Research Article

Applications of a New q -Difference Operator in Janowski-Type Meromorphic Convex Functions

Bakhtiar Ahmad ¹, Muhammad Ghaffar Khan ², Mohamed Kamal Aouf,³
Wali Khan Mashwani ², Zabidin Salleh ⁴, and Huo Tang⁵

¹Govt. Degree College Mardan, 23200 Mardan, Pakistan

²Institute of Numerical Sciences, Kohat University of Science and Technology, Kohat, Pakistan

³Department of Mathematics, Faculty of Science, Mansoura 35516, Egypt

⁴Department of Mathematics, Faculty of Ocean Engineering Technology and Informatics, Universiti Malaysia Terengganu, 21030 Kuala Nerus, Terengganu, Malaysia

⁵School of Mathematics and Computer Sciences, Chifeng University, Chifeng 624000, China

Correspondence should be addressed to Zabidin Salleh; zabidin@umt.edu.my

Received 20 February 2021; Accepted 31 March 2021; Published 15 April 2021

Academic Editor: Sivasubramanian Srikandan

Copyright © 2021 Bakhtiar Ahmad et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The main aim of the present article is the introduction of a new differential operator in q -analogue for meromorphic multivalent functions which are analytic in punctured open unit disc. A subclass of meromorphic multivalent convex functions is defined using this new differential operator in q -analogue. Furthermore, we discuss a number of useful geometric properties for the functions belonging to this class such as sufficiency criteria, coefficient estimates, distortion theorem, growth theorem, radius of starlikeness, and radius of convexity. Also, algebraic property of closure is discussed of functions belonging to this class. Integral representation problem is also proved for these functions.

1. Introduction and Definitions

Let $\mathfrak{A}_p$ denote the family of all meromorphic p -valent functions f that are analytic in the punctured disc $\mathbb{D} = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and obeying the normalization

$$f(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \quad (1)$$

Also, let $\mathcal{MC}_p(\alpha)$ denote the well-known family of meromorphic p -valent convex functions of order α ($0 \leq \alpha < p$) and defined as

$$f(z) \in \mathcal{MC}_p(\alpha) \Leftrightarrow \operatorname{Re} \frac{(zf'(z))'}{f'(z)} < -\alpha. \quad (2)$$

For $0 < q < 1$, the q -difference operator or q -derivative of

a function f is defined by

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \quad z \neq 0, q \neq 1. \quad (3)$$

It can easily be seen that for $n \in \mathbb{N}$, where $\mathbb{N}$ stands for the set of natural numbers and $z \in \mathbb{D}$,

$$\partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n, q] a_n z^{n-1}, \quad (4)$$

where

$$[n, q] = \frac{1 - q^n}{1 - q} = 1 + \sum_{l=1}^n q^l, \quad (5)$$

$$[0, q] = 0.$$

For any nonnegative integer n , the q -number shift factorial is defined by

$$[n, q]! = \begin{cases} 1, & n = 0, \\ [1, q][2, q][3, q] \cdots [n, q], & n \in \mathbb{N}. \end{cases} \quad (6)$$

Also, the q -generalized Pochhammer symbol for $x \in \mathbb{R}$ is given by

$$[x, q]_n = \begin{cases} 1, & n = 0, \\ [x, q][x+1, q] \cdots [x+n-1, q], & n \in \mathbb{N}. \end{cases} \quad (7)$$

In (3), if $q \rightarrow 1^-$, then this operator becomes the conventional derivative in the classical calculus, so the limits can be generalized by introducing the parameter q , with condition $0 < q < 1$, and all such concepts, which have been developed thus, are known as quantum calculus (q -calculus). Many physical phenomena are better explained using this generalized operator, and as a result, this field attracted a lot of the researchers due to its various applications in the branches of mathematics and physics (see details in [1, 2]). Jackson [3, 4] was the pioneer of this field, who gave some applications of q -calculus and introduced the q -analogues of derivative and integral. Aral and Gupta [1, 2, 5] defined an operator, which is known as q -Baskakov Durrmeyer operator by using q -beta functions. The generalization of complex operators known as q -Picard and q -Gauss-Weierstrass singular integral operators was discussed by Aral and Anastassiou in [6–8]. Later, Kanas and Răducanu [9] introduced the q -analogue of a Ruscheweyh differential operator and studied its various properties. More applications of this operator can be seen in the paper [10]. Huda and Darus [11] utilized the q -analogue of a Liu-Srivastava operator and defined an integral operator. In somewhat similar way, Mohammed and Darus [12] introduced a generalized operator along with investigating a class of functions relating to q -hypergeometric functions. Later, Seoudy [13] estimated coefficient bounds for some q -starlike and q -convex functions of complex order. Recently, Arif and Ahmad defined a new q -differential operator for meromorphic multivalent functions and investigated classes related to q -meromorphic starlike and convex functions in their articles [14, 15].

In this article, we introduce a new q -differential operator for meromorphic functions and use this operator to define and study some properties of a new family of meromorphic multivalent functions associated with circular domain.

We now define the differential operator $\mathcal{D}_{\mu, q} : \mathfrak{A}_p \rightarrow \mathfrak{A}_p$ by

$$\mathcal{D}_{\mu, q} f(z) = (1 + [p, q]\mu)f(z) + \mu q^p z \partial_q f(z), \quad (8)$$

where $\mu \geq 0$.

Using (1), we can easily obtain

$$\mathcal{D}_{\mu, q} f(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} (1 + [p, q]\mu + \mu q^p [n, q]) a_n z^n. \quad (9)$$

We take

$$\begin{aligned} \mathcal{D}_{\mu, q}^0 f(z) &= f(z), \\ \mathcal{D}_{\mu, q}^2 f(z) &= \mathcal{D}_{\mu, q}(\mathcal{D}_{\mu, q} f(z)) = \frac{1}{z^p} \\ &+ \sum_{n=p+1}^{\infty} (1 + [p, q]\mu + \mu q^p [n, q])^2 a_n z^n. \end{aligned} \quad (10)$$

In a similar way, for $m \in \mathbb{N} \cup \{0\}$, we get

$$\mathcal{D}_{\mu, q}^m f(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} (1 + [p, q]\mu + \mu q^p [n, q])^m a_n z^n. \quad (11)$$

From (8) and (11), we get the following identity:

$$\mathcal{D}_{\mu, q}^{m+1} f(z) = \mu q^p z \partial_q \mathcal{D}_{\mu, q}^m f(z) + (1 + [p, q]\mu) \mathcal{D}_{\mu, q}^m f(z). \quad (12)$$

We now define a subfamily $\mathcal{ME}_{\mu, q}(p, m, A, B)$ of $\mathfrak{A}_p$ by using the operator $\mathcal{D}_{\mu, q}^m$ as follows.

