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Research Article

Terminal Value Problem for Implicit Katugampola Fractional Differential Equations in b-Metric Spaces

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This manuscript deals with a class of Katugampola implicit fractional differential equations in b-metric spaces. The results are based on the $\alpha-\varphi$-Geraghty type contraction and the fixed point theory. We express an illustrative example.

1. Introduction and Preliminaries

An interesting extension and unification of fractional derivatives of the type Caputo and the type Caputo-Hadamard is called Katugampola fractional derivative that has been introduced by Katugampola [1, 2]. Some fundamental properties of this operator are presented in [3, 4]. Several results of implicit fractional differential equations have been recently provided (see [4–14] and the references therein). A new class of mixed monotone operators with concavity and applications to fractional differential equations has been considered in [15]. In [16], the authors presented some existence and uniqueness results for a class of terminal value problem for differential equations with Hilfer-Katugampola fractional derivative.

On the other side, a novel extension of b-metric was suggested by Czerwik [17, 18]. Although the b-metric standard looks very similar to the metric definition, it has a quite different structure and properties. For example, in the b-metric topology framework, an open (closed) set is not open (closed). Additionally, the b-metric function is not continuous. These weaknesses make this new structure more interesting (see [19–28]).

Throughout the paper, any mentioned set is nonempty. We consider the following type of terminal value problems of Katugampola implicit differential equations of noninteger orders:

$$\begin{cases} \left({}^{\rho}D_{0^+}^{r}+\vartheta\right)(\tau)=\kappa\left(\tau,\vartheta(\tau),\left({}^{\rho}D_{0^+}^{r}+\vartheta\right)(\tau)\right), & \tau\in I:=[0,T], \\ \vartheta(T)=\vartheta_T\in\mathbb{R}, \end{cases} \tag{1}$$

with $T>0$ and the function $\kappa: I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous. Here, ${}^{\rho}D_{0^+}^{r}$ is the Katugampola fractional derivative of order $r\in(0,1]$.

Set $C(I):=\{h \mid h \text{ real continuous functions on } I:=[0,T]\}$. Then, $C(I)$ forms a Banach space with the norm $\|\vartheta\|_{\infty}=\sup_{\tau\in I}|\vartheta(\tau)|$.

Set $L^1(I):=\{\vartheta: I\to\mathbb{R} \mid \vartheta \text{ is measurable function and Lebesgue integrable}\}$. Then, $L^1(I)$ becomes a Banach space with the norm $\|\vartheta\|_{L^1}=\int_0^T|\vartheta(\tau)|dt$.

Set $C_{r,\rho}(I)=\{\vartheta:(0,T]\to\mathbb{R}|\tau^{\rho(1-r)}\vartheta(\tau)\in C(I)\}$. Then, it forms a Banach space $\|\vartheta\|_C:=\sup_{\tau\in I}\|\tau^{\rho(1-r)}\vartheta(\tau)\|$. Here, $C_{r,\rho}(I)$ is called the weighted space of continuous functions.

Definition 1 (Katugampola fractional integral) [1]. The Katugampola fractional integrals of order $r>0$ and $\rho>0$ of a function $y\in X_c^p(I)$ are defined by

$$ {}^\rho T_{0^+}^r y(\tau)\frac{\rho^{1-r}}{\Gamma(r)}\int_0^t \frac{s^{\rho-1}y(s)}{(\tau^\rho-s^\rho)^{1-r}}ds,\quad \tau\in I. \tag{2}$$

Definition 2 (Katugampola fractional derivatives) [1, 2]. The generalized fractional derivatives of order $r>0$ and $\rho>0$ corresponding to the Katugampola fractional integrals (2) defined for any $\tau\in I$ by

$$ {}^\rho D_{0^+}^r y(\tau)=\left(\tau^{1-\rho}\frac{d}{dt}\right)^n\left({}^\rho T_{0^+}^{n-r}y\right)(\tau)=\frac{\rho^{r-n+1}}{\Gamma(n-r)}\left(\tau^{1-\rho}\frac{d}{dt}\right)^n\int_0^t\frac{s^{\rho-1}y(s)}{(\tau^\rho-s^\rho)^{r-n+1}}ds, \tag{3}$$

where $n=[r]+1$; if the integrals exist.

Remark 1 ([1, 2]). As a basic example, we quote for $r,\rho>0$ and $\theta>-\rho$,

$$ {}^\rho D_{0^+}^r\tau^\theta=\frac{\rho^{r-1}\Gamma(1+(\theta/\rho))}{\Gamma(1-r+(\theta/\rho))}\tau^{\theta-r\rho}. \tag{4}$$

Giving in particular,

$$ {}^\rho D_{0^+}^r\tau^{\rho(r-i)}=0,\quad \text{for each } i=1,2,\cdots,n. \tag{5}$$

In fact, for $r,\rho>0$ and $\theta>-\rho$, we have

$$\begin{aligned} {}^\rho D_{0^+}^r\tau^\theta &=\frac{\rho^{r-n+1}}{\Gamma(n-r)}\left(\tau^{1-\rho}\frac{d}{dt}\right)^n\int_0^t s^{\rho+\theta-1}(\tau^\rho-s^\rho)^{n-r-1}ds\\ &=\frac{\rho^{r-1}\Gamma(1+(\theta/\rho))}{\Gamma(1+n-r+(\theta/\rho))}\left[n-r+\frac{\theta}{\rho}\right]\cdots\left[1-r+\frac{\theta}{\rho}\right]\tau^{\theta-r}\rho\\ &=\frac{\rho^{r-1}\Gamma(1+(\theta/\rho))}{\Gamma(1-r+(\theta/\rho))}\tau^{\theta-r}\rho. \end{aligned} \tag{6}$$

If we put $i=r-(\theta/\rho)$, we obtain from (6):

$$ {}^\rho D_{0^+}^r\tau^{\theta(r-i)}=\rho^{r-1}\frac{\Gamma(r-i+1)}{\Gamma(n-i+1)}(n-i)(n-i-1)\cdots(1-m)\tau^{-\rho i}. \tag{7}$$

So, ${}^\rho D_{0^+}^r\tau^{\rho(r-i)}=0,\forall r,\rho>0$.

