to+\infty} D(u_n, u_{n+1}) = 0$.

Let us next show that $\{u_n\}$ is a D-Cauchy sequence. Suppose that this is not the case. Then, we can construct subsequences $\{u_{n_k}\}$ and $\{u_{m_k}\}$ of $\{u_n\}$ satisfying

$$\begin{gathered} D\left(u_{n_k}, u_{m_k}\right) \geq \varepsilon, \\ m_k > n_k \geq k, \end{gathered} \tag{24}$$

for some $\varepsilon > 0$. Notice that $D(u_{n_k}, u_{m_k}) < +\infty$ for all positive integers k, by (18) and (A3), and $(Px_{n_k}, Qx_{n_k}) \in E_G$ and (Px_{m_k}, Qx_{m_k}) belong to E_G, by (17). Thus,

$$\begin{aligned} D\left(u_{n_k}, Qx_{n_k}\right) &= D\left(u_{m_k}, Qx_{m_k}\right) = D(A, B) = D\left(u_{n_k-1}, Px_{n_k}\right) \\ &= D\left(u_{m_k-1}, Px_{m_k}\right) \end{aligned} \tag{25}$$

for all k. The dominating G-proximal Geraghty property of (P, Q) implies

$$D\left(u_{n_k}, u_{m_k}\right) \leq \gamma\left(M\left(u_{n_k-1}, u_{m_k-1}, u_{n_k}, u_{m_k}\right)\right)M\left(u_{n_k-1}, u_{m_k-1}, u_{n_k}, u_{m_k}\right), \tag{26}$$

where

$$M\left(u_{n_k-1}, u_{m_k-1}, u_{n_k}, u_{m_k}\right) = \max\left\{D\left(u_{n_k-1}, u_{m_k-1}\right), D\left(u_{n_k-1}, u_{n_k}\right), D\left(u_{m_k-1}, u_{m_k}\right)\right\}. \tag{27}$$

Observe that $M(x_{n_k-1}, x_{m_k-1}, x_{n_k}, x_{m_k})$ is clearly neither $D(u_{n_k-1}, u_{n_k})$ nor $D(u_{m_k-1}, u_{m_k})$ for sufficiently large k, as $\lim_{n\to+\infty} D(u_n, u_{n+1}) = 0$. Without loss of generality, we may assume

$$M\left(x_{n_k-1}, x_{m_k-1}, x_{n_k}, x_{m_k}\right) = D\left(u_{n_k-1}, u_{m_k-1}\right), \tag{28}$$

implying

$$D\left(u_{n_k}, u_{m_k}\right) \le \gamma\left(D\left(u_{n_k-1}, u_{m_k-1}\right)\right) D\left(u_{n_k-1}, u_{m_k-1}\right) \tag{29}$$

for all k. Moreover, by induction, we obtain

$$D\left(u_{n_k-i}, u_{m_k-i}\right) \le \gamma\left(D\left(u_{n_k-i-1}, u_{m_k-i-1}\right)\right) D\left(u_{n_k-i-1}, u_{m_k-i-1}\right), \tag{30}$$

where $i = 0, 1, 2, \cdots, n_k - 1$. Therefore,

$$D\left(u_{n_k}, u_{m_k}\right) \le \prod_{i=1}^{n_k} \gamma\left(D\left(u_{n_k-i}, u_{m_k-i}\right)\right) D\left(u_0, u_{m_k-n_k}\right). \tag{31}$$

Define

$$\gamma\left(D\left(u_{n_k-i_k}, u_{m_k-i_k}\right)\right) = \max\left\{\gamma\left(D\left(u_{n_k-i}, u_{m_k-i}\right)\right) : 1 \le i \le n_k\right\},$$
$$\eta = \limsup_{k\to+\infty}\left\{\gamma\left(D\left(u_{n_k-i_k}, u_{m_k-i_k}\right)\right)\right\}. \tag{32}$$

Note that $0 \le \eta \le 1$. If $\eta < 1$, then $\lim_{k\to+\infty}\prod_{i=1}^{n_k}\gamma(D(u_{n_k-i}, u_{m_k-i})) = 0$. If $\eta = 1$, then γ forces $\{D(u_{n_k-i_k}, u_{n_k+m_k-i_k})\}$ to possess a subsequence converging to 0 as $k \to +\infty$. Both cases above contradict the fact that $D(u_{n_k}, u_{m_k}) \ge \varepsilon$ for all k.

