Research Article

Differences of Positive Linear Operators on Simplices

Ana-Maria Acu ¹, Gülen Başcanbaz-Tunca ², and Ioan Rasa ³

¹Lucian Blaga University of Sibiu, Department of Mathematics and Informatics, Str. Dr. I. Ratiu, No. 5-7, RO-550012 Sibiu, Romania

²Ankara University, Faculty of Science, Department of Mathematics, Str. Döğol, 06100 Besevler, Ankara, Turkey

³Technical University of Cluj-Napoca, Faculty of Automation and Computer Science, Department of Mathematics, Str. Memorandumului No. 28, 400114 Cluj-Napoca, Romania

Correspondence should be addressed to Ana-Maria Acu; anamaria.acu@ulbsibiu.ro

Received 7 January 2021; Revised 9 March 2021; Accepted 13 March 2021; Published 25 March 2021

Academic Editor: John R. Akeroyd

Copyright © 2021 Ana-Maria Acu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The aim of the paper is twofold: we introduce new positive linear operators acting on continuous functions defined on a simplex and then estimate differences involving them and/or other known operators. The estimates are given in terms of moduli of smoothness and K -functionals. Several applications and examples illustrate the general results.

1. Introduction

Differences of positive linear operators were intensively investigated in the last years; see [1–14] and the references therein. The operators involved in these studies act usually on continuous functions defined on real intervals, and the differences are estimated in terms of moduli of smoothness and K -functionals. In some papers, operators having equal central moments up to a certain order are considered. Other articles deal with operators constructed with the same fundamental functions and different functionals in front of them.

The study of differences of positive linear operators is important from a theoretical point of view, but also from a practical one. Let (U_n) and (V_n) be certain positive linear operators. If we know that $|U_n(f) - V_n(f)|$ is small, we can choose (U_n) or (V_n) taking into account other qualities of them like shape-preserving properties and smoothness/Lipschitz preserving properties.

This paper is concerned with differences of positive linear operators acting on continuous functions defined on simplices. For the sake of simplicity, we consider only the case of the canonical simplex in $\mathbb{R}^2$, where the notation is simpler, but the results can be easily translated to an arbitrary simplex in $\mathbb{R}^n$.

We consider the bivariate versions of some classical operators like Bernstein, Durrmeyer, Kantorovich, and genuine Bernstein-Durrmeyer operators. These bivariate versions

were already studied in literature from other points of view. We introduce the bivariate versions of other operators: U_n^p (see [15, 16]) and the operators defined in [17]. All these operators are constructed with the fundamental Bernstein polynomials on the two-dimensional simplex. A different kind of operator is the bivariate version of the univariate Beta operator of Mühlbach and Lupas (see [18–20]); we introduce it and use it in composition with the Bernstein operator to get a useful representation of U_n^p .

We get estimates of differences of the abovementioned operators, in terms of suitable moduli of smoothness and K -functionals.

To resume, the aim of our paper is twofold: we introduce new operators on a simplex and then estimate differences involving them and other known operators.

The list of applications and examples can be enlarged. In particular, we will be interested for a future work in studying differences of bivariate versions of operators, which preserve exponential functions (see [21–23]). We also intend to deepen the study of the newly introduced Beta operators on the simplex and to consider the composition of it with other operators, leading to new applications and why not new theoretical aspects/problems. Given a Markov operator (i.e., a positive linear operator which preserves the constant functions), the study of its iterate is important not only in Approximation Theory but also in Ergodic Theory and other areas of research. We intend to investigate from this point of

view the newly introduced operators, which are in fact Markov operators.

We end this Introduction by presenting some notation and a fundamental inequality expressed in Lemma 1. Section 2 contains the main theoretical results, while Section 3 is devoted to applications and examples.

Let $S := \{(x, y) \in \mathbb{R}^2 | x, y \geq 0, x + y \leq 1\}$ be the canonical simplex in $\mathbb{R}^2$ and $E(S)$ denote a space of real-valued continuous functions of two variables defined on S , containing the polynomials. Throughout the paper, we will denote by $\mathbf{1}$ the constant function, namely,

$$\mathbf{1} : S \longrightarrow \mathbb{R}, \mathbf{1}(x, y) = 1, (x, y) \in S, \quad (1)$$

and $pr_i : S \longrightarrow \mathbb{R}, i = 1, 2$, will denote the i th coordinate functions restricted on S , which are given by

$$pr_1(x, y) = x \text{ and } pr_2(x, y) = y, (x, y) \in S. \quad (2)$$

Let $F : E(S) \longrightarrow \mathbb{R}$ be a positive linear functional such that $F(\mathbf{1}) = 1$. Set

$$\begin{aligned} b_1^F &:= F(pr_1), b_2^F := F(pr_2), \\ \mu_{i,j}^F &:= F\left(\left(pr_1 - b_1^F \mathbf{1}\right)^i \left(pr_2 - b_2^F \mathbf{1}\right)^j\right), i, j \in \mathbb{N}. \end{aligned} \quad (3)$$

Then, one has

$$\begin{aligned} \mu_{1,0}^F &= 0, \mu_{2,0}^F = F(pr_1^2) - (b_1^F)^2 \geq 0, \\ \mu_{0,1}^F &= 0, \mu_{0,2}^F = F(pr_2^2) - (b_2^F)^2 \geq 0. \end{aligned} \quad (4)$$

Let $C^2(S)$ be the space of all real-valued (continuous) functions, differentiable on $\text{int}(S)$ and whose partial derivatives of order ≤ 2 can be continuously extended to S , having

$$\|f\| = \sup \{|f(x, y)| : (x, y) \in S\} < \infty. \quad (5)$$

Lemma 1. *If $f \in C^2(S)$, then*

$$\begin{aligned} |F(f) - f(b_1^F, b_2^F)| &\leq M_f \{\mu_{2,0}^F + \mu_{0,2}^F\}, \\ \text{where } M_f &:= \max \left\{ \|f_{xx}\|, \|f_{xy}\|, \|f_{yy}\| \right\}. \end{aligned} \quad (6)$$

Proof. Consider the line segment connecting (b_1^F, b_2^F) with $(t_1, t_2) \in S$. From Taylor's formula (see [24], p.245), there is a point (c_1, c_2) on this line segment, different from (b_1^F, b_2^F)

and (t_1, t_2) , such that

$$\begin{aligned} f(t_1, t_2) &= f(b_1^F, b_2^F) + f_x(b_1^F, b_2^F)(t_1 - b_1^F) \\ &\quad + f_y(b_1^F, b_2^F)(t_2 - b_2^F) \\ &\quad + \frac{1}{2} \left\{ f_{xx}(c_1, c_2)(t_1 - b_1^F)^2 \right. \\ &\quad + 2f_{xy}(c_1, c_2)(t_1 - b_1^F)(t_2 - b_2^F) \\ &\quad \left. + f_{yy}(c_1, c_2)(t_2 - b_2^F)^2 \right\}. \end{aligned} \quad (7)$$

Therefore, we can write

$$\begin{aligned} &|f - f(b_1^F, b_2^F)\mathbf{1} - f_x(b_1^F, b_2^F)(pr_1 - b_1^F \mathbf{1}) \\ &\quad - f_y(b_1^F, b_2^F)(pr_2 - b_2^F \mathbf{1})| \\ &\leq \frac{1}{2} \left\{ \|f_{xx}\| (pr_1 - b_1^F \mathbf{1})^2 + 2\|f_{xy}\| (pr_1 - b_1^F \mathbf{1}) \right. \\ &\quad \cdot (pr_2 - b_2^F \mathbf{1}) + \|f_{yy}\| (pr_2 - b_2^F \mathbf{1})^2 \left. \right\} \\ &\leq \frac{1}{2} \left\{ (\|f_{xx}\| + \|f_{xy}\|) (pr_1 - b_1^F \mathbf{1})^2 \right. \\ &\quad \left. + (\|f_{xy}\| + \|f_{yy}\|) (pr_2 - b_2^F \mathbf{1})^2 \right\} \\ &\leq M_f \left((pr_1 - b_1^F \mathbf{1})^2 + (pr_2 - b_2^F \mathbf{1})^2 \right), \end{aligned} \quad (8)$$

which gives the result.