Theorem 1 ([2]). *Let $r,\rho,c\in\mathbb{R}$, be such that $r,\rho>0$. Then, for any $\kappa,\omega\in X_c^p(I)$, where $1\le p\le\infty$, we have*

(1) Inverse property:

$$ {}^\rho D_{0^+}^r\,{}^\rho I_{0^+}^r\kappa(\tau)=\kappa(\tau),\quad \textit{for all } r\in(0,1]. \tag{8}$$

(2) Linearity property: for all $r\in(0,1)$, we have

$$\begin{cases} {}^\rho D_{0^+}^r(\kappa+\omega)(\tau)={}^\rho D_{0^+}^r\kappa(\tau)+{}^\rho D_{0^+}^r\omega(\tau).\\ {}^\rho I_{0^+}^r(\kappa+\omega)(\tau)={}^\rho I_{0^+}^r\kappa(\tau)+{}^\rho I_{0^+}^r\omega(\tau).\end{cases} \tag{9}$$

Lemma 1 ([2]). *Let $r,\rho>0$. If $\vartheta\in C(I)$; then the fractional differential equation ${}^\rho D_{0^+}+\vartheta(\tau)=0$, has a unique solution*

$$\vartheta(\tau)=C_1\tau^{\rho(r-1)}+C_2\tau^{\rho(r-2)}+\cdots+C_n\tau^{\rho(r-n)}, \tag{10}$$

where $C_i\in\mathbb{R}$ with $i=1,2,\cdots,n$.

Proof. Let $r,\rho>0$. from Remark 1, we have

$$ {}^\rho D_{0^+}^r\tau^{\rho(r-i)}=0,\quad \text{for each } i=1,2,\cdots,n. \tag{11}$$

Then, the fractional equation ${}^\rho D_{0^+}^r\vartheta(\tau)=0$ has a particular solution as follows:

$$\vartheta(\tau)=C_i\tau^{\rho(r-i)},\quad C_i\in\mathbb{R}, \text{for each } i=1,2,\cdots,n. \tag{12}$$

Thus, the general solution of ${}^\rho D_{0^+}^r\vartheta(\tau)=0$ is a sum of particular solutions (12), i.e.

$$\vartheta(\tau)=C_1\tau^{\rho(r-1)}+C_2\tau^{\rho(r-2)}+\cdots+C_n\tau^{\rho(r-n)}, C_i\in\mathbb{R}\,; (i=1,2,\cdots,n). \tag{13}$$

Lemma 2. *Let $r,\rho>0$. If $\vartheta,{}^\rho D_{0^+}^r\vartheta\in C(I)$ and $0<r\le 1$, then*

$$ {}^\rho I_{0^+}^r\,{}^\rho D_{0^+}^r\vartheta(\tau)=\vartheta(\tau)+c\tau^{\rho(r-1)}, \tag{14}$$

for some constant $c\in\mathbb{R}$.

Proof. Let ${}^\rho D_{0^+}^r\vartheta\in C(I)$ be the fractional derivative (3) of order $0<r\le 1$. If we apply the operator ${}^\rho D_{0^+}^r$ to ${}^\rho I_{0^+}^r\,{}^\rho D_{0^+}^r\vartheta(\tau)-\vartheta(\tau)$ and use the properties (8) and (9), we get

$$\begin{aligned} {}^\rho D_{0^+}^r\left[{}^\rho I_{0^+}^r\,{}^\rho D_{0^+}^r\vartheta(\tau)-\vartheta(\tau)\right]&={}^\rho D_{0^+}^r\,{}^\rho I_{0^+}^r\,{}^\rho D_{0^+}^r\vartheta(\tau)-{}^\rho D_{0^+}^r\vartheta(\tau)\\ &={}^\rho D_{0^+}^r\vartheta(\tau)-{}^\rho D_{0^+}^r\vartheta(\tau)=0. \end{aligned} \tag{15}$$

From the proof of Lemma 1, there exists $c\in\mathbb{R}$, such that

$$ {}^\rho I_{0^+}^r\,{}^\rho D_{0^+}^r\vartheta(\tau)-\vartheta(\tau)=c\tau^{\rho(r-1)}, \tag{16}$$

which implies (14).