We have shown that $\{u_n\}$ is a D-Cauchy sequence. By the assumption (A1), we have $\lim_{n\to+\infty} u_n = u$ for some $u \in A_0$. Notice that, from (18), we may write

$$D(u_{n+1}, Qx_{n+1}) = D(A, B) = D(u_n, Px_{n+1}) \tag{33}$$

for all integers $n \ge 0$. Since the pair (P, Q) commutes proximally and is G-proximal edge preserving, we have

$$Pu_{n+1} = Qu_n,$$
$$(u_n, u_{n+1}) \in E_G, \tag{34}$$

for all n. The G-continuity of P and Q implies

$$Pu = \lim_{n\to+\infty} Pu_n = \lim_{n\to+\infty} Qu_{n-1} = Qu. \tag{35}$$

Lemma 11 then guarantees a common best proximity point.

For uniqueness, we assume that any common proximity point of (P, Q) satisfies the property $(Px, Qx) \in E_G$. Let x^*, $y^* \in CP(P, Q)$. Then,

$$D(x^*, Px^*) = D(y^*, Py^*) = D(A, B) = D(x^*, Qx^*) = D(y^*, Qy^*). \tag{36}$$

As seen in the proof of Lemma 11, we have $D(x^*, x^*) = D(y^*, y^*) = 0$ and

$$M(x^*, y^*, x^*, y^*) = \max\{D(x^*, y^*), D(x^*, x^*), D(y^*, y^*)\} = D(x^*, y^*) < +\infty. \tag{37}$$

Since (P, Q) is dominating G-proximal Geraghty, we obtain

$$D(x^*, y^*) \le \gamma(D(x^*, y^*)) D(x^*, y^*) \le D(x^*, y^*). \tag{38}$$

The inequalities above together with the property of γ yield $D(x^*, y^*) = 0$, as required.

The following is a modification of Theorem 12. Note that the G-continuity of P and Q is dropped and replaced by (iv), which helps facilitate the existence of a common best proximity point.

Theorem 13. *Assume (A0)–(A5) and (i), (ii), and (iii) as in Theorem12. In addition, if the following holds*

(iv) For any sequence $\{x_n\}$ in A that D-converges to $x \in A$ and satisfies $(Qx_n, Px_{n+1}) \in E_G$, there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$D\left(x, Qx_{n_k}\right) = D(A, B) = D\left(x, Px_{n_k}\right), \tag{39}$$

then $CP(P, Q) \neq \varnothing$. Moreover, if $(Px, Qx) \in E_G$, for all $x \in CP(P, Q)$, then (P, Q) has a unique common best proximity point.

Proof. The conditions (i) and (ii) are used to construct $\{x_n\}$ and $\{u_n\}$, as in the proof of Theorem 12. Let us assume that $\lim_{n\to+\infty} u_n = u$ for some $u \in A_0$. By (18), we have

$$D(u_n, Qx_n) = D(A, B) = D(u_{n-1}, Px_n), \tag{40}$$

implying, since (P, Q) commutes proximally, that

$$Pu_n = Qu_{n-1} \tag{41}$$

for all integers $n \ge 1$. Since $(Qu_n, Pu_{n+1}) = (Qu_n, Qu_n) \in E_G$ and the condition (iv), there exists subsequence $\{u_{n_k}\}$ of

$\{u_n\}$ such that

$$D\left(u, Qu_{n_k}\right) = D(A, B) = D\left(u, Pu_{n_k}\right). \tag{42}$$

This, again, yields $Pu = Qu$. Lemma 11 shows $CP(P, Q) \neq \varnothing$.

The uniqueness part is shown in the same fashion as in Theorem 12.