2. Difference of Bivariate Positive Linear Operators

Denote by $C(S)$ the space of real-valued continuous functions on S with the norm $\|f\| = \max_{(x,y) \in S} |f(x, y)|, f \in C(S)$. Let

K be a set of nonnegative integers and for $k, l \in K$ let $p_{k,l} \in C(S), p_{k,l} \geq 0$, satisfy $\sum_{k,l \in K} p_{k,l} = \mathbf{1}$. Let $F_{k,l} : E(S) \longrightarrow \mathbb{R}$ and $G_{k,l} : E(S) \longrightarrow \mathbb{R}, k, l \in K$, be positive linear functionals such that $F_{k,l}(\mathbf{1}) = 1$ and $G_{k,l}(\mathbf{1}) = 1$. Moreover, let $D(S)$ be the set of all $f \in E(S)$ for which

$$\begin{aligned} \sum_{k,l \in K} F_{k,l}(f) p_{k,l} &\in C(S), \\ \sum_{k,l \in K} G_{k,l}(f) p_{k,l} &\in C(S). \end{aligned} \quad (9)$$

Now, consider the bivariate positive linear operators V

and W acting from $D(S)$ into $C(S)$ defined, for $f \in D(S)$, by

$$\begin{aligned} V(f)(x, y) &= \sum_{k,l \in K} F_{k,l}(f) p_{k,l}(x, y), \\ W(f)(x, y) &= \sum_{k,l \in K} G_{k,l}(f) p_{k,l}(x, y), \end{aligned} \quad (10)$$

respectively. For future correspondences, we denote

$$\sigma(x, y) := \sum_{k,l \in K} \left(\mu_{2,0}^{F_{k,l}} + \mu_{2,0}^{G_{k,l}} + \mu_{0,2}^{F_{k,l}} + \mu_{0,2}^{G_{k,l}} \right) p_{k,l}(x, y), \quad (11)$$

$$\delta := \sup_{k,l \in K} \left| \left(b_1^{F_{k,l}}, b_2^{F_{k,l}} \right) - \left(b_1^{G_{k,l}}, b_2^{G_{k,l}} \right) \right|, \quad (12)$$

where $|\cdot|$ is the l_1 -norm in $\mathbb{R}^2$.

In the following, we adopt the definitions of K -functional and modulus of smoothness from [25, 26]. Let

$$S(h) = \{x \in S \mid x + ht \in S \text{ for } 0 \leq t \leq 1\}, h \in \mathbb{R}^2. \quad (13)$$

For $r \in \mathbb{N}$, r th order differences on the subset $S(rh)$ are defined as

$$\Delta_h^r f(x) = \sum_{k=0}^r (-1)^{r-k} \binom{r}{k} f(x + kh). \quad (14)$$

The r th order modulus of smoothness of f is a function $\omega_r : C(S) \times (0, \infty) \rightarrow [0, \infty)$ given by

$$\omega_r(f, \alpha) = \sup_{0 < |h| \leq \alpha} \|\Delta_h^r f\|, \alpha > 0. \quad (15)$$

Let $C^r(S)$ be the space of all real-valued (continuous) functions, differentiable on $\text{int}(S)$ and whose partial derivatives of order $\leq r$ can be continuously extended to S , with the seminorm

$$|g|_{C^r(S)} = \sum_{\gamma_1 + \gamma_2 = r} \left\| \frac{\partial^r g}{\partial x^{\gamma_1} \partial y^{\gamma_2}} \right\| < \infty, \gamma_i \geq 0, i = 1, 2, \gamma_1 + \gamma_2 = r. \quad (16)$$

For $f \in C(S)$, we shall use the following K -functional:

$$K_r(f, t) = \inf \left\{ \|f - g\| + t |g|_{C^r(S)} : g \in C^r(S) \right\}. \quad (17)$$

Then, there exist $c_1, c_2 > 0$ such that for any $t > 0$ (see [25, 26])

$$c_1 K_r(f, t^r) \leq \omega_r(f, t) \leq c_2 K_r(f, t^r). \quad (18)$$

Here, c_2 depends only on r (for the general definition on the L_p , $1 \leq p \leq \infty$, spaces of functions on bounded domains, see [25] or, on unbounded domains see [27], p.341.

Theorem 2. If $f \in D(S) \cap C^2(S)$, then

$$|(V - W)(f)(x, y)| \leq M_f \sigma(x, y) + \omega_1(f, \delta), \quad (19)$$

where M_f is defined in Lemma 1.

Proof. Let $(x, y) \in S$. From Lemma 1, we get

$$\begin{aligned} |(V - W)(f)(x, y)| &\leq \sum_{k,l \in K} |F_{k,l}(f) - G_{k,l}(f)| p_{k,l}(x, y) \\ &\leq \sum_{k,l \in K} p_{k,l}(x, y) \left\{ \left| F_{k,l}(f) - f(b_1^{F_{k,l}}, b_2^{F_{k,l}}) \right| \right. \\ &\quad + \left| G_{k,l}(f) - f(b_1^{G_{k,l}}, b_2^{G_{k,l}}) \right| + \left| f(b_1^{F_{k,l}}, b_2^{F_{k,l}}) \right. \\ &\quad \left. - f(b_1^{G_{k,l}}, b_2^{G_{k,l}}) \right\} \leq M_f \sum_{k,l \in K} p_{k,l}(x, y) \left[\mu_{2,0}^{F_{k,l}} + \mu_{0,2}^{F_{k,l}} + \mu_{2,0}^{G_{k,l}} + \mu_{0,2}^{G_{k,l}} \right] \\ &\quad + \omega_1(f; \left| (b_1^{F_{k,l}}, b_2^{F_{k,l}}) - (b_1^{G_{k,l}}, b_2^{G_{k,l}}) \right|) \leq M_f \sigma(x, y) + \omega_1(f, \delta). \end{aligned} \quad (20)$$

Theorem 3. If $f \in C(S)$, then

$$|(V - W)(f)(x, y)| \leq \eta_1 \omega_1(f, \delta) + \eta_2 \omega_2(f, \sqrt{\sigma(x, y)}), \quad (21)$$

where $\eta_1, \eta_2 > 0$, and $\delta = \sup_{k,l \in K} \{|b_1^{F_{k,l}} - b_1^{G_{k,l}}| + |b_2^{F_{k,l}} - b_2^{G_{k,l}}|\}$.

Proof. Let $g \in C^2(S)$. From Theorem 2, we get

$$\begin{aligned} |(V - W)(f)(x, y)| &\leq |V(f - g)(x, y)| + |W(g - f)(x, y)| \\ &\quad + |(V - W)(g)(x, y)| \leq 2 \|f - g\| + M_g \sigma(x, y) \\ &\quad + \left| g(b_1^{F_{k,l}}, b_2^{F_{k,l}}) - g(b_1^{G_{k,l}}, b_2^{G_{k,l}}) \right|, \end{aligned} \quad (22)$$

where M_g is the same notation as in Lemma 1 for g . Since partial derivatives of g exist and are continuous everywhere in S , it follows that g is differentiable at every point of the line segment connecting the points $(b_1^{F_{k,l}}, b_2^{F_{k,l}})$ and $(b_1^{G_{k,l}}, b_2^{G_{k,l}})$ in S , $k, l \in K$. By the mean value theorem (see, e.g., [24], p. 239), there is a point (a_1, a_2) on this line segment such that

$$\begin{aligned} g(b_1^{F_{k,l}}, b_2^{F_{k,l}}) - g(b_1^{G_{k,l}}, b_2^{G_{k,l}}) &= g_x(a_1, a_2) (b_1^{F_{k,l}} - b_1^{G_{k,l}}) \\ &\quad + g_y(a_1, a_2) (b_2^{F_{k,l}} - b_2^{G_{k,l}}). \end{aligned} \quad (23)$$