Lemma 3. *Let $h \in L^1(I, \mathbb{R})$ and $0 < r \leq 1$ and $\rho > 0$. A function $\vartheta \in C(I)$ forms a solution for*

$$\begin{cases} ({}^{\rho}D^{r}_{0^+}\vartheta)(\tau) = z(\tau), & \tau \in I, \\ \vartheta(T) = \vartheta_T, \end{cases} \tag{17}$$

if and only if ϑ fulfills

$$\vartheta(\tau) = (\vartheta_T - {}^{\rho}I^{r}_{0^+}z(T))\left(\frac{\tau}{T}\right)^{\rho(r-1)} + \frac{\rho^{r-1}}{\Gamma(r)}\int_0^{\tau} s^{\rho-1}(\tau^{\rho} - s^{\rho})^{r-1}z(s)ds. \tag{18}$$

Proof. Let $r, \rho > 0$. and $0 < r \leq 1$. Suppose that ϑ satisfies (17). Employing the operator ${}^{\rho}I^{r}_{0^+}$ to the each side of the equation

$$({}^{\rho}D^{r}_{0^+}\vartheta)(\tau) = z(\tau), \tag{19}$$

we find

$${}^{\rho}I^{r}_{0^+}{}^{\rho}D^{r}_{0^+}\vartheta(\tau) = {}^{\rho}I^{r}_{0^+}z(\tau). \tag{20}$$

From Lemma 2, we get

$$\vartheta(\tau) + c\tau^{\rho(r-1)} = {}^{\rho}I^{r}_{0^+}z(\tau), \tag{21}$$

for some $c \in \mathbb{R}$. If we use the terminal condition $\vartheta(T) = \vartheta_T$ in (21), we find

$$\vartheta(T) = \vartheta_T = {}^{\rho}I^{r}_{0^+}z(T) - cT^{\rho(r-1)}, \tag{22}$$

which shows

$$c = ({}^{\rho}I^{r}_{0^+}z(T) - \vartheta_T)T^{\rho(1-r)}. \tag{23}$$

Henceforth, we deduce (18).

Contrariwise, if ϑ achieves (18), then $({}^{\rho}D^{r}_{0^+}\vartheta)(\tau) = z(\tau)$; for $\tau \in I$ and $\vartheta(\tau) = \vartheta_T$.

Lemma 4. *Contemplate the problem (1), and set $g \in C(I)$, and $\omega(\tau) = \varkappa(\tau, \vartheta(\tau), \omega(\tau))$.*

We presume ϑ achieves

$$\vartheta(\tau) = (\vartheta_T - {}^{\rho}I^{r}_{0^+}\omega(T))\left(\frac{\tau}{T}\right)^{\rho(r-1)} + \frac{\rho^{1-r}}{\Gamma(r)}\int_0^{\tau} s^{\rho-1}(\tau^{\rho} - s^{\rho})^{r-1}\omega(s)ds. \tag{24}$$

Then, ϑ forms a solution of (1).

Definition 3 [29, 30]. A function $d : S \times S \longrightarrow [0,\infty)$ is called b-metric if there is $c \geq 1$ and d fulfills

(i) $(bM1)$ $d(\nu, \vartheta) = 0$ if and only if $\nu = \vartheta$

(ii) $(bM2)$ $d(\nu, \mu) = d(\mu, \nu)$

(iii) $(bM3)$ $d(\mu, \vartheta) \leq c[d(\mu, \nu) + d(\nu, \vartheta)]$

for all $\mu, \nu, \vartheta \in S$. We say that the tripled (S, d, c) is b -metric space (in short, b.m.s.).

Example 1 [29, 30]. *Let $d : C(I) \times C(I) \longrightarrow [0,\infty)$ be described as*

$$d(\nu, \vartheta) = \left\|(\nu - \vartheta)^2\right\|_{\infty} \coloneqq \sup_{\tau \in I}\|\nu(\tau) - \vartheta(\tau)\|^2, \quad \text{for all } \nu, \vartheta E C(I). \tag{25}$$

Ergo, $(C(I), d, 2)$ is b-metric space.

Example 2 [29, 30]. Set $S = [0, 1]$ and $d : S \times S \longrightarrow [0,\infty)$ be designated by

$$d(\nu, \vartheta) = |\nu^r - \vartheta^r|, \quad \text{for all } \nu, \vartheta \in S. \tag{26}$$

Henceforth, (S, d, r) with $r \geq 2$ is b-metric space.

We set the following: $\{\phi : [0,\infty) \to [0,\infty) | \phi$ is continuous, increasing, $\phi(0) = 0$ and $\phi(c\mu) \leq c\phi(\mu) \leq c\mu$ for $c > 1\}$.

For some $c \geq 1$, we set $\mathscr{F} \coloneqq \{\lambda : [0,\infty) \to [0, (1/c^2)) | \lambda$ is nondecreasing$\}$.

Definition 4 [29, 30]. A self-operator T, on a *b.m.s.* (S, d, c), is called a generalized $\alpha - \phi -$ Geraghty contraction whenever there exists $\alpha : S \times S \longrightarrow [0,\infty)$, and some $L \geq 0$ such that for

$$D(\nu, \vartheta) = \max\left\{d(\nu, \vartheta), d(\vartheta, T(\vartheta)), d(\nu, T(\nu)), \frac{d(\nu, T(\vartheta)) + d(\vartheta, T(\nu))}{2s}\right\}, \tag{27}$$

$$N(\nu, \vartheta) = \min\{d(\nu, \vartheta), d(\vartheta, T(\vartheta)), d(\nu, T(\nu))\}, \tag{28}$$

we have

$$\alpha(\mu, \nu)\varphi\left(c^3 d(T(\mu), T(\nu))\right) \leq \lambda(\varphi(D(\mu, \nu))(\varphi(D(\mu, \nu)) + L\psi(N(\mu, \nu), \tag{29}$$

for all $\mu, \nu, \vartheta \in S$, where $\lambda \in \mathscr{F}, \varphi, \psi \in \Phi$.