Example 14. Let $X = \mathbb{R}^3$ be equipped with the JS-metric D given by

$$D((x_1, y_1, z_1), (x_2, y_2, z_2)) = \begin{cases} \dfrac{|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|}{3}, & (y_1, y_2) = (0, 0) \text{ or } (z_1, z_2) = (0, 0), \\ |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|, & \text{otherwise.} \end{cases} \tag{43}$$

Let $A = \{(x, 1, 1) : 0 \leq x \leq 3\}$ and $B = \{(x, -1, 1) : 0 \leq x \leq 3\}$. It is easy to see that $D(A, B) = 2$.

Define the mappings $P, Q : A \to B$ by

$$\begin{aligned} P(x, 1, 1) &= (x, -1, 1), \\ Q(x, 1, 1) &= (\ln(2 + x), -1, 1) \end{aligned} \tag{44}$$

for all $(x, 1, 1) \in A$. Notice that P and Q are continuous.

Let

$$E_G = \left\{((x, y, z), (u, v, w)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x \geq u \text{ and } y \leq v \text{ and } z \geq w\right\}. \tag{45}$$

We will show that (P, Q) is G-proximal edge preserving.

(1) Let $(x^*, 1, 1), (y^*, 1, 1) \in A$ and

$$(P(x^*, 1, 1), P(y^*, 1, 1)) = ((x^*, -1, 1), (y^*, -1, 1)) \in E_G, \tag{46}$$

we have $x^* \geq y^*$ and $\ln(2 + x^*) \geq \ln(2 + y^*)$. Thus,

$$(Q(x^*, 1, 1), Q(y^*, 1, 1) = ((\ln(2 + x^*), -1, 1), (\ln(2 + y^*), -1, 1))) \in E_G. \tag{47}$$

(2) Let $x, u, v \in A$. Observe that they must have the following forms:

$$\begin{aligned} x &= (\widehat{x}, 1, 1), \\ u &= (\widehat{u}, 1, 1), \\ v &= (\widehat{v}, 1, 1), \end{aligned} \tag{48}$$

such that $(Px, Qx) \in E_G$ and

$$D(u, Qx) = D(A, B) = D(v, Px). \tag{49}$$

Thus, $\widehat{x} \geq \ln(2 + \widehat{x})$ and

$$D((\widehat{u}, 1, 1), (\ln(2 + \widehat{x}), -1, 1)) = 2 = D((\widehat{v}, 1, 1), (\widehat{x}, -1, 1)). \tag{50}$$

We have $\widehat{u} = \ln(2 + \widehat{x})$, $\widehat{v} = \widehat{x}$ implying that $\widehat{v} \geq \widehat{u}$; that is, $(v, u) \in E_G$.

To show that the pair (P, Q) is dominating G-proximal Geraghty, define the mapping $\gamma : [0, +\infty] \to [0, 1]$ by

$$\gamma(t) = \begin{cases} 1, & t = 0, \\ \dfrac{\ln(1 + t)}{t}, & 0 < t, \\ 0, & t = +\infty. \end{cases} \tag{51}$$

Then, $\gamma \in \Gamma$.

Let $x_1, x_2, u_1, u_2, v_1, v_2 \in A$. Notice that they must have the following forms:

$$\begin{aligned} x_1 &= (\widehat{x}_1, 1, 1), \\ x_2 &= (\widehat{x}_2, 1, 1), \\ u_1 &= (\widehat{u}_1, 1, 1), \\ u_2 &= (\widehat{u}_2, 1, 1), \\ v_1 &= (\widehat{v}_1, 1, 1), \\ v_2 &= (\widehat{v}_2, 1, 1), \end{aligned} \tag{52}$$

such that $(Px_1, Qx_1), (Px_2, Qx_2) \in E_G$ and

$$D(u_1, Qx_1) = D(u_2, Qx_2) = D(A, B) = D(v_1, Px_1) = D(v_2, Px_2). \tag{53}$$

Thus, $\widehat{x}_1 \geq \ln(2 + \widehat{x}_1)$, $\widehat{x}_2 \geq \ln(2 + \widehat{x}_2)$ and

$$\begin{aligned} \widehat{u}_1 &= \ln(2 + \widehat{x}_1), \\ \widehat{u}_2 &= \ln(2 + \widehat{x}_2), \\ \widehat{v}_1 &= \widehat{x}_1, \\ \widehat{v}_2 &= \widehat{x}_2. \end{aligned} \tag{54}$$