From (16), we get

$$\begin{aligned} \left| g(b_1^{F_{k,l}}, b_2^{F_{k,l}}) - g(b_1^{G_{k,l}}, b_2^{G_{k,l}}) \right| &\leq \|g_x\| |b_1^{F_{k,l}} - b_1^{G_{k,l}}| \\ &\quad + \|g_y\| |b_2^{F_{k,l}} - b_2^{G_{k,l}}| \leq |g|_{C^1(S)} \sup_{k,l \in K} \left\{ |b_1^{F_{k,l}} - b_1^{G_{k,l}}| \right. \\ &\quad \left. + |b_2^{F_{k,l}} - b_2^{G_{k,l}}| \right\} \leq \delta |g|_{C^1(S)}, k, l \in K. \end{aligned} \quad (24)$$

Moreover, since $M_g \leq |g|_{C^2(S)}$, (22) gives that

$$|(V - W)(f)(x, y)| \leq 2\|f - g\| + \delta|g|_{C^1(S)} + \sigma(x, y)|g|_{C^2(S)} \leq K_1(f, \delta) + K_2(f, \sigma(x, y)). \quad (25)$$

Finally, from (18), we obtain

$$|(V - W)(f)(x, y)| \leq \eta_1 \omega_1(f, \delta) + \eta_2 \omega_2\left(f, \sqrt{\sigma(x, y)}\right). \quad (26)$$

3. Applications

3.1. Difference of Bivariate Bernstein Operators and Their Durrmeyer Variants. For every $n \geq 1, f \in C(S)$, and $(x, y) \in S$, the n th bivariate Bernstein operator $B_n : C(S) \rightarrow C(S)$ is defined by

$$B_n(f)(x, y) = \sum_{\substack{k, l=0, \dots, n \\ k+l \leq n}} f\left(\frac{k}{n}, \frac{l}{n}\right) p_{n,k,l}(x, y), \quad (27)$$

where

$$p_{n,k,l}(x, y) := \frac{n!}{k!l!(n-k-l)!} x^k y^l (1-x-y)^{n-k-l}, \quad (28)$$

with $k, l = 0, \dots, n, k+l \leq n, (x, y) \in S$, (see, e.g., [28], p. 115).

For $f \in L^1(S)$, the bivariate Durrmeyer operators $M_n : L^1(S) \rightarrow C(S)$ are defined by

$$M_n(f)(x, y) = \sum_{\substack{k, l=0, \dots, n \\ k+l \leq n}} \left((n+1)(n+2) \int_0^1 \int_0^{1-s} p_{n,k,l}(s, t) \cdot f(s, t) dt ds \right) p_{n,k,l}(x, y), \quad (29)$$

see, e.g., [29].

Now, denoting

$$F_{n,k,l}(f) := f\left(\frac{k}{n}, \frac{l}{n}\right), 0 \leq k+l \leq n, \\ G_{n,k,l}(f) := (n+1)(n+2) \int_0^1 \int_0^{1-s} p_{n,k,l}(s, t) f(s, t) dt ds, \quad (30)$$

the bivariate Bernstein operators and bivariate Durrmeyer

operators can be written as

$$B_n(f)(x, y) = \sum_{\substack{k, l=0, \dots, n \\ k+l \leq n}} F_{n,k,l}(f) p_{n,k,l}(x, y), \\ M_n(f)(x, y) = \sum_{\substack{k, l=0, \dots, n \\ k+l \leq n}} G_{n,k,l}(f) p_{n,k,l}(x, y), \quad (31)$$

respectively.

Proposition 4. For bivariate Bernstein operators and their Durrmeyer variants, the following properties hold:

(i) If $f \in C^2(S)$, then

$$|(B_n - M_n)(f)(x, y)| \leq M_f \sigma(x, y) + \omega_1\left(f, \frac{3}{n+3}\right), \quad (32)$$

where M_f is the same as in Lemma 1 and

$$\sigma(x, y) = \frac{(-x^2 - y^2 + x + y)n^2 + (x^2 + y^2 + 2)n + 4}{(n+3)^2(n+4)} \leq \frac{1}{n+4}. \quad (33)$$

(ii) If $f \in C(S)$, then

$$|(B_n - M_n)(f)(x, y)| \leq \eta_1 \omega_1\left(f, \frac{3}{n+3}\right) + \eta_2 \omega_2\left(f, \sqrt{\sigma(x, y)}\right). \quad (34)$$

Proof. We need to evaluate the terms in (11). So, we get the following results:

$$b_1^{F_{n,k,l}} = \frac{k}{n}, b_2^{F_{n,k,l}} = \frac{l}{n}, \\ b_1^{G_{n,k,l}} = \frac{k+1}{n+3}, b_2^{G_{n,k,l}} = \frac{l+1}{n+3}, \quad (35)$$

$0 \leq k+l \leq n$. Therefore, we easily obtain that

$$\mu_{2,0}^{F_{n,k,l}} = 0, \mu_{2,0}^{G_{n,k,l}} = \frac{(k+1)(n+2-k)}{(n+3)^2(n+4)}, \\ \mu_{0,2}^{F_{n,k,l}} = 0, \mu_{0,2}^{G_{n,k,l}} = \frac{(l+1)(n+2-l)}{(n+3)^2(n+4)}. \quad (36)$$

Using Maple, one obtains

$$\begin{aligned}\sigma(x, y) &= \sum_{\substack{k, l=0, \dots, n \\ k+l \leq n}} \left[\frac{(k+1)(n+2-k)}{(n+3)^2(n+4)} + \frac{(l+1)(n+2-l)}{(n+3)^2(n+4)} \right] \\ p_{n,k,l}(x, y) &= \frac{(-x^2 - y^2 + x + y)n^2 + (x^2 + y^2 + 2)n + 4}{(n+3)^2(n+4)}.\end{aligned}\quad (37)$$

It is easy to verify that $\sigma(x, y) \leq 1/(n+4)$.

Now, for δ , we obtain

$$\begin{aligned}\delta &= \max_{0 \leq k+l \leq n} \left\{ \left| b_1^{F_{n,k,l}} - b_1^{G_{n,k,l}} \right| + \left| b_2^{F_{n,k,l}} - b_2^{G_{n,k,l}} \right| \right\} \\ &= \max_{0 \leq k+l \leq n} \left\{ \left| \frac{n-3k}{n(n+3)} \right| + \left| \frac{n-3l}{n(n+3)} \right| \right\} = \frac{3}{n+3}.\end{aligned}\quad (38)$$

The rest of the proof follows from Theorems 2 and 3.

3.2. Difference of Bivariate Bernstein Operators and the Bivariate Operators A_n . Let Π_n be the space of polynomials over $[0, 1]$ of degree at most n . In [17], Aldaz et al. introduced a Bernstein operator $A_n : C[0, 1] \rightarrow \Pi_n$ that fixes 1 and x^2 . The operators A_n are given by

$$A_n(f)(x) = \sum_{k=0}^n f\left(\left(\frac{k(k-1)}{n(n-1)}\right)^{1/2}\right) \binom{n}{k} x^k (1-x)^{n-k}.\quad (39)$$

Here, for $f \in C(S)$ and $(x, y) \in S$, we introduce the bivariate form of the operators A_n as follows

$$A_n(f)(x, y) = \sum_{\substack{k, l=0, \dots, n \\ k+l \leq n}} f\left(\sqrt{\frac{k(k-1)}{n(n-1)}}, \sqrt{\frac{l(l-1)}{n(n-1)}}\right) p_{n,k,l}(x, y).\quad (40)$$