Definition 1. For $-1 \leq B < A \leq 1$ and $0 < q < 1$, we define $f \in \mathfrak{A}_p$ to be in the class $\mathcal{ME}_{\mu, q}(p, m, A, B)$, if it satisfies

$$\frac{-q^p \partial_q \left(z \partial_q \mathcal{D}_{\mu, q}^m f(z) \right)}{[p, q] \partial_q \mathcal{D}_{\mu, q}^m f(z)} < \frac{1 + Az}{1 + Bz}. \quad (13)$$

Here, the relation symbol “ $<$ ” is used for the subordinations.

We see that for particular values of p, m, A, B, μ , and q , we get some of the well-known classes few of which are listed below:

- (1) For $m=0$ and $q \rightarrow 1^-$, we get the class of meromorphic multivalent convex functions associated with Janowski functions denoted by $\mathcal{ME}_p^*[A, B]$
- (2) For $A=1, B=-1$, and $m=0$, we get $\mathcal{ME}_{p, q}^*$, the class of meromorphic multivalent convex functions in q -analogue
- (3) For $A=1, B=-1, m=0$, and $q \rightarrow 1^-$, we get the class of meromorphic multivalent convex functions denoted by $\mathcal{ME}_p^*$
- (4) For $A=1, B=-1, m=0, p=1$, and $q \rightarrow 1^-$, we get $\mathcal{ME}^*$, the class of meromorphic convex functions

It can easily be verified that a function $f \in \mathfrak{A}_p$ will be in the class $\mathcal{MC}_{\mu,q}(p, m, A, B)$, if and only if

$$\left| \frac{\left(q^p \partial_q \left(z \partial_q \mathcal{D}_{\mu,q}^m f(z) \right) \right) / \left([p, q] \partial_q \mathcal{D}_{\mu,q}^m f(z) \right) + 1}{A + B \left(\left(q^p \partial_q \left(z \partial_q \mathcal{D}_{\mu,q}^m f(z) \right) \right) q^p \partial_q \left(z \partial_q \mathcal{D}_{\mu,q}^m f(z) \right) / \left([p, q] \partial_q \mathcal{D}_{\mu,q}^m f(z) \right) [p, q] \partial_q \mathcal{D}_{\mu,q}^m f(z) \right)} \right| < 1. \quad (14)$$

The following lemma is used in our main results.

Lemma 2 (see [16]). *Let $h(z)$ be analytic in $\mathbb{D}$ and have the form*

$$h(z) = 1 + \sum_{n=1}^{\infty} d_n z^n, \quad (15)$$

and $k(z)$ is analytic and convex in $\mathbb{D}$ with series representation

$$k(z) = 1 + \sum_{n=1}^{\infty} k_n z^n. \quad (16)$$

So if $h(z) < k(z)$, then $|d_n| \leq |k_1|$, for $n \in \mathbb{N} = \{1, 2, \dots\}$.

2. Main Results and Their Consequences

In this section, we start with sufficiency criteria for this newly defined class and then, we give the coefficient estimates for the functions belonging to this class. The following lemma is proved which will be used in this section.

Lemma 3. *Suppose that the sequence $\{A_{p+n}\}_{n=1}^{\infty}$ is defined by*

$$\begin{cases} A_{p+n} = \frac{[p, q](A-B)\psi(0)}{\phi(1)\psi(1)} & (n=1), \\ A_{p+n} = \frac{[p, q](A-B)}{\phi(n)\psi(n)} \left(\sum_{k=0}^{n-1} \psi(k)A_{p+k} \right) & (n \geq 2). \end{cases} \quad (17)$$

Then,

$$A_{p+n} = \frac{\psi(0)}{\psi(n)} \cdot \frac{[p, q](A-B)}{\phi(1)} \prod_{k=1}^{n-1} \frac{[p, q](A-B) + \phi(k)}{\phi(k+1)} \quad (n \geq 2). \quad (18)$$

Proof. From (17), we have

$$\phi(n)\psi(n)A_{p+n} = [p, q](A-B) \left(\sum_{k=0}^{n-1} \psi(k)A_k \right). \quad (19)$$

Thus, we obtain that

$$\begin{aligned} \phi(n+1)\psi(n+1)A_{p+n+1} &= [p, q](A-B) \left(\sum_{k=0}^n \psi(k)A_k \right) \\ &= [p, q](A-B)\psi(n)A_{p+n} + [p, q](A-B) \sum_{k=0}^{n-1} \psi(k)A_k \\ &= [p, q](A-B)\psi(n)A_{p+n} + \phi(n)\psi(n)A_{p+n} \\ &= ([p, q](A-B) + \phi(n))\psi(n)A_{p+n}. \end{aligned} \quad (20)$$

From (20), we find that

$$\frac{A_{p+n+1}}{A_{p+n}} = \frac{([p, q](A-B) + \phi(n))\psi(n)}{\phi(n+1)\psi(n+1)} \quad (n \geq 1). \quad (21)$$

Thus,

$$\begin{aligned} A_{p+n} &= \frac{A_{p+n}}{A_{p+n-1}} \cdot \frac{A_{p+n-1}}{A_{p+n-2}} \cdots \frac{A_{p+2}}{A_{p+1}} \\ &\cdot A_{p+1} = \frac{([p, q](A-B) + \phi(n-1))\psi(n-1)}{\phi(n)\psi(n)} \\ &\cdots \frac{([p, q](A-B) + \phi(1))\psi(1)}{\phi(2)\psi(2)} \cdot \frac{[p, q](A-B)\psi(0)}{\phi(1)\psi(1)} \\ &= \frac{\psi(0)}{\psi(n)} \cdot \frac{[p, q](A-B)}{\phi(1)} \prod_{k=1}^{n-1} \frac{[p, q](A-B) + \phi(k)}{\phi(k+1)} \quad (n \geq 2). \end{aligned} \quad (22)$$

In conjunction with (17), we complete the proof of Lemma 3.