To obtain the inequality (11), if $u_1 = u_2$ or $v_1 = v_2$, then we are done. Assume that $u_1 \neq u_2$. Then, $\widehat{u}_1, \widehat{u}_2, \widehat{v}_1, \widehat{v}_2$ are all distinct. As a consequence, $M(v_1, v_2, u_1, u_2) > 0$. Thus, we have that

$$\begin{aligned} D(u_1, u_2) &= |\widehat{u}_1 - \widehat{u}_2| = |\ln(2 + \widehat{v}_1) - \ln(2 + \widehat{v}_2)| \\ &= \left| \ln\left(\frac{2 + \widehat{v}_2 + \widehat{v}_1 - \widehat{v}_2}{2 + \widehat{v}_2}\right) \right| \le \ln(1 + |\widehat{v}_1 - \widehat{v}_2|) \\ &\le \ln(1 + M(v_1, v_2, u_1, u_2)) \\ &= \left[\frac{\ln(1 + M(v_1, v_2, u_1, u_2))}{M(v_1, v_2, u_1, u_2)}\right] M(v_1, v_2, u_1, u_2) \\ &= \gamma(M(v_1, v_2, u_1, u_2)) M(v_1, v_2, u_1, u_2). \end{aligned} \tag{55}$$

Therefore, the pair (P, Q) is dominating G-proximal Geraghty.

Next, consider, by the definition of A_0 and B_0, that $A_0 = A$ and $B_0 = B$. Additionally,

$$\begin{aligned} Q(A_0) &= \{(x, -1, 1) : \ln 2 \le x \le \ln 5\} \subseteq \{(x, -1, 1) : 0 \le x \le 3\} \\ &= B_0 = P(A_0). \end{aligned} \tag{56}$$

Now, it remains to show that (P, Q) commutes proximally. Let $x, u, v \in A$ be such that

$$D(u, Qx) = D(A, B) = D(v, Px). \tag{57}$$

Consequently, $x = (\widehat{x}, 1, 1), u = (\widehat{u}, 1, 1), v = (\widehat{v}, 1, 1)$, where $\widehat{u} = \ln(2 + \widehat{x})$ and $\widehat{v} = \widehat{x}$. Thus,

$$Qv = (\ln(2 + \widehat{v}), -1, 1) = (\ln(2 + \widehat{x}), -1, 1) = (\widehat{u}, -1, 1) = Pu. \tag{58}$$

Thus, (P, Q) commutes proximally.

Finally, by Theorem 12, we can conclude that there is a unique common best proximity point of the pair (P, Q). In fact, the point $(0, 1, 1)$ is the unique common best proximity point of (P, Q).

4. Some Special Cases

Recall that

$$\Gamma = \left\{\gamma : [0, +\infty] \to [0, 1] \,\Big|\, \lim_{n\to+\infty} \gamma(t_n) = 1 \text{ implies } \lim_{n\to+\infty} t_n = 0\right\}. \tag{59}$$

In this section, we present some existence results where functions in Γ are concretely chosen. These results are direct consequences of Theorems 12 and 13.

First of all, let (X, D, G) be a complete JS-metric space endowed with a directed graph, A, B be nonempty subsets of X, and $P, Q : A \to B$ be mappings.

Corollary 15. *Assume (A0)–(A4) and (i), (ii), and (iii) as in Theorem12. In addition, if the following are satisfied*

(i) either of the following holds

(a) P and Q are G-continuous

(b) For any sequence $\{x_n\}$ in A that D-converges to $x \in A$ and satisfies $(Qx_n, Px_{n+1}) \in E_G$, there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$D(x, Qx_{n_k}) = D(A, B) = D(x, Px_{n_k}) \tag{60}$$

(ii) there exists $k \in [0, 1)$ such that for any $x_1, x_2, u_1, u_2, v_1, v_2 \in A$ with $(Px_i, Qx_i) \in E_G$ and

$$D(u_1, Qx_1) = D(u_2, Qx_2) = D(A, B) = D(v_1, Px_1) = D(v_2, Px_2), \tag{61}$$

it follows

$$D(u_1, u_2) \le kM(v_1, v_2, u_1, u_2); \tag{62}$$

then, $CP(P, Q) \neq \emptyset$. Moreover, if $(Px, Qx) \in E_G$, for all $x \in CP(P, Q)$, then (P, Q) has a unique common best proximity point.