Remark 2. In the case when $L = 0$ in Definition 4 and the fact that

$$d(\mu, \nu) \leq D(\mu, \nu), \quad \text{for all } \mu, \nu \in S, \tag{30}$$

the inequality (29) becomes

$$\alpha(\mu, \nu)\varphi\left(c^3 d(T(\mu), T(\nu))\right) \leq \lambda(\varphi(d(\mu, \nu))\varphi(d(\mu, \nu))). \tag{31}$$

Definition 5 [29, 30]. Set $\alpha : S \times S \longrightarrow [0,\infty)$. An operator $T : S \longrightarrow S$, is $\alpha -$ admissible if

$$\alpha(\mu, \nu) \geq 1 \Rightarrow \alpha(T(\mu), T(\nu)) \geq 1, \tag{32}$$

for all $\mu, \nu \in S$.

Definition 6 [29, 30]. Let (S, d, c) with $c \geq 1$ be a *b.m.s* and $\alpha : S \times S\mathbb{R}_+^*$.

We say that S is $\alpha-$ regular if for any sequence $\{v_n\}_{n\in\mathbb{N}}$ in S such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$ and $\alpha(v_n, v_{n+1}) \geq 1$ for each n; there exists a subsequence $\{v_n(\kappa)\}_{\kappa\in\mathbb{N}}$ of $\{v_n\}n$ with $\alpha(v_{n(\kappa)}, x) \geq 1$ for all k.

Theorem 2 [29, 30]. *We presume that a self-operator T over a complete b.m.s.*

(S, d, c) with $c \geq 1$ forms a generalized $\alpha - \varphi -$ Geraghty contraction. Furthermore,

(i) T is $\alpha-$ admissible with initial value $\alpha(\mu 0, T(\mu 0)) \geq 1$ for some $\mu 0 \in M$

(ii) either T is continuous or M is $\alpha-$ regular

Then T possesses a fixed point. Furthermore, if

(iii) for all fixed points μ, v of T, either $\alpha(\mu, v) \geq 1$ or $\alpha(v, \mu) \geq 1$, then the found fixed point is unique

This manuscript launches the study of Katugampola implicit fractional differential equations on b.m.s.

2. Main Results

Observe that $(C_{r,\rho}(I), d, 2)$ is a complete b.m.s. with $d : C_{r,\rho}(I) \times C_{r,\rho}(I) \longrightarrow [0,\infty)$ described as

$$d(v, \vartheta) = \left\|(v-\vartheta)^2\right\|C := \sup_{\tau\in I} \tau^{\rho(1-r)} |v(\tau) - \vartheta(\tau)|^2. \quad (33)$$

A function $\vartheta \in C_{r,\rho}(I)$ is called a solution of (1) if it archives

$$\vartheta(\tau) = \left(\vartheta_T - {}^\rho I_{0^+}^r \omega(T)\right)\left(\frac{\tau}{T}\right)^{\rho(r-1)} + \frac{\rho^{1-r}}{\Gamma(r)} \int_0^\tau s^{\rho-1} \left(\tau^\rho - s^\rho\right)^{r-1} \omega(s) ds, \quad (34)$$

with $\omega(\tau) = \kappa(\tau, \vartheta(\tau), \omega(\tau)) \in C(I)$.

In the sequel, we shall need the following hypotheses:

(H_1) There exist $\varphi \in \Phi, p : C(I) \times C(I) \longrightarrow (0,\infty)$ and $q : I \longrightarrow (0, 1)$ so that for each $\vartheta, v, \vartheta_1, v_1 \in C_{r,\rho}(I)$, and $\tau \in I$

$$|\kappa(\tau, \vartheta, v) - \kappa(\tau, \vartheta_1, v_1)| \leq \tau^{\rho/2(1-r)} p(\vartheta, v) |\vartheta - \vartheta_1| + q(\tau)|v - v_1|, \quad (35)$$

with

$$\left\|\frac{\rho^{1-r}}{\Gamma(r)} \int_0^T s^{\rho-1} \left(T^\rho - s^\rho\right)^{r-1} \frac{p(\vartheta, v)}{1 - q*} ds\right\|_C^2 + \left\|\frac{\rho^{1-r}}{\Gamma(r)} \int_0^\tau s^{\rho-1} \left(\tau^\rho - s^\rho\right)^{r-1} \frac{p(\vartheta, v)}{1 - q*} ds\right\|_C^2 \leq \varphi\left(\left\|(\vartheta - v)^2\right\|C\right) \quad (36)$$

(H_2) There are $\mu_0 \in C_{r,\rho}(\mathrm{I})$ and $\theta : C_{r,\rho}(I) \times C_{r,\rho}(I) \longrightarrow \mathbb{R}$, so that

$$\theta\left(\mu_0(\tau), \left(\vartheta_T - {}^\rho I_{0+}^r \omega(T)\right)\left(\frac{\tau}{T}\right)^{\rho(r-1)} + \frac{\rho^{1-r}}{\Gamma(r)} \int_0^\tau s^{\rho-1} \left(\tau^\rho - s^\rho\right)^{r-1} \omega(s) ds\right) \geq 0, \quad (37)$$

with $g \in C(I)$ and $\omega(\tau) = \kappa(\tau, \mu 0(\tau), \omega(\tau))$

(H_3) For any $\tau \in I$, and $\vartheta, v \in C_{r,\rho}(I)$, $\theta(\vartheta(\tau), v(\tau)) \geq 0$ implies

$$\theta\left(\frac{\rho^{1-r}}{\Gamma(r)} \int_0^\tau s^{\rho-1} \left(\tau^\rho - s^\rho\right)^{r-1} \omega(s) ds, \frac{\rho^{1-r}}{\Gamma(r)} \int_0^\tau s^{\rho-1} \left(\tau^\rho - s^\rho\right)^{r-1} \mathfrak{z}(s) ds\right) \geq 0, \quad (38)$$

with $\omega, \mathfrak{z} \in C(I)$ so that

$$\begin{cases} \mathfrak{z}(\tau) = \kappa(\tau, v(\tau), \mathfrak{z}(\tau)), \\ \omega(\tau) = \kappa(\tau, \vartheta(\tau), \omega(\tau)). \end{cases} \quad (39)$$