Denoting

$$\begin{aligned}F_{n,k,l}(f) &= f\left(\frac{k}{n}, \frac{l}{n}\right), G_{n,k,l}(f) \\ &= f\left(\sqrt{\frac{k(k-1)}{n(n-1)}}, \sqrt{\frac{l(l-1)}{n(n-1)}}\right),\end{aligned}\quad (41)$$

for $k, l = 0, \dots, n, k+l \leq n$, we get

$$\begin{aligned}b_1^{F_{n,k,l}} &= \frac{k}{n}, b_2^{F_{n,k,l}} = \frac{l}{n}, \\ b_1^{G_{n,k,l}} &= \sqrt{\frac{k(k-1)}{n(n-1)}}, b_2^{G_{n,k,l}} = \sqrt{\frac{l(l-1)}{n(n-1)}}, \\ \mu_{2,0}^{F_{n,k,l}} &= \mu_{0,2}^{F_{n,k,l}} = \mu_{2,0}^{G_{n,k,l}} = \mu_{0,2}^{G_{n,k,l}} = 0, \\ \delta &= \max_{\substack{k, l=0, \dots, n \\ k+l \leq n}} \left\{ \left| b_1^{F_{n,k,l}} - b_1^{G_{n,k,l}} \right| + \left| b_2^{F_{n,k,l}} - b_2^{G_{n,k,l}} \right| \right\} = \frac{2}{n}.\end{aligned}\quad (42)$$

Proposition 5. For bivariate Bernstein operators and bivariate operators A_n , the following properties hold:

(i) If $f \in C^2(S)$, then

$$|(B_n - A_n)(f)(x, y)| \leq \omega_1\left(f, \frac{2}{n}\right).\quad (43)$$

(ii) If $f \in C(S)$, then

$$|(B_n - A_n)(f)(x, y)| \leq \eta_1 \omega_1\left(f, \frac{2}{n}\right).\quad (44)$$

3.3. Difference of Bivariate Bernstein Operators and Bivariate Genuine Bernstein-Durrmeyer Operators. In 1987, Chen [30] and Goodman and Sharma [31] constructed the following positive linear operators

$$\begin{aligned}U_{n,1}(f)(x) &= f(0)p_{n,0}(x) + f(1)p_{n,n}(x) \\ &\quad + \sum_{k=1}^{n-1} p_{n,k}(x)(n-1) \\ &\quad \cdot \int_0^1 p_{n-2,k-1}(t)f(t)dt,\end{aligned}\quad (45)$$

where $n \in \mathbb{N}, f \in C[0, 1]$, and

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, x \in [0, 1], 0 \leq k \leq n.\quad (46)$$

For the historical background of these operators, we refer to [32]. In 1991, Goodman and Sharma [33] constructed and studied the multivariate form of the operators $U_{n,1}$ on a simplex. In [34], Sauer deeply studied the multivariate genuine Bernstein-Durrmeyer operators. Here, for $f \in L^1(S)$, we

consider the bivariate form given by

$$\begin{aligned}
 U_n(f)(x, y) = & f(0, 0)(1 - x - y)^n + f(1, 0)x^n + f(0, 1)y^n \\
 & + \sum_{l=1}^{n-1} p_{n,0,l}(x, y)(n-1) \int_0^1 p_{n-2,l-1}(t)f(0, t)dt \\
 & + \sum_{k=1}^{n-1} p_{n,k,0}(x, y)(n-1) \int_0^1 p_{n-2,k-1}(s)f(s, 0)ds \\
 & + \sum_{k=1}^{n-1} p_{n,k,n-k}(x, y)(n-1) \\
 & \cdot \int_0^1 p_{n-2,k-1}(t)f(t, 1-t)dt \\
 & + \sum_{\substack{k+l \leq n-1 \\ k \geq 1, l \geq 1}}^{n-1} p_{n,k,l}(x, y)(n-1)(n-2) \\
 & \cdot \int_0^1 \int_0^{1-t} p_{n-3,k-1,l-1}(s, t)f(s, t)ds dt
 \end{aligned} \tag{47}$$

with the bivariate Bernstein's fundamental functions given by (28) (see [33], Formula 1.7). These operators satisfy $U_n(f)(x, y) = f(x, y)$ at the vertices of S .

Proposition 6. *For bivariate Bernstein operators and bivariate genuine Bernstein-Durrmeyer operators, the following properties hold:*

(i) *If $f \in C^2(S)$, then*

$$|(B_n - U_n)(f)(x, y)| \leq M_f \sigma(x, y), \tag{48}$$

where M_f is the same as in Lemma 1 and

$$\sigma(x, y) = \frac{(x + y - x^2 - y^2)(n-1)}{n(n+1)} \leq \frac{1}{2(n+1)}. \tag{49}$$

(ii) *If $f \in C(S)$, then*

$$|(B_n - U_n)(f)(x, y)| \leq \eta_2 \omega_2\left(f; \sqrt{\sigma(x, y)}\right). \tag{50}$$

Proof. If we denote

$$F_{n,k,l}(f) := f\left(\frac{k}{n}, \frac{l}{n}\right), 0 \leq k + l \leq n,$$

$$G_{n,0,0}(f) = f(0, 0), k = l = 0,$$

$$G_{n,n,0}(f) = f(1, 0), k = n, l = 0,$$

$$G_{n,0,n}(f) = f(0, 1), k = 0, l = n,$$

$$\begin{aligned}
 G_{n,k,0}(f) &= (n-1) \int_0^1 p_{n-2,k-1}(s)f(s, 0)ds, 1 \leq k \leq n-1, l = 0, \\
 G_{n,0,l}(f) &= (n-1) \int_0^1 p_{n-2,l-1}(t)f(0, t)dt, k = 0, 1 \leq l \leq n-1, \\
 G_{n,k,n-k}(f) &= (n-1) \int_0^1 p_{n-2,k-1}(t)f(t, 1-t) \\
 &\quad \cdot dt, 1 \leq k \leq n-1, l = n-k, \\
 G_{n,k,l}(f) &= (n-1)(n-2) \int_0^1 \int_0^{1-t} p_{n-3,k-1,l-1}(s, t)f(s, t) \\
 &\quad \cdot ds dt, 1 \leq k + l \leq n-1,
 \end{aligned} \tag{51}$$

then for the bivariate Bernstein operators, we have

$$B_n(f)(x, y) = \sum_{\substack{k,l=0,\dots,n \\ k+l \leq n}} F_{n,k,l}(f) p_{n,k,l}(x, y). \tag{52}$$

The bivariate genuine Bernstein-Durrmeyer operators are given by

$$U_n(f)(x, y) = \sum_{\substack{k,l=0,\dots,n \\ k+l \leq n}} G_{n,k,l}(f) p_{n,k,l}(x, y). \tag{53}$$

Now, for $0 \leq k + l \leq n$, we get

$$b_1^{F_{n,k,l}} = \frac{k}{n}, b_2^{F_{n,k,l}} = \frac{l}{n}, b_1^{G_{n,k,l}} = \frac{k}{n}, b_2^{G_{n,k,l}} = \frac{l}{n}. \tag{54}$$

Hence, we obtain

$$\begin{aligned}
 \mu_{2,0}^{F_{n,k,l}} &= 0, \mu_{2,0}^{G_{n,k,l}} = \frac{k(n-k)}{n^2(n+1)}, \\
 \mu_{0,2}^{F_{n,k,l}} &= 0, \mu_{0,2}^{G_{n,k,l}} = \frac{l(n-l)}{n^2(n+1)}.
 \end{aligned} \tag{55}$$

Therefore, $\delta = 0$ and

$$\begin{aligned}
 \sigma(x, y) &= \sum_{\substack{k,l=0,\dots,n \\ k+l \leq n}} \left[\frac{k(n-k)}{n^2(n+1)} + \frac{l(n-l)}{n^2(n+1)} \right] p_{n,k,l}(x, y) \\
 &= \frac{(x + y - x^2 - y^2)(n-1)}{n(n+1)}, (x, y) \in S.
 \end{aligned} \tag{56}$$

The proof is concluded by using Theorems 2 and 3.