Theorem 4. *If $f \in \mathfrak{A}_p$ is of the form (1), then it will be in the class $\mathcal{MC}_{\mu,q}(p, m, A, B)$ if and only if the inequality*

$$\begin{aligned} &\sum_{n=p+1}^{\infty} (q^p [n, q]^2 (1+B) + (1+A) [n, q] [p, q]) \\ &\cdot (1 + [p, q]\mu + \mu q^p [n, q]^m |a_n|) \leq \frac{[p, q]^2 (A-B)}{q^p} \end{aligned} \quad (23)$$

is satisfied.

Proof. For $f \in \mathcal{MC}_{\mu,q}(p, m, A, B)$, we need to prove the inequality (14). For this, consider

$$\begin{aligned} H &:= \left| \frac{\left(q^p \partial_q \left(z \partial_q \mathcal{D}_{\mu,q}^m f(z) \right) \right) / \left([p, q] \partial_q \mathcal{D}_{\mu,q}^m f(z) \right) + 1}{A + B \left(\left(q^p \partial_q \left(z \partial_q \mathcal{D}_{\mu,q}^m f(z) \right) \right) q^p \partial_q \left(z \partial_q \mathcal{D}_{\mu,q}^m f(z) \right) / \left([p, q] \partial_q \mathcal{D}_{\mu,q}^m f(z) \right) [p, q] \partial_q \mathcal{D}_{\mu,q}^m f(z) \right)} \right| \\ &= \left| \frac{q^p \partial_q \left(z \partial_q \mathcal{D}_{\mu,q}^m f(z) \right) + [p, q] \partial_q \mathcal{D}_{\mu,q}^m f(z)}{A [p, q] \partial_q \mathcal{D}_{\mu,q}^m f(z) + B q^p \partial_q \left(z \partial_q \mathcal{D}_{\mu,q}^m f(z) \right)} \right|. \end{aligned} \quad (24)$$

By using (8) and with the help of (3) and (11),

$$\begin{aligned} H &:= \left| \frac{\sum_{n=p+1}^{\infty} (1 + [p, q] \mu + \mu q^p [n, q])^m (q^p [n, q]^2 + [p, q] [n, q]) a_n z^{n-1}}{-((A - B)[p, q]^2)(A - B)[p, q]^2 / q^p z^{p+1} + \sum_{n=p+1}^{\infty} (1 + [p, q] \mu + \mu q^p [n, q])^m (A[p, q][n, q] + B q^p [n, q]^2) a_n z^{n-1}} \right| \\ &= \left| \frac{\sum_{n=p+1}^{\infty} (1 + [p, q] \mu + \mu q^p [n, q])^m (q^p [n, q]^2 + [p, q] [n, q]) a_n z^{n+p}}{-((A - B)[p, q]^2)(A - B)[p, q]^2 / q^p + \sum_{n=p+1}^{\infty} (1 + [p, q] \mu + \mu q^p [n, q])^m (A[p, q][n, q] + B q^p [n, q]^2) a_n z^{n+p}} \right| \\ &\leq \frac{\sum_{n=p+1}^{\infty} (1 + [p, q] \mu + \mu q^p [n, q])^m (q^p [n, q]^2 + [p, q] [n, q]) |a_n|}{((A - B)[p, q]^2)(A - B)[p, q]^2 / q^p - \sum_{n=p+1}^{\infty} (1 + [p, q] \mu + \mu q^p [n, q])^m (A[p, q][n, q] + B q^p [n, q]^2) |a_n|}. \end{aligned} \quad (25)$$

Now, if we use the inequality (23), then

$$H < 1, \quad (26)$$

Conversely, let $f \in \mathcal{MC}_{\mu,q}(p, m, A, B)$ and be of the form (1); then, from (14), we have for $z \in \mathbb{D}$,

and this completes the direct part of the proof.

$$\begin{aligned} &\left| \frac{\left(q^p \partial_q \left(z \partial_q \mathcal{D}_{\mu,q}^m f(z) \right) \right) / \left([p, q] \partial_q \mathcal{D}_{\mu,q}^m f(z) \right) + 1}{A + B \left(\left(q^p \partial_q \left(z \partial_q \mathcal{D}_{\mu,q}^m f(z) \right) \right) q^p \partial_q \left(z \partial_q \mathcal{D}_{\mu,q}^m f(z) \right) / \left([p, q] \partial_q \mathcal{D}_{\mu,q}^m f(z) \right) [p, q] \partial_q \mathcal{D}_{\mu,q}^m f(z) \right)} \right| \\ &= \left| \frac{\sum_{n=p+1}^{\infty} (1 + [p, q] \mu + \mu q^p [n, q])^m (q^p [n, q]^2 + [p, q] [n, q]) a_n z^{n+p}}{-((A - B)[p, q]^2)(A - B)[p, q]^2 / q^p + \sum_{n=p+1}^{\infty} (1 + [p, q] \mu + \mu q^p [n, q])^m (A[p, q][n, q] + B q^p [n, q]^2) a_n z^{n+p}} \right|. \end{aligned} \quad (27)$$

Since $|\Re z| \leq |z|$, we have

$$\Re \left\{ \frac{\sum_{n=p+1}^{\infty} (1 + [p, q] \mu + \mu q^p [n, q])^m (q^p [n, q]^2 + [p, q] [n, q]) a_n z^{n+p}}{((A - B)[p, q]^2)(A - B)[p, q]^2 / q^p + \sum_{n=p+1}^{\infty} (1 + [p, q] \mu + \mu q^p [n, q])^m (A[p, q][n, q] + B q^p [n, q]^2) a_n z^{n+p}} \right\} < 1. \quad (28)$$

Now if the values of z are chosen on the real axis, then $(q^p \partial_q (z \partial_q \mathcal{D}_{\mu,q}^m f(z))) / ([p, q] \partial_q \mathcal{D}_{\mu,q}^m f(z))$ is real. Using some calculations in the inequality (28) and letting $z \rightarrow 1^-$ through real values, we finally get (23).