Proof. Define $\gamma : [0, +\infty] \to [0, 1]$ by $\gamma(t) = k$ for some $k \in [0, 1)$. Clearly, $\gamma \in \Gamma$, and hence, (A5) is satisfied.

Corollary 16. *Assume (A0)–(A4) and (i), (ii), (iii), and (iv) as in Corollary15. In addition, if the following holds*

(i) for any $x_1, x_2, u_1, u_2, v_1, v_2 \in A$ with $(Px_i, Qx_i) \in E_G$ and

$$D(u_1, Qx_1) = D(u_2, Qx_2) = D(A, B) = D(v_1, Px_1) = D(v_2, Px_2), \tag{63}$$

it follows

$$D(u_1, u_2) \le \frac{M(v_1, v_2, u_1, u_2)}{1 + M(v_1, v_2, u_1, u_2)}; \tag{64}$$

then $CP(P, Q) \neq \emptyset$. Moreover, if $(Px, Qx) \in E_G$, for all $x \in CP(P, Q)$, then (P, Q) has a unique common best proximity point.

Proof. Define $\psi : [0, +\infty] \to [0, 1]$ by $\psi(t) = 1/(1 + t)$ for all $t \in [0, +\infty)$ and $\psi(+\infty) = 0$. For any sequence $\{t_n\}$ with $\lim_{n\to\infty} \psi(t_n) = 1$, we easily have $\lim_{n\to\infty} t_n = 0$. Thus, $\psi \in \Gamma$, and hence, (A5) is satisfied.

5. Application on a JS-Metric Space Endowed with an Arbitrary Relation

In this section, it is shown that our result gives rise to a common best proximity point theorem for a mapping on a JS-metric space endowed with a binary relation $\mathscr{R}$ on X denoted by $(X, D, \mathscr{R})$. To begin with, let us introduce some terminology.

Definition 17. *Let A, B be nonempty subsets of $(X, D, \mathscr{R})$ and $x_0 \in X$. A mapping $P : A \to B$ is called $\mathscr{R}$ -continuous at x_0 if for any sequence $\{x_n\}$ in A that D -converges to x_0 and $x_n \mathscr{R} x_{n+1}$ for all n , the sequence $Px_n D$ -converges to Px_0.*

Definition 18. *Let A, B be nonempty subsets of $(X, D, \mathscr{R})$ and $P, Q : A \to B$ be mappings. A pair (P, Q) is said to be $\mathscr{R}$ -proximally comparative preserving if the following assertions hold:*

(i) If $Px\mathscr{R}Py$, then $Qx\mathscr{R}Qy$

(ii) For any $x, u, v \in A$ such that $Px\mathscr{R}Qx$ and

$$D(u, Qx) = D(A, B) = D(v, Px), \tag{65}$$

it follows $v\mathscr{R}u$.

Definition 19. *Let A, B be nonempty subsets of $(X, D, \mathscr{R})$ and $P, Q : A \to B$ be mappings. A pair (P, Q) is said to be dominating $\mathscr{R}$ -proximally comparative Geraghty if there exists a function $\gamma \in \Gamma$ such that for any $x_1, x_2, u_1, u_2, v_1, v_2 \in A$ satisfying $Px_1\mathscr{R}Qx_1$, $Px_2\mathscr{R}Qx_2$, and*

$$D(u_1, Qx_1) = D(u_2, Qx_2) = D(A, B) = D(v_1, Px_1) = D(v_2, Px_2), \tag{66}$$

it follows that

$$D(u_1, u_2) \le \gamma(M(v_1, v_2, u_1, u_2))M(v_1, v_2, u_1, u_2), \tag{67}$$

where $M(v_1, v_2, u_1, u_2) = \max\{D(v_1, v_2), D(v_1, u_1), D(v_2, u_2)\}$.

