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Research Article

Sharp Bounds of the Coefficient Results for the Family of Bounded Turning Functions Associated with a Petal-Shaped Domain

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The goal of this article is to determine sharp inequalities of certain coefficient-related problems for the functions of bounded turning class subordinated with a petal-shaped domain. These problems include the bounds of first three coefficients, the estimate of Fekete-Szegö inequality, and the bounds of second- and third-order Hankel determinants.

1. Preliminary Concepts

Let the family of holomorphic (or analytic) functions in the region (or domain) of unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be described by the symbol $\mathscr{H}(\mathbb{D})$ and let $\mathscr{A}$ be the subfamily of $\mathscr{H}(\mathbb{D})$ which is defined by

$$\mathscr{A} = \left\{ f \in \mathscr{H}(\mathbb{D}) \colon f(z) = \sum_{k=1}^{\infty} a_k z^k \, (\text{with } a_1 = 1) \right\}. \tag{1}$$

Further, the set $\mathscr{S} \subset \mathscr{A}$ contains all normalized univalent functions in $\mathbb{D}$. For two functions $F_1, F_2 \in \mathscr{H}(\mathbb{D})$, we say that F_1 is subordinate to F_2, written symbolically by $F_1 \prec F_2$, if there exists a Schwarz function v with $v(0) = 0$ and $|v(z)| < 1$ that is analytic in $\mathbb{D}$ such that $f(z) = g(v(z)), z \in \mathbb{D}$. However, if F_2 is univalent in $\mathbb{D}$, then the following relation holds:

$$F_1(z) \prec F_2(z), (z \in \mathbb{D}) \Leftrightarrow F_1(0) = F_2(0) \text{ and } F_1(\mathbb{D}) \subset F_2(\mathbb{D}). \tag{2}$$

In geometric function theory, the most basic and important subfamilies of the set $\mathscr{S}$ are the family $\mathscr{S}^*$ of starlike functions and the family $\mathscr{C}$ of convex functions which are defined as follows:

$$\mathscr{C} = \left\{ f \in \mathscr{A} : \frac{\left(zf'(z)\right)'}{f'(z)} \prec \Lambda(z) \, (z \in \mathbb{D}) \right\} \coloneqq \mathscr{C}(\Lambda),$$
$$\mathscr{S}^* = \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \Lambda(z) \, (z \in \mathbb{D}) \right\} \coloneqq \mathscr{S}^*(\Lambda), \tag{3}$$

with

$$\Lambda(z) = 1 + 2\sum_{n=2}^{\infty} z^n \coloneqq \frac{1+z}{1-z}, (z \in \mathbb{D}). \tag{4}$$

By varying the function $\Lambda(z)$ in (18), we get some subfamilies of the set $\mathscr{S}^*$ which have significant geometric sense. For example,

(i) If we take $\Lambda(z) = (1 + Mz)/(1 + Nz)$ with $-1 \le N < M \le 1$, then the deduced family

$$\mathcal{S}^*[M,N] \equiv \mathcal{S}^*\left(\frac{1+Mz}{1+Nz}\right) \quad (5)$$

is described by the functions of the Janowski starlike family established in [1] and later studied in different directions in [2, 3]

(ii) The family $\mathcal{S}_L^* \equiv \mathcal{S}^*(\Lambda(z))$ with $\Lambda(z)=\sqrt{1+z}$ was developed in [4] by Sokól and Stankiewicz. The image of the function $\Lambda(z)=\sqrt{1+z}$ demonstrates that the image domain is bounded by the Bernoulli's lemniscate right-half plan specified by $|w^2-1|<1$

(iii) By selecting $\Lambda(z)=1+\sin z$, the class $\mathcal{S}^*(\Lambda(z))$ lead to the family $\mathcal{S}_{\sin}^*$ which was explored in [5] while $\mathcal{S}_e^* \equiv \mathcal{S}^*(e^z)$ has been produced in the article [6] and later studied in [7]