Theorem 5. If $f \in \mathcal{MC}_{\mu,q}(p, m, A, B)$ and is of the form (1), then

$$|a_{p+1}| \leq \frac{p, q](A - B)\psi(0)}{\phi(1)\psi(1)}, \quad (29)$$

$$|a_{p+n}| \leq \frac{\psi(0)}{\psi(n)} \cdot \frac{p, q](A - B)}{\phi(1)} \prod_{k=1}^{n-1} \frac{[p, q](A - B) + \phi(k)}{\phi(k+1)} \quad (n \geq 2), \quad (30)$$

where

$$\phi(n) := q^p[p + n, q]^2 + [p, q][p + n, q], \quad (31)$$

$$\psi(n) := (1 + [p, q]\mu + \mu q^p[p, q][p + n, q])^m. \quad (32)$$

Proof. If $f \in \mathfrak{A}_p$ is in the class $\mathcal{MC}_{\mu,q}(p, m, A, B)$, then it satisfies

$$\frac{-q^p \partial_q (z \partial_q \mathcal{D}_{\mu,q}^m f(z))}{[p, q] \partial_q \mathcal{D}_{\mu,q}^m f(z)} < \frac{1 + Az}{1 + Bz}. \quad (33)$$

Now, let

$$h(z) = \frac{-q^p \partial_q (z \partial_q \mathcal{D}_{\mu,q}^m f(z))}{[p, q] \partial_q \mathcal{D}_{\mu,q}^m f(z)}. \quad (34)$$

Since

$$\operatorname{Re} h(z) > 0, \quad (35)$$

so $h(z)$ is in the class P with its representation which is given by

$$h(z) = 1 + \sum_{n=1}^{\infty} d_n z^n. \quad (36)$$

Now,

$$h(z) < \frac{1 + Az}{1 + Bz}. \quad (37)$$

But

$$\frac{1 + Az}{1 + Bz} = 1 + (A - B)z + \dots \quad (38)$$

Now, using Lemma 2, we get

$$|d_n| \leq (A - B), \quad (39)$$

now putting the series expansions of $h(z)$ and $f(z)$ in (34), simplifying and comparing the coefficients of z^{n+p} on both sides

$$\begin{aligned} & -q^p(1 + [p, q]\mu + \mu q^p[n + p, q])^m [p + n, q]^2 a_{p+n} \\ &= [p, q][p + n, q](1 + [p, q]\mu + \mu q^p[n + p, q])^m a_{p+n} \\ &+ [p, q] \sum_{i=0}^{n-1} (1 + [p, q]\mu + \mu q^p[p + i, q])^m [p + i, q] a_{p+i} d_{n-i}, \end{aligned} \quad (40)$$

which implies that

$$\begin{aligned} & -(1 + [p, q]\mu + \mu q^p[n + p, q])^m \\ & \cdot (q^p[p + n, q]^2 + [p, q][p + n, q]) a_{p+n} \\ &= [p, q] \sum_{i=0}^{n-1} (1 + [p, q]\mu + \mu q^p[p + i, q])^m [p + i, q] a_{p+i} d_{n-i}. \end{aligned} \quad (41)$$

Now, by taking absolute on both sides with using the triangle inequality and using (39), we obtain

$$\begin{aligned} & (1 + [p, q]\mu + \mu q^p[p + n, q])^m \\ & \cdot (q^p[p + n, q]^2 + [p, q][p + n, q]) |a_{p+n}| \\ & \leq [p, q](A - B) \sum_{i=0}^{n-1} (1 + [p, q]\mu + \mu q^p[p + i, q])^m |a_{p+i}|. \end{aligned} \quad (42)$$

Using the notation (31) and (32) implies that

$$|a_{p+1}| \leq \frac{p, q](A - B)\psi(0)}{\phi(1)\psi(1)}, \quad (43)$$

$$|a_{p+n}| \leq \frac{p, q](A - B)}{\phi(n)\psi(n)} \left(\sum_{k=0}^{n-1} \psi(k) |a_{p+k}| \right) \quad (n \geq 2). \quad (44)$$

Now, we define the sequence $\{A_{p+n}\}_{n=1}^{\infty}$ as follows:

$$\begin{cases} A_{p+1} = \frac{[p, q](A - B)\psi(0)}{\phi(1)\psi(1)} & (n = 1), \\ A_{p+n} = \frac{[p, q](A - B)}{\phi(n)\psi(n)} \left(\sum_{k=0}^{n-1} \psi(k) A_{p+k} \right) & (n \geq 2). \end{cases} \quad (45)$$

In order to prove that

$$|a_{p+n}| \leq A_{p+n} \quad (n \geq 2), \quad (46)$$

we use the principle of mathematical induction. It is easy to verify that

$$|a_{p+1}| \leq A_{p+1} = \frac{[p, q](A - B)\psi(0)}{\phi(1)\psi(1)}. \quad (47)$$

Thus, assuming that

$$|a_{p+j}| \leq A_{p+j} \quad (j = 2, 3, \dots, n), \quad (48)$$

we find from (44) and (48) that

$$\begin{aligned} |a_{p+n+1}| &\leq \frac{[p, q](A-B)}{\phi(n)\psi(n)} \left(\sum_{k=0}^n \psi(k) |a_{p+k}| \right) \\ &\leq \frac{[p, q](A-B)}{\phi(n)\psi(n)} \left(\sum_{k=0}^n \psi(k) A_{p+k} \right) = A_{p+n+1}. \end{aligned} \quad (49)$$

Therefore, by the principle of mathematical induction, we have

$$|a_{p+n}| \leq A_{p+n} \quad (n \geq 2). \quad (50)$$

By means of Lemma 3 and (45), we know that

$$A_{p+n} = \frac{\psi(0)}{\psi(n)} \cdot \frac{[p, q](A-B)}{\phi(1)} \prod_{k=1}^{n-1} \frac{[p, q](A-B) + \phi(k)}{\phi(k+1)} \quad (n \geq 2). \quad (51)$$

Combining (50) and (51), we readily get the coefficient estimates (30).