Corollary 20. *Let A, B be nonempty subsets of a complete JS-metric space $(X, D, \mathscr{R})$ and let $P, Q : A \to B$ be mappings. Suppose that the pair (P, Q) is dominating $\mathscr{R}$ -proximally comparative Geraghty. Assume that A_0 and B_0 are nonempty such that A_0 is closed and bounded. If the following assertions hold:*

(i) $Q(A_0) \subseteq B_0$ and $Q(A_0) \subseteq P(A_0)$

(ii) P and Q commute proximally

(iii) For each $u, v \in A$, if there exist $x, y \in A$ such that

$$D(u, Qx) = D(A, B) = D(v, Py), \tag{68}$$

then $D(u, v) < +\infty$

(iv) There exists $x_0 \in A_0$ such that $Px_0\mathscr{R}Qx_0$

(v) (P, Q) is $\mathscr{R}$-proximally comparative preserving

(vi) Suppose that one of the following holds

(a) P and Q are $\mathscr{R}$-continuous

(b) For sequence $\{x_n\}$ in A that D-converges to $x \in A$ and satisfies $Qx_n\mathscr{R}Px_{n+1}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$D(x, Qx_{n_k}) = D(A, B) = D(x, Px_{n_k}), \tag{69}$$

then $CP(P, Q) \neq \varnothing$. Moreover, if $Px\mathscr{R}Qx$ for all $x \in CP(P, Q)$, then the pair (P, Q) has a unique common best proximity point.

Proof. We define a directed graph $G = (V_G, E_G)$ with $V_G = X$ and

$$E_G = \{(x, y) \in X \times X : x\mathscr{R}y\}. \tag{70}$$

In order to apply Theorem 12 or 13, all the hypotheses must hold.

(1) We will show that (P, Q) is dominating G-proximal Geraghty. To this end, let $x_1, x_2, u_1, u_2, v_1, v_2 \in A$ be such that

$$\begin{gathered}(Px_1, Qx_1) \in E_G,\\ (Px_2, Qx_2) \in E_G,\\ D(u_1, Qx_1) = D(u_2, Qx_2) = D(A, B) = D(v_1, Px_1) = D(v_2, Px_2).\end{gathered} \tag{71}$$

Then, we have $Px_1\mathscr{R}Qx_1$ and $Px_2\mathscr{R}Qx_2$. Since (P, Q) is dominating $\mathscr{R}$-proximally comparative Geraghty, the pair (P, Q) is dominating G-proximally Geraghty.

(2) The condition (iv) implies that there exists $x_0 \in A_0$ such that

$$(Px_0, Qx_0) \in E_G \tag{72}$$

(3) Let (P, Q) be $\mathscr{R}$-proximally comparative preserving. We will show that (P, Q) is G-proximal edge preserving. Notice that $(Px, Py) \in E_G$ implies $Px\mathscr{R}Py$. Since (P, Q) is $\mathscr{R}$-proximally comparative preserving, we get $Qx\mathscr{R}Qy$ implying that $(Qx, Qy) \in E_G$

Let $x, u, v \in A$ such that $(Px, Qx) \in E_G$ and

$$D(u, Qx) = D(A, B) = D(v, Px). \tag{73}$$

By the definition of E_G, we have $Px\mathscr{R}Qx$. Since (P, Q) is

$\mathscr{R}$-proximally comparative preserving, we obtain $v\mathscr{R}u$. This yields $(v, u) \in E_G$.

(4) The condition (a) implies that P and Q are G-continuous on A. Applying Theorem 12, we achieve CP $(P, Q) \neq \emptyset$

Suppose the condition (b) holds. Let $\{x_n\}$ be a sequence in A that D-converges to $x \in A$ and satisfy $(Qx_n, Px_{n+1}) \in E_G$. Then, $Qx_n\mathscr{R}Px_{n+1}$. By (b), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$D\left(x, Qx_{n_k}\right) = D(A, B) = D\left(x, Px_{n_k}\right). \tag{74}$$

Theorem 13 now applies and gives $\mathrm{CP}(P, Q) \neq \emptyset$.