(iv) The family $\mathcal{S}_c^* := \mathcal{S}^*(\Lambda(z))$ with $\Lambda(z)=1+(4/3)z+(2/3)z^2$ was contributed by Sharma and his coauthors [8] which contains function $f \in \mathcal{A}$ such that $z f'(z)/f(z)$ is located in the region bounded by the cardioid given by

$$\left(9x^2+9y^2-18x+5\right)^2-16\left(9x^2+9y^2-6x+1\right)=0 \quad (6)$$

(v) The family $\mathcal{S}_R^* \equiv \mathcal{S}^*(\Lambda(z))$ with $\Lambda(z)=1+(z/(\sqrt{2}+1))((\sqrt{2}+1+z)/(\sqrt{2}+1-z))$ is studied in [9] while $\mathcal{S}_{\cos}^* := \mathcal{S}^*(\cos(z))$ and $\mathcal{S}_{\cosh}^* := \mathcal{S}^*(\cosh(z))$ were recently examined by Bano and Raza [10] and Alotaibi et al. [11], respectively

(vi) If we consider $\Lambda(z)=1\sinh^{-1}z$, then the class $\mathcal{S}_\rho^* := \mathcal{S}^*(1+\sinh^{-1}z)$ was provided by Kumar and Arora [12] and is defined as a function $f \in \mathcal{A}$ which is in the family $\mathcal{S}_\rho^*$ if (18) holds for the function $\Lambda(z)=\rho(z)$, where

$$\rho(z)=1+\sinh^{-1}z \quad (7)$$

Clearly, the function $\rho(z)$ is a multivalued function and has the branch cuts about the line segments $(-i\infty,-i)\cup(i,i\infty)$, on the imaginary axis, and hence, it is holomorphic in $\mathbb{D}$. In a geometric point of view, the function $\rho(z)$ maps the unit disc $\mathbb{D}$ onto a petal-shaped region Ω_ρ,

$$\Omega_\rho=\{w\in\mathbb{C}:|\sinh(w-1)|<1\}. \quad (8)$$

Using this idea, we now consider a subfamily $\mathcal{BT}_s$ of analytic functions as

$$\mathcal{BT}_s=\left\{f\in\mathcal{A}: f'(z)\prec\tilde{\Lambda}(z),\text{ and }\tilde{\Lambda}(z)\text{ is given by (8)}\right\}. \quad (9)$$

If we take the function $\Lambda(z)$, given by (4), instead of $\tilde{\Lambda}(z)$ in (9), we get the familiar class $\mathcal{R}$ of bounded turning functions. From the statement of the Nashiro-Warschowski theorem, it follows that the functions in $\mathcal{R}$ are univalent in $\mathbb{D}$. The properties of this class was studied extensively by the researchers, see [13–16].

The Hankel determinant $\mathcal{HD}_{q,n}(f)$(with $q,n\in\mathbb{N}=\{1,2,\cdots\}$ and $a_1=1$) for a function $f\in\mathcal{S}$ of the series form (1) was given by Pommerenke [17, 18] as

$$\mathcal{HD}_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \quad (10)$$

Specifically, the first-, second-, and third-order Hankel determinants, respectively, are

$$\begin{aligned} \mathcal{HD}_{2,1}(f) &= \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3-a_2^2, \\ \mathcal{HD}_{2,2}(f) &= \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4-a_3^2, \\ \mathcal{HD}_{3,1}(f) &= \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \\ &= a_3\left(a_2a_4-a_3^2\right)-a_4\left(a_4-a_2a_3\right)+a_5\left(a_3-a_2^2\right). \end{aligned} \quad (11)$$

In literature, there are relatively few findings in relation to the Hankel determinant for the function f belongs to the general family $\mathcal{S}$. For the function $f\in\mathcal{S}$, the best established sharp inequality is $|\mathcal{HD}_{2,n}(f)|\le\lambda\sqrt{n}$, where λ is absolute constant, which is due to Hayman [19]. Further, for the same class $\mathcal{S}$, it was obtained in [20] that

$$\begin{aligned} \left|\mathcal{HD}_{2,2}(f)\right| &\le \lambda,\text{ for } 1\le\lambda\le\frac{11}{3}, \\ \left|\mathcal{HD}_{3,1}(f)\right| &\le \mu,\text{ for } \frac{4}{9}\le\mu\le\frac{32+\sqrt{285}}{15}. \end{aligned} \quad (12)$$