3. Closure Theorems

Let the functions $f_k(z)$, $(k = 1, 2, 3, \dots, l)$ be defined by

$$f_k(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} a_{n,k} z^n \quad (z \in \mathbb{D}, a_{n,k} \geq 0). \quad (52)$$

Theorem 6. Let the functions $f_k(z)$ ($k = 1, 2, 3, \dots, l$) defined by (52) be in the class $\mathcal{MC}_{\mu,q}(p, m, A, B)$. Then, the function $F \in \mathcal{MC}_{\mu,q}(p, m, A, B)$, where

$$F(z) = \sum_{k=1}^l \lambda_k f_k(z) \left(\lambda_k \geq 0, \sum_{k=1}^l \lambda_k = 1 \right). \quad (53)$$

Proof. From (53), we have

$$F(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} \left(\sum_{k=1}^l \lambda_k a_{n,k} \right) z^n. \quad (54)$$

By Theorem 4, we have

$$\begin{aligned} &\sum_{n=p+1}^{\infty} (q^p[n, q]^2(1+B) + (1+A)[n, q][p, q]) \\ &\quad \cdot (1 + [p, q]\mu + \mu q^p[n, q])^m \left(\sum_{k=1}^l \lambda_k a_{n,k} \right) \\ &= \sum_{k=1}^l \lambda_k \left(\sum_{n=p+1}^{\infty} (q^p[n, q]^2(1+B) + (1+A)[n, q][p, q]) \right. \\ &\quad \cdot (1 + [p, q]\mu + \mu q^p[n, q])^m a_{n,k} \\ &\leq \sum_{k=1}^l \lambda_k \left(\frac{[p, q]^2(A-B)}{q^p} \right) = \frac{[p, q]^2(A-B)}{q^p}. \end{aligned} \quad (55)$$

Hence, by Theorem 4, $F \in \mathcal{MC}_{\mu,q}(p, m, A, B)$.

Theorem 7. The class $\mathcal{MC}_{\mu,q}(p, m, A, B)$ is closed under convex combination.

Proof. Let the function $f_k(z)$ ($k = 1, 2$) given by (52) be in the class $\mathcal{MC}_{\mu,q}(p, m, A, B)$. It is enough to show that

$$h(z) = \alpha f_1(z) + (1-\alpha)f_2(z), \quad 0 \leq \alpha \leq 1, \quad (56)$$

is in the class $\mathcal{MC}_{\mu,q}(p, m, A, B)$. Since for $0 \leq \alpha \leq 1$,

$$h(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} [\alpha a_{n,1} + (1-\alpha)a_{n,2}] z^n, \quad (57)$$

by Theorem 4, we have

$$\begin{aligned} &\sum_{n=p+1}^{\infty} (q^p[n, q]^2(1+B) + (1+A)[n, q][p, q]) \\ &\quad \cdot (1 + [p, q]\mu \alpha a_{n,1} + (1-\alpha)a_{n,2}) \\ &= \alpha \sum_{n=p+1}^{\infty} (q^p[n, q]^2(1+B) + (1+A)[n, q][p, q]) \\ &\quad \cdot (1 + [p, q]\mu + \mu q^p[n, q])^m a_{n,1} + (1-\alpha) \sum_{n=p+1}^{\infty} \\ &\quad \cdot (q^p[n, q]^2(1+B) + (1+A)[n, q][p, q]) \\ &\quad \cdot (1 + [p, q]\mu + \mu q^p[n, q])^m a_{n,2} \\ &\leq \alpha \frac{[p, q]^2(A-B)}{q^p} + (1-\alpha) \frac{[p, q]^2(A-B)}{q^p} = \frac{[p, q]^2(A-B)}{q^p}. \end{aligned} \quad (58)$$

Hence, by Theorem 4, $h(z) \in \mathcal{MC}_{\mu,q}(p, m, A, B)$.

Theorem 8. Let the function $f_k(z)$ ($k = 1, 2$) given by (52) belong to $\mathcal{MC}_{\mu,q}(p, m, A, B)$; then, their weighted mean $h_j(z)$ is also in the class $\mathcal{MC}_{\mu,q}(p, m, A, B)$, where $h_j(z)$ is defined by

$$h_j(z) = \left\{ \frac{(1-j)f_1(z) + (1+j)f_2(z)}{2} \right\}. \quad (59)$$

Proof. From (59), one can easily write

$$h_j(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} \left[\frac{(1-j)a_{n,1} + (1+j)a_{n,2}}{2} \right] z^n. \quad (60)$$

To prove $h_j(z) \in \mathcal{MC}_{\mu,q}(p, m, A, B)$, we consider

$$\begin{aligned} & \sum_{n=p+1}^{\infty} (q^p[n, q]^2(1+B) + (1+A)(n, q)[p, q]) \\ & \cdot (1 + [p, q]\mu + \mu q^p[n, q])^m \left[\frac{(1-j)a_{n,1} + (1+j)a_{n,2}}{2} \right] \\ &= \frac{(1-j)}{2} \sum_{n=p+1}^{\infty} (q^p[n, q]^2(1+B) + (1+A)[n, q][p, q]) \\ & \cdot (1 + [p, q]\mu + \mu q^p[n, q])^m a_{n,1} \\ &+ \frac{(1+j)}{2} \sum_{n=p+1}^{\infty} (q^p[n, q]^2(1+B) + (1+A)[n, q][p, q]) \\ & \cdot (1 + [p, q]\mu + \mu q^p[n, q])^m a_{n,2} \leq \frac{(1-j)}{2} \frac{[p, q]^2(A-B)}{q^p} \\ &+ \frac{(1+j)}{2} \frac{[p, q]^2(A-B)}{q^p} = \frac{[p, q]^2(A-B)}{q^p}. \end{aligned} \quad (61)$$

Hence, by Theorem 4, $h_j(z) \in \mathcal{MC}_{\mu,q}(p, m, A, B)$.