Finally, if $Px\mathscr{R}Qx$ for all $x \in \mathrm{CP}(P, Q)$, then $(Px, Qx) \in E_G$ for all $x \in \mathrm{CP}(P, Q)$. Therefore, (P, Q) has a unique common best proximity point.

6. Conclusion

This work has proposed some existence and uniqueness theorems for common best proximity points of any two mappings in a JS-metric space endowed with a directed graph G. The results obtained were mainly due to the assumptions that the pair of mappings is G-proximal edge preserving and dominated G-proximal Geraghty. The results have been further applied to a situation where the JS-metric space enjoys a binary relation.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors have no conflict of interests regarding the publication of this paper.

Acknowledgments

This research was partially supported by Chiang Mai University.

References

[1] K. Fan, "Extensions of two fixed point theorems of F.E. Browder," *Mathematische zeitschrift*, vol. 122, pp. 234–240, 1969.

[2] S. Reich, "Approximate selections, best approximations, fixed points, and invariant sets," *Journal of Mathematical Analysis and Applications*, vol. 62, no. 1, pp. 104–113, 1978.

[3] S. S. Basha and P. Veeramani, "Best proximity pair theorems for multifunctions with open fibres," *Journal of Approximation Theory*, vol. 103, no. 1, pp. 119–129, 2000.

[4] W. A. Kirk, S. Reich, and P. Veeramani, "Proximinal retracts and best proximity pair theorems," *Numerical Functional Analysis and Optimization*, vol. 24, no. 7-8, pp. 851–862, 2003.

[5] A. A. Eldred and P. Veeramani, "Existence and convergence of best proximity points," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 2, pp. 1001–1006, 2006.

[6] C. Mongkolkeha, Y. J. Cho, and P. Kumam, "Best proximity points for Geraghty's proximal contraction mappings," *Fixed Point Theory and Applications*, vol. 2013, no. 1, p. 17, 2013.

[7] R. Suparatulatorn and S. Suantai, "A new hybrid algorithm for global minimization of best proximity points in Hilbert spaces," *Carpathian Journal of Mathematics*, vol. 35, no. 1, pp. 95–102, 2019.

[8] H. Işık, H. Aydi, N. Mlaiki, and S. Radenović, "Best proximity point results for Geraghty type Ƶ-proximal contractions with an application," *Axioms*, vol. 8, no. 3, 2019.

[9] H. Aydi, H. Lakzian, Z. D. Mitrović, and S. Radenović, "Best proximity points of *MT*-cyclic contractions with property UC," *Numerical Functional Analysis and Optimization*, vol. 41, no. 7, pp. 871–882, 2020.

[10] H. Scarf, "Fixed-point theorems and economic analysis: mathematical theorems can be used to predict the probable effects of changes in economic policy," *American Scientist*, vol. 71, no. 3, pp. 289–296, 1983.

[11] A. Amini-Harandi, N. Hussain, and F. Akbar, "Best proximity point results for generalized contractions in metric spaces," *Fixed Point Theory and Applications*, vol. 2013, no. 1, p. 113, 2013.

[12] N. Hussain, A. R. Khan, and I. Iqbal, "Common best proximity results for multivalued proximal contractions in metric space with applications," *Journal of Nonlinear Sciences and Applications*, vol. 9, no. 6, pp. 4814–4828, 2016.

[13] A. H. Ansari, A. Razani, and N. Hussain, "New best proximity point results in G–metric space," *Journal of Linear and Topological Algebra*, vol. 6, pp. 73–89, 2017.

[14] V. Parvaneh, N. Hussain, A. Mukheimer, and H. Aydi, "On fixed point results for modified JS-contractions with applications," *Axioms*, vol. 8, no. 3, p. 84, 2019.

[15] H. H. Al-Sulami, N. Hussain, and J. Ahmad, "Best proximity results with applications to nonlinear dynamical systems," *Mathematics*, vol. 7, pp. 1–20, 2019.

[16] S. S. Basha, "Common best proximity points: global minimization of multi-objective functions," *Journal of Global Optimization*, vol. 54, no. 2, pp. 367–373, 2012.