4. The Difference $U_n^\rho - U_n^r$

Let $\rho > 0$ and $n \in \mathbb{N}$. The operators $U_{n,1}^\rho : C[0, 1] \longrightarrow \prod_n$ are introduced by Paltanea in [35] (see also [15, 16]). These operators are defined by

$$U_{n,1}^\rho(f; x) := \sum_{k=1}^{n-1} \left(\int_0^1 \frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{\beta(k\rho, (n-k)\rho)} f(t) dt \right) \cdot p_{n,k}(x) + f(0)(1-x)^n + f(1)x^n, \quad (57)$$

where $f \in C[0, 1]$, $x \in [0, 1]$, and $\beta(\cdot, \cdot)$ are Euler's Beta function.

Here, for $f \in L^1(S)$, we consider the bivariate form of these operators, given by

$$\begin{aligned} U_n^\rho f(x, y) &= f(0, 0)(1-x-y)^n + f(1, 0)x^n + f(0, 1)y^n \\ &+ \sum_{l=1}^{n-1} F_{n,0,l}^\rho(f) p_{n,0,l}(x, y) \\ &+ \sum_{k=1}^{n-1} F_{n,k,0}^\rho(f) p_{n,k,0}(x, y) \\ &+ \sum_{k=1}^{n-1} F_{n,k,n-k}^\rho(f) p_{n,k,n-k}(x, y) \\ &+ \sum_{\substack{k \geq 1, l \geq 1 \\ k+l \leq n-1}} F_{n,k,l}^\rho(f) p_{n,k,l}(x, y), \end{aligned} \quad (58)$$

where

$$\begin{aligned} F_{n,0,l}^\rho(f) &:= \frac{\int_0^1 t^{l\rho-1}(1-t)^{(n-l)\rho-1} f(0, t) dt}{B(l\rho, (n-l)\rho)}, \\ F_{n,k,0}^\rho(f) &:= \frac{\int_0^1 s^{k\rho-1}(1-s)^{(n-k)\rho-1} f(s, 0) ds}{B(k\rho, (n-k)\rho)}, \\ F_{n,k,n-k}^\rho(f) &:= \frac{\int_0^1 t^{k\rho-1}(1-t)^{(n-k)\rho-1} f(t, 1-t) dt}{B(k\rho, (n-k)\rho)}, \\ F_{n,k,l}^\rho(f) &:= \frac{\iint_S s^{k\rho-1} t^{l\rho-1} (1-s-t)^{(n-k-l)\rho-1} f(s, t) ds dt}{\iint_S s^{k\rho-1} t^{l\rho-1} (1-s-t)^{(n-k-l)\rho-1} ds dt}. \end{aligned} \quad (59)$$

It can be easily seen that, for $\rho = 1$, we obtain the genuine Bernstein-Durrmeyer operators U_n . On the other hand, these operators have the following limiting behavior.

Theorem 7. For any $f \in C(S)$, one has $\lim_{\rho \rightarrow \infty} U_n^\rho(f) = B_n(f)$, uniformly.

Proof. Let $f = pr_1^j$, $j = 0, 1, \dots$. Then,

$$F_{n,k,l}^\rho(pr_1^j) = \frac{(k\rho + j - 1) \cdots (k\rho)}{(n\rho + j - 1) \cdots (n\rho)}, \quad \text{for } k, l \geq 1, \quad (60)$$

$$\lim_{\rho \rightarrow \infty} F_{n,k,l}^\rho(pr_1^j) = \left(\frac{k}{n}\right)^j = pr_1^j\left(\frac{k}{n}, \frac{l}{n}\right).$$

Since

$$\begin{aligned} F_{n,0,l}^\rho(pr_1^j) &= \begin{cases} 1, j=0, \\ 0, j>0, \end{cases} \quad F_{n,k,0}^\rho(pr_1^j) = F_{n,k,n-k}^\rho(pr_1^j) \\ &= \frac{(k\rho + j - 1) \cdots (k\rho)}{(n\rho + j - 1) \cdots (n\rho)}, \end{aligned} \quad (61)$$

we get

$$\begin{aligned} \lim_{\rho \rightarrow \infty} F_{n,0,l}^\rho(pr_1^j) &= pr_1^j\left(0, \frac{l}{n}\right), \quad \lim_{\rho \rightarrow \infty} F_{n,k,0}^\rho(pr_1^j) = \left(\frac{k}{n}\right)^j = pr_1^j\left(\frac{k}{n}, 0\right), \\ \lim_{\rho \rightarrow \infty} F_{n,k,n-k}^\rho(pr_1^j) &= \left(\frac{k}{n}\right)^j = pr_1^j\left(\frac{k}{n}, \frac{n-k}{n}\right). \end{aligned} \quad (62)$$

Similar results can be obtained for pr_2^j , $j = 0, 1, \dots$.

Using Korovkin's theorem (see [36], p. 534, C.4.3.3), it follows $\lim_{\rho \rightarrow \infty} F_{n,\mu,\nu}^\rho(f) = f(\mu/n, \nu/n)$. Therefore,

$$\lim_{\rho \rightarrow \infty} U_n^\rho(f) = B_n(f), \quad f \in C(S). \quad (63)$$

Proposition 8. For the bivariate operators U_n^ρ , the following properties hold:

(i) If $f \in C^2(S)$, then

$$|(U_n^\rho - U_n^r)(f)(x, y)| \leq M_f \sigma(x, y), \quad (64)$$

where M_f is the same as in Lemma 1 and

$$\sigma(x, y) = \frac{(nr + n\rho + 2)(x + y - x^2 - y^2)(n-1)}{n(n\rho + 1)(nr + 1)}. \quad (65)$$

(ii) If $f \in C(S)$, then

$$|(U_n^\rho - U_n^r)(f)(x, y)| \leq \eta_2 \omega_2\left(f; \sqrt{\sigma(x, y)}\right). \quad (66)$$

Proof. Since $b_1^{F_{n,k,l}^\rho} = k/n$, $b_1^{F_{n,k,l}^\rho} = l/n$, we get

$$\mu_{2,0}^{F_{n,k,l}^\rho} = \frac{k(n-k)}{n^2(n\rho + 1)}, \quad \mu_{0,2}^{F_{n,k,l}^\rho} = \frac{l(n-l)}{n^2(n\rho + 1)}. \quad (67)$$

Therefore, $\delta = 0$ and

$$\sigma(x, y) = \frac{(nr + n\rho + 2)(x + y - x^2 - y^2)(n - 1)}{n(n\rho + 1)(nr + 1)}. \quad (68)$$

5. Difference of Bivariate Bernstein Operators and Their Kantorovich Variants

In 2017, F. Altomare et al. [37] introduced Kantorovich operators on S as follows

$$\mathbb{C}_n(f)(x, y) = \sum_{\substack{k, l=0, \dots, n \\ k+l \leq n}} p_{n,k,l}(x, y) 2 \iint_S f\left(\frac{k+as}{n+a}, \frac{l+at}{n+a}\right) ds dt, \quad (69)$$

where $p_{n,k,l}(x, y)$ is given by (28). It can be easily seen that, for $a = 1$, we obtain Kantorovich operators K_n introduced in [38].