4. Distortion Theorem

In the next two results, we shall discuss the growth and distortion theorems for our newly defined class of functions.

Theorem 9. *If f is in the class $\mathcal{MC}_{\mu,q}(p, m, A, B)$ and has the form (1), then for $|z| = r$, we have*

$$\frac{1}{r^p} - \vartheta_1 r^p \leq |f(z)| \leq \frac{1}{r^p} + \vartheta_1 r^p, \quad (62)$$

where

$$\vartheta_1 = \frac{[p, q]^2(A-B)}{q^p(q^p[p+1, q]^2(1+B) + (1+A)[p+1, q][p, q])(1 + [p, q]\mu + \mu q^p[p+1, q])^m}. \quad (63)$$

Proof. As

$$|f(z)| = \left| \frac{1}{z^p} + \sum_{n=p+1}^{\infty} a_n z^n \right| \leq \frac{1}{|z|^p} + \sum_{n=p+1}^{\infty} |a_n| |z|^n, \quad (64)$$

for $|z| = r < 1$, we have $r^n < r^p$ for $n \geq p+1$ and

$$|f(z)| \leq \frac{1}{r^p} + r^p \sum_{n=p+1}^{\infty} |a_n|. \quad (65)$$

Similarly,

$$|f(z)| \geq \frac{1}{r^p} - r^p \sum_{n=p+1}^{\infty} |a_n|. \quad (66)$$

Now, if $f \in \mathcal{MC}_{\mu,q}(p, m, A, B)$, then by (23),

$$\begin{aligned} & \sum_{n=p+1}^{\infty} (q^p[n, q]^2(1+B) + (1+A)[n, q][p, q]) \\ & \cdot (1 + [p, q]\mu + \mu q^p[n, q])^m |a_n| \leq \frac{[p, q]^2(A-B)}{q^p}. \end{aligned} \quad (67)$$

But we know that

$$\begin{aligned} & (q^p[p+1, q]^2(1+B) + (1+A)[p+1, q][p, q]) \\ & \cdot (1 + [p, q]\mu + \mu q^p[p+1, q])^m \sum_{n=p+1}^{\infty} |a_n| \\ & \leq \sum_{n=p+1}^{\infty} (q^p[n, q]^2(1+B) + (1+A)[n, q][p, q]) \\ & \cdot (1 + [p, q]\mu + \mu q^p[n, q])^m |a_n|. \end{aligned} \quad (68)$$

Hence,

$$\begin{aligned} & (q^p[p+1, q]^2(1+B) + (1+A)[p+1, q][p, q]) \\ & \cdot (1 + [p, q]\mu + \mu q^p[p+1, q])^m \sum_{n=p+1}^{\infty} |a_n| \leq \frac{[p, q]^2(A-B)}{q^p}, \end{aligned} \quad (69)$$

which implies that

$$\sum_{n=p+1}^{\infty} |a_{n+p}| \leq \frac{[p, q]^2(A-B)}{q^p(q^p[q^p[p+1, q]^2(1+B) + (1+A)[p+1, q][p, q]](1 + [p, q]\mu + \mu q^p[p+1, q])^m}. \quad (70)$$

Now, by putting this value in (65) and (66), we get the required proof.

Theorem 10. *Let $f \in \mathcal{MC}_{\mu,q}(p, m, A, B)$ and have the form (1). Then, for $|z| = r$,*

$$\frac{p, q]_m}{q^{mp+\zeta} r^{m+p}} - \vartheta_2 r^p \leq \left| \partial_q^m f(z) \right| \leq \frac{p, q]_m}{q^{mp+\zeta} r^{m+p}} + \vartheta_2 r^p, \quad (71)$$

where

$$\begin{aligned} \vartheta_2 = & \frac{[p, q]^2(A-B)}{q^p(q^p[p+1, q](1+B) + (1+A)[p, q])(1 + [p, q]\mu + \mu q^p[p+1, q])^m}, \\ & \zeta = \sum_{n=1}^m n. \end{aligned} \quad (72)$$

Proof. From the help of (3) and (4), we can write

$$\partial_q^m f(z) = \frac{(-1)^m [p, q]_m}{q^{mp+\zeta} z^{p+m}} + \sum_{n=p+1}^{\infty} [n - (m-1), q]_{m+1} a_n z^{n-m}. \quad (73)$$

Since $|z| = r < 1$ implies that $r^{n-m} \leq r^p$ for $m \leq n$ and $n \geq p+1$, hence

$$\left| \partial_q^m f(z) \right| \leq \frac{p, q]_m}{q^{mp+\zeta} r^{m+p}} + r^p \sum_{n=p+1}^{\infty} [n - (m-1), q]_{m+1} |a_n|. \quad (74)$$

Similarly,

$$\left| \partial_q^m f(z) \right| \geq \frac{p, q]_m}{q^{mp+\zeta} r^{m+p}} - r^p \sum_{n=p+1}^{\infty} [n - (m-1), q]_{m+1} |a_n|. \quad (75)$$

Since f is in the class $\mathcal{MC}_{\mu, q}(p, m, A, B)$, so by (23), we have the inequality

$$\begin{aligned} & (q^p [p+1, q] (1+B) + (1+A) [p, q]) \\ & \cdot (1 + [p, q] \mu + \mu q^p [p+1, q])^m \\ & \cdot \sum_{n=p+1}^{\infty} [n, q] |a_n| \leq \frac{[p, q]^2 (A-B)}{q^p}, \end{aligned} \quad (76)$$

from which it can be deduced that

$$\begin{aligned} & \sum_{n=p+1}^{\infty} [n, q] |a_n| \\ & \leq \frac{[p, q]^2 (A-B)}{q^p (q^p [p+1, q] (1+B) + (1+A) [p, q]) (1 + [p, q] \mu + \mu q^p [p+1, q])^m}, \end{aligned} \quad (77)$$

but it can easily be seen that

$$\sum_{n=p+1}^{\infty} [n - (m-1), q] |a_{p+n}| \leq \sum_{n=p+1}^{\infty} [n, q] |a_n|, \quad (78)$$

which implies

$$\begin{aligned} & \sum_{n=p+1}^{\infty} [n - (m-1), q] |a_{p+n}| \\ & \leq \frac{[p, q]^2 (A-B)}{q^p (q^p [p+1, q] (1+B) + (1+A) [p, q]) (1 + [p, q] \mu + \mu q^p [p+1, q])^m}. \end{aligned} \quad (79)$$

Now, using this inequality in (74) and (75), we obtain the required proof.