(H_4) If $\vartheta_{nn\in N} \subset C(I)$ with $\vartheta_n \longrightarrow \vartheta$ and $\theta(\vartheta_n, \vartheta_{n+1}) \geq$, then

$$\theta(\vartheta_n, \vartheta) \geq 1. \quad (40)$$

Theorem 3. *We presume (H_1)–(H_4). Then, the problem (1) possesses at least a solution on I.*

Proof. Take the operator $N : C_r, \rho(I) \longrightarrow C_r, \rho(I)$ into account that is described as

$$(N\vartheta)(\tau) = \left(\vartheta T - {}^\rho I_{0+}^r \omega(T)\right)\left(\frac{\tau}{T}\right)^{\rho(r-1)} + \frac{\rho^{1-r}}{\Gamma(r)} \int_0^\tau s^{\rho-1} \left(\tau^\rho s^\rho\right)^{r-1} \omega(s) ds, \quad (41)$$

where $\omega \in C(I)$, with $\omega(\tau) = \kappa(\tau, \vartheta(\tau), \omega(\tau))$.

On account of Lemma 4, we deduce that solutions of (1) are the fixed points of N.

Let $C_{r,\rho}(I) \times C_{r,\rho}(I) \longrightarrow (0,\infty)$ be the function defined by

$$\begin{cases} \alpha(\vartheta, v) = 1, & \text{if } \theta(\vartheta(\tau) v(\tau)) \geq 0, \tau \in I, \\ \alpha(\vartheta, v) = 0, & \text{otherwise.} \end{cases} \quad (42)$$

First, we demonstrate that N form a generalized $\alpha - \varphi$ -Geraghty operator. For any $\tau \in I$ and each $\vartheta, v \in C(I)$, we derive that

$$\begin{aligned} &\left|\tau^{\rho(1-r)}(N\vartheta)(\tau) - \tau^{\rho(1-r)}(Nv)(\tau)\right| \\ &\quad \leq \tau^{\rho(1-r)} \left|{}^\rho I_{0+}^r (g-h)(T)\right| \left(\frac{\tau}{T}\right)^{\rho(r-1)} \\ &\qquad + \frac{\rho^{1-r} \tau^{\rho(1-r)}}{\Gamma(r)} \int_0^\tau s^{\rho-1} \left(\tau^\rho - s^\rho\right)^{r-1} |\omega(s) - \mathfrak{z}(s)| ds, \end{aligned} \quad (43)$$

where ω, $\mathfrak{z}\in C(I)$, with

$$\omega(\tau) = \kappa(\tau, \vartheta(\tau), \omega(\tau)), \tag{44}$$

$$\mathfrak{z}(\tau) = \kappa(\tau, \nu(\tau), \mathfrak{z}(\tau)). \tag{45}$$

From (H_1), we have

$$\begin{aligned}|\omega(\tau) - \mathfrak{z}(\tau)| &= |\kappa(\tau, \vartheta(\tau), \omega(\tau)) - \kappa(\tau, \upsilon(\tau), \mathfrak{z}(\tau))| \\ &\le p(\vartheta, \upsilon)\tau^{\rho/2(1-r)}|\vartheta(\tau) - \upsilon(\tau)| + q(\tau)|\omega(\tau) - \mathfrak{z}(\tau)| \\ &\le p(\vartheta, \upsilon)\left(\tau^{\rho(1-r)}|\vartheta(\tau) - \upsilon(\tau)|^2\right)^{1/2} + q(\tau)|\omega(\tau) - \mathfrak{z}(\tau)|.\end{aligned} \tag{46}$$

Thus,

$$|\omega(\tau) - \mathfrak{z}(\tau)| \frac{p(\vartheta, \upsilon)}{1 - q*} \left\|(\vartheta - \upsilon)^2\right\|_C^{1/2}, \tag{47}$$

where $q* = \sup_{\tau\in I}|q(\tau)|$.

Next, we have

$$\begin{aligned}\left|\tau^{\rho(1-r)}(N\vartheta)(\tau) - \tau^{\rho(1-r)}(N\upsilon)(\tau)\right| &\le \tau^{\rho(1-r)}|^{\rho}I^{r}_{0+}(g-h)(T)|\left(\frac{\tau}{T}\right)^{\rho(r-1)} \\ &+ \frac{\rho^{1-r}\tau^{\rho(1-r)}}{\Gamma(r)}\int_0^{\tau} s^{\rho-1}(\tau^{\rho} - s^{\rho})^{r-1}\frac{p(\vartheta, \upsilon)}{1 - q*}\left\|(\vartheta - \upsilon)^2\right\|_C^{1/2} ds \\ &\le \frac{\rho^{1-r}\tau^{\rho(1-r)}}{\Gamma(r)}\int_0^{T} s^{\rho-1}(T^{\rho} - s^{\rho})^{r-1}\frac{p(\vartheta, \upsilon)}{1 - q*}\left\|(\vartheta - \upsilon)^2\right\|_C^{1/2} ds \\ &+ \frac{\rho^{1-r}\tau^{\rho(1-r)}}{\Gamma(r)}\int_0^{\tau} s^{\rho-1}(\tau^{\rho} - s^{\rho})^{r-1}\frac{p(\vartheta, \upsilon)}{1 - q*}\left\|(\vartheta - \upsilon)^2\right\|_C^{1/2} ds.\end{aligned} \tag{48}$$