[17] S. S. Basha, N. Shahzad, and R. Jeyaraj, "Common best proximity points: global optimization of multi- objective functions," *Applied Mathematics Letters*, vol. 24, no. 6, pp. 883–886, 2011.

[18] P. Kumam and C. Mongkolekeha, "Common best proximity points for proximity commuting mapping with Geraghty's functions," *Carpathian Journal of Mathematics*, vol. 31, no. 3, pp. 359–364, 2015.

[19] L. Chen, "Common best proximity points theorems," *Journal of Management Research and Analysis*, vol. 39, no. 3, pp. 289–294, 2019.

[20] A. Khemphet, P. Chanthorn, and N. Phudolsitthiphat, "Common best proximity coincidence point theorem for dominating proximal generalized Geraghty in complete metric spaces," vol. 2020, Article ID 9620254, Journal of Function Spaces, 2020.

[21] M. I. Ayari, "A best proximity point theorem for α-proximal Geraghty non-self mappings," *Fixed Point Theory and Applications*, vol. 2019, no. 1, Article ID 661, 2019.

[22] M. A. Geraghty, "On contractive mappings," *Proceedings of the American Mathematical Society*, vol. 40, no. 2, pp. 604–608, 1973.

[23] J. Jachymski, “The contraction principle for mappings on a metric space with a graph,” *Proceedings of American Mathematical Society*, vol. 136, pp. 1359–1373, 2008.

[24] P. Cholamjiak, “Fixed point theorems for Banach type contraction on Tvs-cone metric spaces endowed with a graph,” *Journal of Computational Analysis and Applications*, vol. 16, pp. 338–345, 2014.

[25] R. Suparatulatorn, W. Cholamjiak, and S. Suantai, “A modified S-iteration process for G-nonexpansive mappings in Banach spaces with graphs,” *Numerical Algorithms*, vol. 77, no. 2, pp. 479–490, 2018.

[26] H. A. Hammad, W. Cholamjiak, D. Yambangwai, and H. Dutta, “A modified shrinking projection methods for numerical reckoning fixed points of G–nonexpansive mappings in Hilbert spaces with graphs,” *Miskolc Mathematical Notes*, vol. 20, no. 2, pp. 941–956, 2019.

[27] R. Suparatulatorn, S. Suantai, and W. Cholamjiak, “Hybrid methods for a finite family of G–nonexpansive mappings in Hilbert spaces endowed with graphs,” *AKCE international journal of graphs and combinatorics*, vol. 14, no. 2, pp. 101–111, 2017.

[28] S. Suantai, M. Donganont, and W. Cholamjiak, “Hybrid methods for a countable family of G–nonexpansive mappings in Hilbert spaces endowed with graphs,” *Mathematics*, vol. 7, no. 10, p. 936, 2019.

[29] W. Atiponrat, S. Dangskul, and A. Khemphet, “Coincidence point theorems for KC-contraction mappings in JS–metric spaces endowed with a directed graph,” *Carpathian Journal of Mathematics*, vol. 35, no. 3, pp. 263–272, 2019.

[30] P. Charoensawan and W. Atiponrat, “Common fixed point and coupled coincidence point theorems for Geraghty’s type contraction mapping with two metrics endowed with a directed graph,” *Journal of Mathematics*, vol. 2017, Article ID 5746704, 9 pages, 2017.

[31] C. Klanarong and S. Suantai, “Best proximity point theorems for G–proximal generalized contraction in complete metric spaces endowed with graphs,” *Thai Journal of Mathematics*, vol. 15, no. 1, pp. 261–276, 2017.

[32] M. Jleli and B. Samet, “A generalized metric space and related fixed point theorems,” *Fixed Point Theory and Applications*, vol. 2015, no. 1, Article ID 312, p. 14, 2015.

[33] K. P. R. Sastry, G. A. Naidu, C. Srinivasa Rao, and B. Ramunaidu, “Fixed point theorems for weak k-quasi contractions on a generalized metric space,” *Bulletin of Mathematics and Statistics Research*, vol. 4, no. 1, pp. 119–127, 2016.