5. Integral Representation

Theorem 11. Let the function f given by (1) be in the class $\mathcal{MC}_{\mu, q}(p, m, A, B)$. Then, the function $G(z)$ represented by

$$G(z) = (1 + \gamma) \frac{1}{z^p} + \gamma p \int_0^z \frac{f(t)}{t} dt \quad (\gamma \geq 0, z \in \mathbb{D}), \quad (80)$$

is in the class $\mathcal{MC}_{\mu, q}(p, m, A, B)$.

Proof. From (1),

$$f(z) = z^{-p} + \sum_{n=p+1}^{\infty} a_n z^n. \quad (81)$$

Then,

$$\begin{aligned} G(z) &= (1 + \gamma) \frac{1}{z^p} + \gamma p \int_0^z \frac{t^{-p} + \sum_{n=p+1}^{\infty} a_n t^n}{t} dt \\ &= (1 + \gamma) \frac{1}{z^p} + \gamma p \left(\frac{-1}{p} \frac{1}{z^p} + \sum_{n=p+1}^{\infty} \frac{a_n}{n} z^n \right) \\ &= z^{-p} + \sum_{n=p+1}^{\infty} \frac{a_n \gamma p}{n} z^n. \end{aligned} \quad (82)$$

Consider

$$\begin{aligned} & \sum_{n=p+1}^{\infty} (q^p [n, q]^2 (1+B) + (1+A) [n, q] [p, q]) \\ & \cdot (1 + [p, q] \mu + \mu q^p [n, q])^m \left| \frac{a_n \gamma p}{n} \right| \\ & \leq \sum_{n=p+1}^{\infty} (q^p [n, q]^2 (1+B) + (1+A) [n, q] [p, q]) \\ & \cdot (1 + [p, q] \mu + \mu q^p [n, q])^m \gamma p |a_n| \leq \frac{[p, q]^2 (A-B)}{q^p}, \end{aligned} \quad (83)$$

since $\gamma p \leq 1$.

Therefore, by Theorem 4, $G(z) \in \mathcal{MC}_{\mu, q}(p, m, A, B)$.

6. Radius Problems

The following results are about the radii of convexity and starlikeness for the functions of the class $\mathcal{MC}_{\mu, q}(p, m, A, B)$.

Theorem 12. If $f \in \mathcal{MC}_{\mu, q}(p, m, A, B)$, then $f \in \mathcal{MC}_p(\alpha)$ for $|z| < r_1$, where

$$r_1 = \left(\frac{q^p p (p - \alpha) (q^p [n + p, q]^2 (1+B) + (1+A) [n + p, q] [p, q]) (1 + [p, q] \mu + \mu q^p [n + p, q])^m}{(p + n) (n + p + \alpha) [p, q]^2 (A-B)} \right)^{1/(n+2p)}. \quad (84)$$

Proof. Let $f \in \mathcal{MC}_{\mu, q}(p, m, A, B)$. To prove $f \in \mathcal{MC}_p(\alpha)$, we

only need to show

$$\left| \frac{zf''(z) + (p+1)f'(z)}{zf''(z) + (1+2\alpha-p)f'(z)} \right| < 1. \quad (85)$$

Using (1) along with some simple computation yields

$$\sum_{n=1}^{\infty} \frac{(p+n)(p+n+\alpha)}{p(p-\alpha)} |a_{n+p}| |z|^{n+2p} < 1. \quad (86)$$

As f is in the class $\mathcal{MC}_{\mu,q}(p, m, A, B)$, so we have from (23)

$$\begin{aligned} & \sum_{n=p+1}^{\infty} (q^p[n, q]^2(1+B) + (1+A)[n, q][p, q]) \\ & \cdot (1 + [p, q]\mu + \mu q^p[n, q])^m |a_n| \\ & \leq \frac{[p, q]^2(A-B)}{q^p} \Rightarrow \sum_{n=p+1}^{\infty} q^p (q^p[n, q]^2(1+B) \\ & + (1+A)[n, q][p, q]) (1 + [p, q]\mu + \mu q^p[n, q])^m |a_n| < 1. \end{aligned} \quad (87)$$

Equivalently,

$$\begin{aligned} & \sum_{n=1}^{\infty} q^p (q^p[n+p, q]^2(1+B) + (1+A)[n+p, q][p, q]) \\ & \cdot (1 + [p, q]\mu + \mu q^p[n+p, q])^m |a_{n+p}| < 1. \end{aligned} \quad (88)$$

Now, inequality (86) will be true, if the following holds:

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(p+1)(n+p+\alpha)}{p(p-\alpha)} |a_{n+p}| |z|^{n+2p} \\ & < \sum_{n=1}^{\infty} q^p \frac{(q^p[n+p, q]^2(1+B) + (1+A)[n+p, q][p, q]) (1 + [p, q]\mu + \mu q^p[n+p, q])^m}{[p, q]^2(1-A)} |a_{n+p}|, \end{aligned} \quad (89)$$

which implies that

$$|z|^{n+2p} < \frac{q^p p(p-\alpha) (q^p[n+p, q]^2(1+B) + (1+A)[n+p, q][p, q]) (1 + [p, q]\mu + \mu q^p[n+p, q])^m}{(p+n)(n+p+\alpha)[p, q]^2(A-B)}, \quad (90)$$

and so

$$|z| < \left(\frac{q^p p(p-\alpha) (q^p[n+p, q]^2(1+B) + (1+A)[n+p, q][p, q]) (1 + [p, q]\mu + \mu q^p[n+p, q])^m}{(p+n)(n+p+\alpha)[p, q]^2(A-B)} \right)^{1/(n+2p)} = r_1. \quad (91)$$

We get the required condition.