Thus,

$$\begin{aligned}&\alpha(\vartheta, \upsilon)\left|\tau^{\rho(1-r)}(N\vartheta)(\tau) - \tau^{\rho(1-r)}(N\upsilon)(\tau)\right|^2 \\ &\le \left\|(\vartheta - \upsilon)^2\right\|C\alpha(\vartheta, \upsilon)\left\|\frac{\rho^{1-r}}{\Gamma(r)}\int_0^{T} s^{\rho-1}(T^{\rho} - s^{\rho})^{r-1}\frac{p(\vartheta, \upsilon)}{1 - q*}ds\right\|_C^2 \\ &+ \left\|(\vartheta - \upsilon)^2\right\|C\alpha(\vartheta, \upsilon)\left\|\frac{\rho^{1-r}}{\Gamma(r)}\int_0^{T} s^{\rho-1}(\tau^{\rho} - s^{\rho})^{r-1}\frac{p(\vartheta, \upsilon)}{1 - q*}ds\right\|_C^2 \\ &\le \left\|(\vartheta - \upsilon)^2\right\|C\phi\left(\left\|(\vartheta - \upsilon)^2\right\|C\right).\end{aligned} \tag{49}$$

Hence,

$$\alpha(\vartheta, \nu)\varphi\left(2^3 d(N(\vartheta), N(\nu)\right) \le \lambda(\varphi(d(\vartheta, \nu))\varphi(d(\vartheta, \nu), \tag{50}$$

where $\lambda \in \digamma$, $\varphi \in \Phi$, with $\lambda(\tau) = 1/8t$, and $\varphi(\tau) = \tau$.

So, N is generalized $\alpha - \varphi -$ Geraghty operator.

Let $\vartheta, \nu \in C_{r},_{\rho}(I)$ such that

$$\alpha(\vartheta, \nu) \ge 1. \tag{51}$$

Accordingly, for any $t \in I$, we find

$$\theta(\vartheta(\tau), \nu(\tau)) \ge 0. \tag{52}$$

This implies from (H_3) that

$$\theta(Nu(\tau), N\nu(\tau)) \ge 0, \tag{53}$$

which gives $\alpha(N(\vartheta), N(\nu)) \ge 1$.

Ergo, N is a α-admissible.

Now, from (H_2), there exists $\mu_0 \in C_{r},_{\rho}(I)$ such that

$$\alpha(\mu_0, N(\mu_0)) \ge 1. \tag{54}$$

Finally, from (H_4), if $\mu_{nn\in}N \subset M$ with $\mu_n \longrightarrow \mu$ and $\alpha(\mu_n, \mu_n + 1) \ge 1$, then,

$$\alpha(\mu_n, \mu) \ge 1. \tag{55}$$

Theorem 2 implies that fixed point ϑ of N forms a solution for (1).

3. An Example

The tripled $(C_{r},_{\rho}([0, 1]), d, 2)$ is a complete b.m.s. with $d : C_{r},_{\rho}([0, 1]) \times C_{r,\rho}([0, 1]) \longrightarrow [0,\infty)$ such that

$$d(\mu, \vartheta) = \left\|(\mu - \vartheta)^2\right\|C. \tag{56}$$

We take the following fractional differential problem into consideration

$$\begin{cases}({}^{\rho}D^{r}_{0+}\mu)(\tau) = \kappa(\tau, \mu(\tau), (\rho D^{r}_{0+}\mu)(\tau)), & \tau \in [0, 1], \\ \mu(1) = 2,\end{cases} \tag{57}$$

with

$$\kappa(\tau, \mu(\tau), \vartheta(\tau)) = \frac{\tau\rho/2(1-r)(1 + \sin(|\mu(\tau)|))}{4(1 + |\mu(\tau)|)} + \frac{e^{-\tau}}{2(1 + |\vartheta(\tau)|)}; \tau \in [0, 1]. \tag{58}$$

Let $\tau \in (0, 1]$, and $\mu, \vartheta \in C_{r,\rho}([0, 1])$. If $|\mu(\tau)| \le |\vartheta(\tau)|$, then