If we denote

$$G_{k,l}(f) := 2 \iint_S f\left(\frac{k+as}{n+a}, \frac{l+at}{n+a}\right) ds dt, \quad (70)$$

the bivariate Kantorovich operators can be written as

$$\mathbb{C}_n(f)(x, y) = \sum_{\substack{k, l=0, \dots, n \\ k+l \leq n}} G_{k,l}(f) p_{n,k,l}(x, y). \quad (71)$$

Proposition 9. *For bivariate Bernstein operators and bivariate Bernstein-Kantorovich operators, the following properties hold:*

(i) *If $f \in C^2(S)$, then*

$$|(B_n - \mathbb{C}_n)(f)(x, y)| \leq M_f \sigma(x, y) + \omega_1\left(f, \frac{4a}{3(n+a)}\right), \quad (72)$$

where M_f is the same as in Lemma 1 and

$$\sigma(x, y) = \frac{a^2}{9(n+a)^2}. \quad (73)$$

(ii) *If $f \in C(S)$, then*

$$|(B_n - \mathbb{C}_n)(f)(x, y)| \leq \eta_1 \omega_1\left(f, \frac{4a}{3(n+a)}\right) + \eta_2 \omega_2\left(f, \sqrt{\sigma(x, y)}\right). \quad (74)$$

Proof. As in the previous examples, taking Bernstein opera-

tors as

$$B_n(f)(x, y) = \sum_{\substack{k, l=0, \dots, n \\ k+l \leq n}} F_{k,l}(f) p_{n,k,l}(x, y), \quad (75)$$

we get

$$\begin{aligned} b_1^{F_{k,l}} &= \frac{k}{n}, b_2^{F_{k,l}} = \frac{l}{n}, \\ b_1^{G_{k,l}} &= \frac{3k+a}{3(n+a)}, b_2^{G_{k,l}} = \frac{3l+a}{3(n+a)}. \end{aligned} \quad (76)$$

Therefore, we easily obtain that

$$\begin{aligned} \mu_{2,0}^{F_{k,l}} &= 0, \mu_{2,0}^{G_{k,l}} = \frac{a^2}{18(n+a)^2}, \\ \mu_{0,2}^{F_{k,l}} &= 0, \mu_{0,2}^{G_{k,l}} = \frac{a^2}{18(n+a)^2}. \end{aligned} \quad (77)$$

Then

$$\sigma(x, y) = \sum_{\substack{k, l=0, \dots, n \\ k+l \leq n}} \frac{a^2}{9(n+a)^2} p_{n,k,l}(x, y) = \frac{a^2}{9(n+a)^2}. \quad (78)$$

Moreover, we have

$$\begin{aligned} \delta &= \max_{0 \leq k+l \leq n} \left\{ \left| b_1^{F_{k,l}} - b_1^{G_{k,l}} \right| + \left| b_2^{F_{k,l}} - b_2^{G_{k,l}} \right| \right\} \\ &= \max_{0 \leq k+l \leq n} \left\{ \frac{a|3k-n|}{3n(n+a)} + \frac{a|3l-n|}{3n(n+a)} \right\} \leq \frac{4a}{3(n+a)}. \end{aligned} \quad (79)$$

Then, the proof follows from Theorems 2 and 3.

6. A Beta Operator on $C(S)$

For $\rho \in (0, \infty)$, $f \in C(S)$, and $(x, y) \in S$, let us define

$$\mathcal{B}_\rho(f)(x, y) = \begin{cases} f(x, y), (x, y) \in \{(0, 0), (1, 0), (0, 1)\}, \\ \frac{\int_0^1 s^{\rho x-1} (1-s)^{\rho(1-x)-1} f(s, 0) ds}{B(\rho x, \rho(1-x))}, x \in (0, 1), y = 0, \\ \frac{\int_0^1 t^{\rho y-1} (1-t)^{\rho(1-y)-1} f(0, t) dt}{B(\rho y, \rho(1-y))}, x = 0, y \in (0, 1), \\ \frac{\int_0^1 u^{\rho x-1} (1-u)^{\rho(1-x)-1} f(u, 1-u) du}{B(\rho x, \rho(1-x))}, y = 1-x, x \in (0, 1), \\ \frac{\iint_S s^{\rho x-1} t^{\rho y-1} (1-s-t)^{\rho-\rho x-\rho y-1} f(s, t) ds dt}{\iint_S s^{\rho x-1} t^{\rho y-1} (1-s-t)^{\rho-\rho x-\rho y-1} ds dt}, (x, y) \in \text{int}(S). \end{cases} \quad (80)$$

For $\rho = n \in \mathbb{N}$, this is the bivariate version of the operator $\mathbb{B}_n$; see [39] and the references therein.

Theorem 10. $\mathcal{B}_\rho$ is a positive linear operator acting between $C(S)$ and $C(S)$. Moreover,

$$\mathcal{B}_\rho(1) = 1, \quad (81)$$

and if $\varphi_{i,j}(x, y) = pr_1^i pr_2^j(x, y)$, $(x, y) \in S$, $i \geq 0$, $j \geq 0$, integers, then

$$\mathcal{B}_\rho(\varphi_{i,j})(x, y) = \frac{\rho x(\rho x + 1) \cdots (\rho x + i - 1)}{\rho(\rho + 1) \cdots (\rho + i - 1)} \cdot \frac{\rho y(\rho y + 1) \cdots (\rho y + j - 1)}{\rho(\rho + 1) \cdots (\rho + j - 1)}. \quad (82)$$

Proof. It is easy to prove (81) and (82) by direct calculation. It remains to prove that if $f \in C(S)$, then, $\mathcal{B}_\rho(f) \in C(S)$. To do this, it suffices to verify that $\mathcal{B}_\rho(f)$ is continuous at each point of the boundary of S . Let us prove that if $0 < a < 1$ then

$$\lim_{\substack{(x,y) \rightarrow (a,0) \\ (x,y) \in \text{int}(S)}} \mathcal{B}_\rho(f)(x, y) = \mathcal{B}_\rho(f)(a, 0). \quad (83)$$

Let $V_{(a,0)} : C(S) \rightarrow \mathbb{R}$, $V_{(a,0)}(g) = \mathcal{B}_\rho(g)(a, 0)$. For $(x, y) \in \text{int}(S)$ define $U_{(x,y)} : C(S) \rightarrow \mathbb{R}$, $U_{(x,y)}(g) = \mathcal{B}_\rho(g)(x, y)$, $g \in C(S)$. Then, $U_{(x,y)}$ and $V_{(a,0)}$ are positive linear functionals of norm 1.

Let $\varepsilon > 0$. Then, there exists a polynomial function p on S such that $\|f - p\|_\infty \leq \varepsilon/4$. Using (82), it is easy to verify that

$$\lim_{\substack{(x,y) \rightarrow (a,0) \\ (x,y) \in \text{int}(S)}} U_{(x,y)}(p) = V_{(a,0)}(p). \quad (84)$$

Consequently, there exists $\delta > 0$ with

$$\left| U_{(x,y)}(p) - V_{(a,0)}(p) \right| \leq \frac{\varepsilon}{2}, \quad \|(x, y) - (a, 0)\|_1 \leq \delta. \quad (85)$$

So, if $\|(x, y) - (a, 0)\|_1 \leq \delta$, we have

$$\begin{aligned} \left| U_{(x,y)}(f) - V_{(a,0)}(f) \right| &\leq \left| U_{(x,y)}(f) - U_{(x,y)}(p) \right| \\ &\quad + \left| U_{(x,y)}(p) - V_{(a,0)}(p) \right| + \left| V_{(a,0)}(p) - V_{(a,0)}(f) \right| \\ &\leq \|f - p\|_\infty + \frac{\varepsilon}{2} + \|p - f\|_\infty \leq \varepsilon. \end{aligned} \quad (86)$$

This shows that

$$\lim_{\substack{(x,y) \rightarrow (a,0) \\ (x,y) \in \text{int}(S)}} U_{(x,y)}(f) = V_{(a,0)}(f), \quad f \in C(S), \quad (87)$$

and then (83) is proved.

The continuity of $\mathcal{B}_\rho(f)$ at the other boundary points can be proved similarly.

Proposition 11. For each $f \in C(S)$, one has

$$\lim_{\rho \rightarrow \infty} \mathcal{B}_\rho(f) = f. \quad (88)$$

Proof. Using Theorem 10, it is easy to verify that (88) is valid for the functions $1, pr_1, pr_2, pr_1^2 + pr_2^2$. But these functions form a Korovkin test system (see [36], p. 534, C.4.3.3), so that (88) holds for each $f \in C(S)$.

In what follows, we formulate a

Conjecture 12. If $f \in C(S)$ is convex and $(x, y) \in S$, then, the function $\rho \rightarrow \mathcal{B}_\rho(f)(x, y)$ is decreasing on $(0, \infty)$.