Theorem 13. Let $f \in \mathcal{MC}_{\mu,q}(p, m, A, B)$. Then, $f \in \mathcal{MS}_p^*(\alpha)$ for $|z| < r_2$, where

$$r_2 = \left(\frac{q^p(p-\alpha) (q^p[n, q]^2(1+B) + (1+A)[n, q][p, q]) (1 + [p, q]\mu + \mu q^p[n, q])^m}{(n+p+\alpha)[p, q]^2(A-B)} \right)^{1/(n+2p)}. \quad (92)$$

Proof. We know that $f \in \mathcal{MS}_p^*(\alpha)$, if and only if

$$\left| \frac{zf'(z) + pf(z)}{zf'(z) - (p-2\alpha)f(z)} \right| < 1. \quad (93)$$

Using (1) and with some simplification, we get

$$\sum_{n=1}^{\infty} \left(\frac{n+p+\alpha}{p-\alpha} \right) |a_{n+p}| |z|^{n+2p} < 1. \quad (94)$$

Now, from (23), we can easily obtain that

$$\sum_{n=p+1}^{\infty} \frac{q^p (q^p[n, q]^2(1+B) + (1+A)[n, q][p, q]) (1 + [p, q]\mu + \mu q^p[n, q])^m}{[p, q]^2(A-B)} |a_n| < 1. \quad (95)$$

For inequality (94) to hold, it will be enough if

$$\begin{aligned} & \sum_{n=1}^{\infty} \left(\frac{n+p+\alpha}{p-\alpha} \right) |a_{n+p}| |z|^{n+2p} \\ & < \sum_{n=1}^{\infty} \frac{q^p (q^p[n, q]^2(1+B) + (1+A)[n, q][p, q]) (1 + [p, q]\mu + \mu q^p[n, q])^m}{[p, q]^2(A-B)}, \end{aligned} \quad (96)$$

which implies that

$$|z|^{n+2p} < \frac{q^p(p-\alpha) (q^p[n, q]^2(1+B) + (1+A)[n, q][p, q]) (1 + [p, q]\mu + \mu q^p[n, q])^m}{(n+p+\alpha)[p, q]^2(A-B)}, \quad (97)$$

and hence,

$$|z| < \left(\frac{q^p(p-\alpha) (q^p[n, q]^2(1+B) + (1+A)[n, q][p, q]) (1 + [p, q]\mu + \mu q^p[n, q])^m}{(n+p+\alpha)[p, q]^2(A-B)} \right)^{1/(n+2p)} = r_2. \quad (98)$$

Thus, we obtain the required result.

7. Conclusion

The applications of q -calculus have been the focus point in the recent times in various branches of mathematics. This article introduces a new operator in q -analogue for meromorphic multivalent functions. Then, a new subclass of multivalent convex functions is defined and studied for some of its geometric properties like sufficient conditions, coefficient estimates, and distortion. Also, problems of closure and integral representation are discussed in detail. Many other classes

can be defined using this operator which will open a lot of new opportunities for research in this and related fields.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors jointly worked on the results, and they read and approved the final manuscript.

References

- [1] A. Aral and V. Gupta, "On the Durrmeyer type modification of the q -Baskakov type operators," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 3-4, pp. 1171–1180, 2010.
- [2] A. Aral and V. Gupta, "On q -Baskakov type operators," *Demonstratio Mathematica*, vol. 42, pp. 109–122, 2009.
- [3] F. H. Jackson, "On q -definite integrals," *The Quarterly Journal of Pure and Applied Mathematics*, vol. 41, pp. 193–203, 1910.
- [4] F. H. Jackson, "XI.—On q -functions and a certain difference operator," *Transactions of the Royal Society of Edinburgh*, vol. 46, no. 2, pp. 253–281, 1909.
- [5] A. Aral and V. Gupta, "Generalized q -Baskakov operators," *Mathematica Slovaca*, vol. 61, pp. 619–634, 2011.
- [6] A. Aral, "On the generalized Picard and Gauss Weierstrass singular integrals," *Journal of Computational Analysis & Applications*, vol. 8, pp. 249–261, 2006.
- [7] G. A. Anastassiou and S. G. Gal, "Geometric and approximation properties of generalized singular integrals," *Journal of the Korean Mathematical Society*, vol. 23, pp. 425–443, 2006.
- [8] G. A. Anastassiou and S. G. Gal, "Geometric and approximation properties of some singular integrals in the unit disk," *Journal of Inequalities and Applications*, vol. 2006, Article ID 17231, 2006.
- [9] S. Kanas and D. Răducanu, "Some class of analytic functions related to conic domains," *Mathematica slovacica*, vol. 64, pp. 1183–1196, 2014.
- [10] H. Aldweby and M. Darus, "Some subordination results on q -analogue of Ruscheweyh differential operator," *Abstract and Applied Analysis*, vol. 2014, Article ID 958563, 2014.
- [11] A. Huda and M. Darus, "Integral operator defined by q -analogue of Liu-Srivastava operator," *Studia Universitatis Babeş-Bolyai Series Mathematica*, vol. 58, pp. 529–537, 2013.
- [12] A. Mohammed and M. Darus, "A generalized operator involving the q -hypergeometric function," *Matematichki Vesnik*, vol. 65, pp. 454–465, 2013.
- [13] T. M. Seoudy and M. K. Aouf, "Coefficient estimates of new classes of q -starlike and q -convex functions of complex order," *Journal of Mathematical Inequalities*, vol. 10, pp. 135–145, 2016.
- [14] M. Arif and B. Ahmad, "New subfamily of meromorphic multivalent starlike functions in circular domain involving q -differential operator," *Mathematica Slovaca*, vol. 68, no. 5, pp. 1049–1056, 2018.
- [15] B. Ahmad and M. Arif, "New subfamily of meromorphic convex functions in circular domain involving q -operator," *International Journal of Analysis and Applications*, vol. 16, pp. 75–82, 2018.
- [16] W. Rogosinski, "On the coefficients of subordinate functions," *Proceedings of the London Mathematical Society*, vol. 48, pp. 48–82, 1943.