$$
\begin{aligned}
&|\kappa(\tau,\mu(\tau),\mu_1(\tau))-\kappa(\tau,\vartheta(\tau),\vartheta_1(\tau))| \\
&\quad= \tau^{\rho/2(1-r)}\left|\frac{1+\sin(|\mu(\tau)|)}{4(1+|\mu(\tau)|)}-\frac{1+\sin(|\vartheta(\tau)|)}{4(1+|\vartheta(\tau)|)}\right| \\
&\qquad+\left|\frac{e^{-\tau}}{2(1+|\mu_1(\tau)|)}-\frac{e^{-\tau}}{2(1+|\vartheta_1(\tau)|)}\right| \\
&\quad\le \frac{\tau^{\rho/2(1-r)}}{4}\,||\mu(\tau)|-|\vartheta(\tau)||+\frac{\tau^{\rho/2(1-r)}}{4}|\sin(|\mu(\tau)|)-\sin(|\vartheta(\tau)|)| \\
&\qquad+\frac{\tau^{\rho/2(1-r)}}{4}\,||\mu(\tau)|\sin(|\vartheta(\tau)|)-|\vartheta(\tau)|\sin(|\mu(\tau)|)| \\
&\qquad+\frac{e^{-\tau}}{2}|\mu_1(\tau)-\vartheta_1(\tau)|\le\frac{\tau^{\rho/2(1-r)}}{4}|\mu(\tau)-\vartheta(\tau)| \\
&\qquad+\frac{\tau^{\rho/2(1-r)}}{4}|\sin(|\mu(\tau)|)-\sin(|\vartheta(\tau)|)| \\
&\qquad+\frac{\tau^{\rho/2(1-r)}}{4}\,||\vartheta(\tau)|\sin(|\vartheta(\tau)|)-|\vartheta(\tau)|\sin(|\mu(\tau)|)| \\
&\qquad+\frac{e^{-\tau}}{2}|\mu_1(\tau)-\vartheta_1(\tau)|=\frac{\tau^{\rho/2(1-r)}}{4}|\mu(\tau)-\vartheta(\tau)| \\
&\qquad+\frac{\tau^{\rho/2(1-r)}}{4}(1+|\upsilon(\tau)|)|\sin(|\mu(\tau)|)-\sin(|\vartheta(\tau)|)| \\
&\qquad+\frac{e^{-\tau}}{2}|\mu_1(\tau)-\vartheta_1(\tau)|\le\frac{\tau^{\rho/2(1-r)}}{4}|\mu(\tau)-\vartheta(\tau)| \\
&\qquad+\frac{\tau^{\rho/2(1-r)}}{2}(1+|\vartheta(\tau)|)\times\left|\sin\left(\frac{||\mu(\tau)|-|\vartheta(\tau)||}{2}\right)\right|\left|\cos\left(\frac{|\mu(\tau)|+|\vartheta(\tau)|}{2}\right)\right| \\
&\qquad+\frac{e^{-\tau}}{2}|\mu_1(\tau)-\vartheta_1(\tau)|\le\frac{\tau^{\rho/2(1-r)}}{4}(2+|\upsilon(\tau)|)|\mu(\tau)-\vartheta(\tau)|+\frac{e^{-\tau}}{2}|\mu_1(\tau)-\vartheta_1(\tau)|.
\end{aligned}
\tag{59}
$$

In the case when $|\vartheta(\tau)|\le|\mu(\tau)|$, we get

$$
|\kappa(\tau,\mu(\tau))-\kappa(\tau,\vartheta(\tau))|\le\frac{\tau^{p/2(1-r)}}{4}\left(2+|\mu(\tau)||\mu(\tau)-\vartheta(\tau)|+\frac{e^{-\tau}}{2}|\mu_1(\tau)-\vartheta_1(\tau)|.\right. \tag{60}
$$

Hence,

$$
\begin{aligned}
&|\kappa(\tau,\mu(\tau))-\kappa(\tau,\vartheta(\tau))| \\
&\quad\le\frac{T^{p/2(1-r)}}{4}\min_{\tau\in I}\{2+|\mu(\tau)|,2+|\vartheta(\tau)|\}|\mu(\tau)-\vartheta(\tau)| \\
&\qquad+\frac{e^{-\tau}}{2}|\mu_1(\tau)-\vartheta_1(\tau)|.
\end{aligned}
\tag{61}
$$

Thus, hypothesis (H_1) is achieved with

$$
p(\mu,\vartheta)=\frac{T^{\rho/2(1-r)}}{4}\min_{r\in I}\{2+|\mu(\tau)|,2+|\vartheta(\tau)|\}, \tag{62}
$$

$$
q(\tau)=\frac{1}{2}e^{-\tau}. \tag{63}
$$

Define the functions $\lambda(\tau)=(1/8)t$, $\phi(\tau)=\tau$, $\alpha: C_{r,\rho}([0,1])\times C_{r,\rho}([0,1])\to\mathbb{R}_+^*$ with

$$
\begin{cases}\alpha(\mu,\vartheta)=1, & \text{if } \delta(\mu(\tau),\vartheta(\tau))\ge 0, \tau\in I,\\ \alpha(\mu,\vartheta)=0, & \text{else}\end{cases} \tag{64}
$$

and $\delta: C_{r,\rho}([0,1])\times C_{r,\rho}([0,1])\longrightarrow R$ with $\delta(\mu,\vartheta)=k\mu-\vartheta k_C$.

Hypothesis (H_2) is satisfied with $\mu_0(\tau)=\mu_0$. Also, (H_3) holds the definition of the function δ. So, Theorem 3 yields that problem (57) admits a solution.

Data Availability

No data is used. No data is available in this work.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] U. N. Katugampola, "A new approach to generalized fractional derivatives," *Bulletin of Mathematical Analysis and Applications*, vol. 6, p. 115, 2014.

[2] U. N. Katugampola, "New approach to a generalized fractional integral," *Applied Mathematics and Computation*, vol. 218, no. 3, pp. 860–865, 2011.

[3] R. Almeida, A. B. Malinowska, and T. Odzijewicz, "Fractional differential equations with dependence on the Caputo-Katugampola derivative," *Journal of Computational and Nonlinear Dynamics*, vol. 11, no. 6, article 061017, 2016.