It is supported by the following facts.

- (i) The unidimensional version of the conjecture is valid: see [14, 40].
- (ii) $\mathcal{B}_\rho$ is a positive linear operator preserving the affine functions; this implies

$$\mathcal{B}_\rho(f) \geq f, \quad f \in C(S) \text{ convex}. \quad (89)$$

Now, (88) combined with (89) support the conjecture.

- (iii) The conjecture is valid for the functions $pr_1^k, pr_2^k, (1 - pr_1 - pr_2)^k, k \in \mathbb{N}$

In the sequel, we present two results under the hypothesis that the conjecture is true. To this end, let us introduce some notation.

Let $f \in C^2(S)$ and

$$\begin{aligned} m_1(f) &:= \min \left\{ f_{xx}(x, y) - |f_{xy}(x, y)| : (x, y) \in S \right\}, \\ m_2(f) &:= \min \left\{ f_{yy}(x, y) - |f_{xy}(x, y)| : (x, y) \in S \right\}, \\ M_1(f) &:= \max \left\{ f_{xx}(x, y) + |f_{xy}(x, y)| : (x, y) \in S \right\}, \\ M_2(f) &:= \max \left\{ f_{yy}(x, y) + |f_{xy}(x, y)| : (x, y) \in S \right\}. \end{aligned} \quad (90)$$

Then, the functions $\varphi(x, y) := f(x, y) - 1/2 m_1(f) x^2 - 1/2 m_2(f) y^2$ and $\psi(x, y) := 1/2 M_1(f) x^2 + 1/2 M_2(f) y^2 - f(x, y)$ are convex on S ; indeed, for each of them, the Hessian matrix is positive semidefinite.

Theorem 13. If $f \in C^2(S)$ and $\sigma > \rho > 0$, then

$$\begin{aligned} &\frac{\sigma - \rho}{2(\sigma + 1)(\rho + 1)} [m_1(f)x(1 - x) + m_2(f)y(1 - y)] \\ &\leq \mathcal{B}_\rho(f)(x, y) - \mathcal{B}_\sigma(f)(x, y) \\ &\leq \frac{\sigma - \rho}{2(\sigma + 1)(\rho + 1)} [M_1(f)x(1 - x) + M_2(f)y(1 - y)], \quad (x, y) \in S. \end{aligned} \quad (91)$$

Proof. Let $\sigma > \rho > 0$. If the Conjecture is true, we have $\mathcal{B}_\rho(\varphi) \geq \mathcal{B}_\sigma(\varphi)$ and $\mathcal{B}_\rho(\psi) \geq \mathcal{B}_\sigma(\psi)$. Thus

$$\begin{aligned} 0 \leq \mathcal{B}_\rho(\varphi)(x, y) - \mathcal{B}_\sigma(\varphi)(x, y) &= \mathcal{B}_\rho(f)(x, y) \\ &- \mathcal{B}_\sigma(f)(x, y) + \frac{x}{2} m_1(f) \frac{(\rho - \sigma)(1 - x)}{(\sigma + 1)(\rho + 1)} \\ &+ \frac{y}{2} m_2(f) \frac{(\rho - \sigma)(1 - y)}{(\sigma + 1)(\rho + 1)}. \end{aligned} \quad (92)$$

Consequently,

$$\begin{aligned} \mathcal{B}_\rho(f)(x, y) - \mathcal{B}_\sigma(f)(x, y) \\ \geq \frac{\sigma - \rho}{2(\sigma + 1)(\rho + 1)} [m_1(f)x(1 - x) + m_2(f)y(1 - y)]. \end{aligned} \quad (93)$$

Moreover,

$$\begin{aligned} 0 \leq \mathcal{B}_\rho(\psi)(x, y) - \mathcal{B}_\sigma(\psi)(x, y) &= \frac{x}{2} M_1(f) \frac{(\sigma - \rho)(1 - x)}{(\sigma + 1)(\rho + 1)} \\ &+ \frac{y}{2} M_2(f) \frac{(\sigma - \rho)(1 - y)}{(\sigma + 1)(\rho + 1)} - \mathcal{B}_\rho(f)(x, y) + \mathcal{B}_\sigma(f)(x, y). \end{aligned} \quad (94)$$

Now,

$$\begin{aligned} \mathcal{B}_\rho(f)(x, y) - \mathcal{B}_\sigma(f)(x, y) \\ \leq \frac{\sigma - \rho}{2(\sigma + 1)(\rho + 1)} [M_1(f)x(1 - x) \\ + M_2(f)y(1 - y)]. \end{aligned} \quad (95)$$

So, combining (93) and (95), we have proved the theorem.

Theorem 14. If $f \in C^2(S)$ and $\sigma > \rho > 0$, then

$$\begin{aligned} \frac{(n - 1)(\sigma - \rho)}{2(n\sigma + 1)(n\rho + 1)} [m_1(f)x(1 - x) \\ + m_2(f)y(1 - y)] \leq U_n^\rho(f)(x, y) - U_n^\sigma(f)(x, y) \\ \leq \frac{(n - 1)(\sigma - \rho)}{2(n\sigma + 1)(n\rho + 1)} [M_1(f)x(1 - x) \\ + M_2(f)y(1 - y)], \quad (x, y) \in S. \end{aligned} \quad (96)$$

Proof. It is easy to verify that $U_n^\rho = B_n \circ \mathcal{B}_{n\rho}$ and, if $u := pr_1^2$, then

$$\mathcal{B}_{n\rho}(u)(x, y) = \frac{(n - 1)\rho x^2 + (\rho + 1)x}{n\rho + 1}. \quad (97)$$

Using these facts and supposing that the conjecture is true, we have

$$U_n^\rho(\varphi) \geq U_n^\sigma(\varphi) \text{ and } U_n^\rho(\psi) \geq U_n^\sigma(\psi), \quad \sigma > \rho > 0. \quad (98)$$

Now,

$$\begin{aligned} 0 \leq U_n^\rho(\varphi)(x, y) - U_n^\sigma(\varphi)(x, y) &= U_n^\rho(f)(x, y) \\ &- U_n^\sigma(f)(x, y) - \frac{(n - 1)(\sigma - \rho)}{2(n\sigma + 1)(n\rho + 1)} x(1 - x)m_1(f) \\ &- \frac{(n - 1)(\sigma - \rho)}{2(n\sigma + 1)(n\rho + 1)} y(1 - y)m_2(f). \end{aligned} \quad (99)$$

Thus,

$$\begin{aligned} U_n^\rho(f)(x, y) - U_n^\sigma(f)(x, y) &\geq \frac{(n - 1)(\sigma - \rho)}{2(n\sigma + 1)(n\rho + 1)} [m_1(f)x(1 - x) \\ &+ m_2(f)y(1 - y)]. \end{aligned} \quad (100)$$

From $U_n^\rho(\psi) \geq U_n^\sigma(\psi)$, we get a similar upper bound for $U_n^\rho(f)(x, y) - U_n^\sigma(f)(x, y)$, which concludes the proof.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no competing financial interests.

Acknowledgments

This work was supported by a Hasso Plattner Excellence Research Grant (LBUS-HPI-ERG-2020-04), financed by the Knowledge Transfer Center of the Lucian Blaga University of Sibiu.