The growth of $|\mathcal{HD}_{q,n}(f)|$ has often been evaluated for different subfamilies of the set $\mathcal{S}$ of univalent functions. For example, the sharp bound of $|\mathcal{HD}_{2,2}(f)|$, for the subfamilies $\mathcal{C}$, $\mathcal{S}^*$, and $\mathcal{R}$ of the set $\mathcal{S}$, was measured by Janteng et al. [21, 22]. These bounds are

$$\left|\mathscr{H}\mathscr{D}_{2,2}(f)\right| \le \begin{cases} \frac{1}{8}, & \text{for } f \in \mathscr{C}, \\ 1, & \text{for } f \in \mathscr{S}^*, \\ \frac{4}{9}, & \text{for } f \in \mathscr{R}. \end{cases} \quad (13)$$

The exact bound for the collection of close-to-convex functions of such a specific determinant is still unavailable (see [23]). On the other hand, for the set of Bazilevič functions, the best estimate of $|\mathscr{H}\mathscr{D}_{2,2}(f)|$ was proved by Krishna and RamReddy [24]. For more work on $|\mathscr{H}\mathscr{D}_{2,2}(f)|$, see References [25–29].

It is very obvious from the formulae provided in (11) that the estimate of $|\mathscr{H}\mathscr{D}_{3,1}(f)|$ is far more complicated compared with finding the bound of $|\mathscr{H}\mathscr{D}_{2,2}(f)|$. In the first paper on $|\mathscr{H}\mathscr{D}_{3,1}(f)|$, published in 2010, Babalola [30] obtained the upper bound of $|\mathscr{H}\mathscr{D}_{3,1}(f)|$ for the families of $\mathscr{C}$, $\mathscr{S}^*$, and $\mathscr{R}$. He obtained the following bounds:

$$\left|\mathscr{H}\mathscr{D}_{3,1}(f)\right| \le \begin{cases} 0.714\cdots, & \text{for } f \in \mathscr{C}, \\ 16, & \text{for } f \in \mathscr{S}^*, \\ 0.742\cdots, & \text{for } f \in \mathscr{R}. \end{cases} \quad (14)$$

Later on, using the same methodology, some other authors [31–35] published their work concerning $|\mathscr{H}\mathscr{D}_{3,1}(f)|$ for different subfamilies of analytic and univalent functions. In 2017, Zaprawa [36] improved Babalola's [30] results by applying a new technique which is given as

$$\left|\mathscr{H}\mathscr{D}_{3,1}(f)\right| \le \begin{cases} \frac{49}{540}, & \text{for } f \in \mathscr{C}, \\ 1, & \text{for } f \in \mathscr{S}^*, \\ \frac{41}{60}, & \text{for } f \in \mathscr{R}. \end{cases} \quad (15)$$

He argues that such limits are indeed not the best ones. After that, in 2018, Kwon et al. [37] enhanced Zaprawa's bound for $f \in \mathscr{S}^*$ and showed that $|\mathscr{H}\mathscr{D}_{3,1}(f)| \le 8/9$, but it is still not the best possible. The firstly examined papers in which the authors obtained the sharp bounds of $|\mathscr{H}\mathscr{D}_{3,1}(f)|$ came to the reader's hands in 2018. Such papers have been written by Kowalczyk et al. [38] and Lecko et al. [39]. These results are given as

$$\left|\mathscr{H}\mathscr{D}_{3,1}(f)\right| \le \begin{cases} \frac{4}{135}, & \text{for } f \in \mathscr{C}, \\ \frac{1}{9}, & \text{for } f \in \mathscr{S}^*\left(\frac{1}{2}\right), \end{cases} \quad (16)$$

where $\mathscr{S}^*(1/2)$ indicates the starlike function family of order 1/2. We would also like to acknowledge the research provided by Mahmood et al. [40] in which they examined the third Hankel determinant in the q-analog for a subfamily of starlike functions and for more contribution of such type families, see [41, 42]. In the present article, our aim is to calculate the sharp bounds of some of the problems related to Hankel determinant for the class $\mathscr{B}\mathscr{T}_s$ of bounded turning functions connected with a petal-shaped domain.