[4] Y. Arioua, B. Basti, and N. Benhamidouche, "Initial value problem for nonlinear implicit fractional differential equations with Katugampola derivative," *Applied Mathematics E - Notes*, vol. 19, pp. 397–412, 2019.

[5] S. Abbas, M. Benchohra, J. R. Graef, and J. Henderson, *Implicit fractional differential and integral equations: existence and stability*, De Gruyter, Berlin, 2018.

[6] S. Abbas, M. Benchohra, and G. M. N'Guérékata, *Topics in fractional differential equations*, Springer, New York, 2012.

[7] S. Abbas, M. Benchohra, and G. M. N'Guerekata, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.

[8] A. Ashyralyev, "A survey of results in the theory of fractional spaces generated by positive operators," *TWMS Journal of Pure and Applied Mathematics*, vol. 6, no. 2, pp. 129–157, 2015.

[9] Z. Baitiche, C. Derbazi, and M. Benchora, "Caputo fractional differential equations with multipoint boundary conditions by topological degree theory," *Results in Nonlinear Analysis*, vol. 3, no. 4, article 167178, 2020.

[10] F. Si Bachir, A. Said, M. Benbachir, and M. Benchohra, "Hilfer-Hadamard fractional differential equations: existence and attractivity," *Advances in the Theory of Nonlinear Analysis and its Application*, vol. 5, no. 1, article 497, 2020.

[11] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V, Amsterdam, 2006.

[12] S. G. Samko, A. A. Kilbas, and O. I. Marichev, "Fractional Integrals and Derivatives," in *Theory and Applications*, English translation from the Russian, Gordon and Breach, Amsterdam, 1987.

[13] V. E. Tarasov, "Fractional dynamics: application of fractional calculus to dynamics of particles," in *Fields and Media*, Springer, Heidelberg; Higher Education Press, Beijing, 2010.

[14] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

[15] H. Shojaat, H. Afshari, and M. S. Asgari, “A new class of mixed monotone operators with concavity and applications to fractional differential equations,” *TWMS Journal of Applied and Engineering Mathematics*, vol. 11, no. 1, pp. 122–133, 2021.

[16] M. Benchohra, S. Bouriah, and J. J. Nieto, “Terminal value problem for differential equations with Hilfer–Katugampola fractional derivative,” *Symmetry*, vol. 11, no. 5, p. 672, 2019.

[17] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, vol. 46, no. 2, 1998Atti del Seminario Matematico e Fisico dell' Universita di Modena, 1998.

[18] S. Czerwik, “Contraction mappings in b-metric spaces,” *Acta mathematica et informatica universitatis ostraviensis*, vol. 1, pp. 5–11, 1993.

[19] H. Afshari, “Solution of fractional differential equations in quasi-b-metric and b-metric-like spaces,” *Advances in Difference Equations*, vol. 2019, no. 1, Article ID 285, 2019.

[20] H. Afshari and E. Karapinar, “A discussion on the existence of positive solutions of the boundary value problems via ψ-Hilfer fractional derivative on b-metric spaces,” *Advances in Difference Equations*, vol. 2020, no. 1, Article ID 616, 2020.

[21] B. Alqahtani, A. Fulga, F. Jarad, and E. Karapinar, “Nonlinear F -contractions on b -metric spaces and differential equations in the frame of fractional derivatives with Mittag-Leffler kernel,” *Chaos Solitons Fractals*, vol. 128, pp. 349–354, 2019.

[22] M.-F. Bota, L. Guran, and A. Petrusel, “New fixed point theorems on b-metric spaces with applications to coupled fixed point theory,” *Journal of Fixed Point Theory and Applications*, vol. 22, no. 3, 2020.

[23] M. Alghamdi, S. Gulyaz-Ozyurt, and E. Karapınar, “A Note on Extended Z-Contraction,” *Mathematics*, vol. 8, no. 2, p. 195, 2020.

[24] H. Aydi, M. F. Bota, E. Karapınar, and S. Mitrović, “A fixed point theorem for set-valued quasicontractions in b-metric spaces,” *Fixed Point Theory and Applications*, vol. 2012, no. 1, Article ID 187, 2012.

[25] H. Aydi and M. F. Bota, “A common fixed point for weak øcontractions on b-metric spaces,” *Fixed Point Theory*, vol. 13, no. 2, pp. 337–346, 2012.

[26] Ş. Cobzaş and S. Czerwik, “The completion of generalized b-metric spaces and fixed points,” *Fixed Point Theory*, vol. 21, no. 1, pp. 133–150, 2020.

[27] D. Derouiche and H. Ramoul, “New fixed point results for F-contractions of Hardy–Rogers type in b-metric spaces with applications,” *Journal of Fixed Point Theory and Applications*, vol. 22, no. 4, p. 86, 2020.

[28] S. K. Panda, E. Karapinar, and A. Atangana, “A numerical schemes and comparisons for fixed point results with applications to the solutions of Volterra integral equations in dislocatedextendedb- metricspace,” *Alexandria Engineering Journal*, vol. 59, no. 2, pp. 815–827, 2020.

[29] H. Afshari, H. Aydi, and E. Karapinar, “Existence of fixed points of set-valued mappings in b-metric spaces,” *East Asian Mathematical Journal*, vol. 32, no. 3, pp. 319–332, 2016.

[30] H. Afshari, H. Aydi, and E. Karapinar, “On generalized α–ψ–Geraghty contractions on b-metric spaces,” *Georgian Mathematical Journal*, vol. 27, no. 1, pp. 9–21, 2018.