References

- [1] A. M. Acu and I. Rasa, "New estimates for the differences of positive linear operators," *Numerical Algorithms*, vol. 73, no. 3, pp. 775–789, 2016.
- [2] A. M. Acu, S. Hodis, and I. Rasa, "A survey on estimates for the differences of positive linear operators," *Constructive Mathematical Analysis*, vol. 1, no. 2, pp. 113–127, 2018.
- [3] A. M. Acu and I. Rasa, "Estimates for the differences of positive linear operators and their derivatives," *Numerical Algorithms*, vol. 85, no. 1, pp. 191–208, 2020.
- [4] A. Aral, D. Inoan, and I. Rasa, "On differences of linear positive operators," *Analysis and Mathematical Physics*, vol. 9, no. 3, pp. 1227–1239, 2019.
- [5] H. Gonska, P. Pitul, and I. Rasa, "On Peano's form of the Taylor remainder, Voronovskaja's theorem and the commutator of positive linear operators," in *Numerical Analysis and Approximation Theory (Proc. Int. Conf. Cluj-Napoca 2006)*, O. Agradini and P. Blaga, Eds., pp. 55–80, Cluj-Napoca, Casa Cartii de Stiintă, 2006.
- [6] H. Gonska, P. Pitul, and I. Raşa, "On differences of positive linear operators," *Carpathian Journal of Mathematics*, vol. 22, no. 1-2, pp. 65–78, 2006.
- [7] H. Gonska and I. Raşa, "Differences of positive linear operators and the second order modulus," *Carpathian Journal of Mathematics*, vol. 24, no. 3, pp. 332–340, 2008.

- [8] V. Gupta, D. Agrawal, and M. T. Rassias, "Quantitative estimates for differences of Baskakov-Type operators," *Complex Analysis and Operator Theory*, vol. 13, no. 8, pp. 4045–4064, 2019.
- [9] V. Gupta, "Estimate for the difference of operators having different basis functions," *Rendiconti del Circolo Matematico di Palermo Series 2*, vol. 69, no. 3, pp. 995–1003, 2020.
- [10] V. Gupta, A. M. Acu, and H. M. Srivastava, "Difference of some positive linear approximation operators for higher-order derivatives," *Symmetry*, vol. 12, no. 6, p. 915, 2020.
- [11] V. Gupta, "On difference of operators with applications to Szász type operators," *RACSAM*, vol. 113, no. 3, pp. 2059–2071, 2019.
- [12] V. Gupta and G. Tachev, "A note on the differences of two positive linear operators," *Constructive Mathematical Analysis*, vol. 2, no. 1, pp. 1–7, 2019.
- [13] V. Gupta, T. M. Rassias, P. N. Agrawal, and A. M. Acu, "Estimates for the differences of positive linear operators," in *Recent Advances in Constructive Approximation Theory*. Springer Optimization and Its Applications, vol 138, Springer, Cham, 2018.
- [14] I. Rasa and E. D. Stanila, "On some operators linking the Bernstein and the genuine Bernstein Durrmeyer operators," *Journal of Applied Functional Analysis*, vol. 9, no. 3-4, pp. 369–378, 2014.
- [15] H. Gonska and R. Paltanea, "Simultaneous approximation by a class of Bernstein-Durrmeyer operators preserving linear functions," *Czechoslovak Mathematical Journal*, vol. 60, no. 3, pp. 783–799, 2010.
- [16] H. Gonska and R. Paltanea, "Quantitative convergence theorems for a class of Bernstein-Durrmeyer operators preserving linear functions," *Ukrainian Mathematical Journal*, vol. 62, pp. 913–922, 2010.
- [17] J. M. Aldaz, O. Kounchev, and H. Render, "Shape preserving properties of generalized Bernstein operators on extended Chebyshev spaces," *Numerische Mathematik*, vol. 114, no. 1, pp. 1–25, 2009.
- [18] A. Lupas, *Die Folge der Betaoperatoren*, Dissertation, Universität Stuttgart, 1972.
- [19] G. Mühlbach, "Verallgemeinerungen der Bernstein- und der Lagrange Polynome, Bemerkungen zu einer Klasse linearer Polynomoperatoren von D. D. Stancu," *Revue Roumaine des Mathématiques Pures et Appliquées*, vol. 15, pp. 1235–1252, 1970.
- [20] G. Mühlbach, "Rekursionsformeln für die zentralen Momente der Pólya- und der Beta-Verteilung," *Metrika*, vol. 19, no. 1, pp. 171–177, 1972.
- [21] T. Acar, A. Aral, and H. Gonska, "On Szász–Mirakyan operators preserving e^{2ax} , $a > 0$," *Mediterranean Journal of Mathematics*, vol. 14, no. 1, p. 6, 2017.
- [22] A. M. Acu, A. Aral, and I. Rasa, "Generalized Bernstein Kantorovich operators: Voronovskaya type results, convergence in variation," *Carpathian J. Math.*, vol. 37, no. 3, 2021.
- [23] A. M. Acu, A. I. Maduta, and I. Rasa, "Voronovskaya type results and operators fixing two functions," *Mathematical Modelling and Analysis* (in press).
- [24] R. G. Bartle, *The Elements of Real Analysis*, 1964, John Wiley & Sons Ltd.
- [25] S. Dekel, D. Leviatan, and M. Sharir, "On bivariate smoothness spaces associated with nonlinear approximation," *Constructive Approximation*, vol. 20, no. 4, pp. 625–646, 2004.
- [26] H. Johnen and K. Scherer, "On the equivalence of the K-functional and the moduli of continuity and some applications," in *Constructive Theory of Functions of Several Variables. Lecture Notes in Math*, vol 571, pp. 119–140, Springer-Verlag, Berlin, 1976.
- [27] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, 1988.
- [28] F. Altomare, "Korovkin-type theorems and approximation by positive linear operators," *Surveys in Approximation Theory*, vol. 5, pp. 92–164, 2010.
- [29] M.-M. Derriennic, "On multivariate approximation by Bernstein-type polynomials," *Journal of Approximation Theory*, vol. 45, no. 2, pp. 155–166, 1985.
- [30] W. Chen, "On the modified Durrmeyer–Bernstein operator," in *Report of the Fifth Chinese Conference on Approximation Theory (in Chinese, Handwritten)*, Zhen Zhou, China, 1987.
- [31] T. N. T. Goodman and A. Sharma, "A modified Bernstein–Schoenberg operator," in *Proceedings of the Conference on Constructive Theory of Functions*, pp. 166–173, Publ. House Bulg. Acad. of Sci., Sofia, 1988.
- [32] H. Gonska, D. Kacsó, and I. Rasa, "The genuine Bernstein–Durrmeyer operators revisited," *Results in Mathematics*, vol. 62, no. 3-4, pp. 295–310, 2012.
- [33] T. N. T. Goodman and A. Sharma, "A Bernstein type operator on the simplex," *Mathematica Balkanica*, vol. 5, pp. 129–145, 1991.
- [34] T. Sauer, "The genuine Bernstein–Durrmeyer operator on a simplex," *Results in Mathematics*, vol. 26, no. 1-2, pp. 99–130, 1994.
- [35] R. Paltanea, "A class of Durrmeyer type operators preserving linear functions," *Annals of the Tiberiu Popoviciu Seminar of Functional Equations, Approximation and Convexity*, vol. 5, pp. 109–117, 2007.
- [36] F. Altomare and M. Campiti, "Korovkin-type approximation theory and its applications," vol. 17, Series: De Gruyter Studies in Mathematics, 1994.
- [37] F. Altomare, M. Cappelletti Montano, V. Leonessa, and I. Rasa, "A generalization of Kantorovich operators for convex compact subsets," *Banach Journal of Mathematical Analysis*, vol. 11, no. 3, pp. 591–614, 2017.
- [38] D.-X. Zhou, "Converse theorems for multidimensional Kantorovich operators," *Analysis Mathematica*, vol. 19, no. 1, pp. 85–100, 1993.
- [39] H. Gonska, D. Kacsó, and I. Rasa, "On the composition and decomposition of positive linear operators (VI)," in *Constructive Theory of Functions, Sozopol 2016*, K. Ivanov, G. Nikolov, and R. Uluchev, Eds., pp. 1–15, Prof. Marin Drinov Academic Publishing House, Sofia, 2018.
- [40] J. A. Adell, F. G. Badia, J. de la Cal, and F. Plo, "On the property of monotonic convergence for beta operators," *Journal of Approximation Theory*, vol. 84, no. 1, pp. 61–73, 1996.