2. A Set of Lemmas

Definition 1. Let $\mathscr{P}$ represent the class of all functions p that are holomorphic in $\mathbb{D}$ with $\mathfrak{Re}(p(z)) > 0$ and has the series representation

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \ (z \in \mathbb{D}). \quad (17)$$

For the proofs of our key findings, we need the following lemma. It contains the well-known formula for c_2, see [43], the formula for c_3 due to Libera and Zlotkiewicz [44], and the formula for c_4 proved in [45].

Lemma 2. *Let $p \in \mathscr{P}$ has the series form ((17)). Then, for x, $\sigma, \rho \in \bar{\mathbb{D}} = \mathbb{D} \cup \{1\}$,*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right), \quad (18)$$

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)\sigma, \quad (19)$$

$$8c_4 = c_1^4 + \left(4 - c_1^2\right)x\left[c_1^2\left(x^2 - 3x + 3\right) + 4x\right] - 4\left(4 - c_1^2\right) \cdot \left(1 - |x|^2\right)\left[c(x - 1)z + \bar{x}\sigma^2 - \left(1 - |\sigma|^2\right)\rho\right]. \quad (20)$$

Lemma 3. *If $p \in \mathscr{P}$ and has the series form ((17)), then*

$$|c_{n+k} - \mu c_n c_k| \le 2 \max\left(1, |2\mu - 1|\right), \quad (21)$$

$$|c_n| \le 2 \text{ for } n \ge 1, \quad (22)$$

$$\left|Jc_1^3 - Kc_1c_2 + Lc_3\right| \le 2|J| + |K - 2J| + 2|J - K + L|, \quad (23)$$

with $J, K, L, \mu \in \mathbb{C}$ and for $B \in [0, 1]$ with $B(2B - 1) \le D \le B$, we have

$$\left|c_3 - 2Bc_1c_2 + Dc_1^3\right| \le 2. \quad (24)$$

The inequalities (21), (22), (23), and (24) in the above lemma are taken from [43, 46], [47–49], and [50], respectively.

3. Coefficient Inequalities for the Class $\mathscr{B}\mathscr{T}_s$

We begin this section by finding the absolute values of the first three initial coefficients for the function of class $\mathscr{B}\mathscr{T}_s$.

Theorem 4. *If $f \in \mathscr{BT}_s$ and has the series representation ((1)), then*

$$|a_2| \leq \frac{1}{2},$$
$$|a_3| \leq \frac{1}{3}, \quad (25)$$
$$|a_4| \leq \frac{1}{4}.$$

These bounds are sharp.

Proof. Let $f \in \mathscr{BT}_s$. Then, (9) can be written in the form of the Schwarz function as

$$f'(z) = 1 + \sinh^{-1}(w(z)), (z \in \mathbb{D}). \quad (26)$$

Now, if $p \in \mathscr{P}$, then it may be written in terms of the Schwarz function w by

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \quad (27)$$

equivalently,

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}. \quad (28)$$

From (1), we easily get

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots. \quad (29)$$

By simplification and using the series expansion (28), we obtain

$$1 + \sinh^{-1}(w(z)) = 1 + \left(\frac{1}{2}c_1\right)z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z^2 + \left(\frac{1}{2}c_3 + \frac{5}{48}c_1^3 - \frac{1}{2}c_1 c_2\right)z^3 \cdot \left(\frac{1}{2}c_4 - \frac{1}{4}c_2^2 - \frac{1}{32}c_1^4 + \frac{5}{16}c_1^2 c_2 - \frac{1}{2}c_1 c_3\right)z^4 + \cdots. \quad (30)$$

Comparing (29) and (30),we get

$$a_2 = \frac{1}{4}c_1, \quad (31)$$

$$a_3 = \frac{1}{3}\left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right), \quad (32)$$

$$a_4 = \frac{1}{4}\left(\frac{1}{2}c_3 + \frac{5}{48}c_1^3 - \frac{1}{2}c_1 c_2\right), \quad (33)$$

$$a_5 = \frac{1}{5}\left(\frac{1}{2}c_4 - \frac{1}{4}c_2^2 - \frac{1}{32}c_1^4 + \frac{5}{16}c_1^2 c_2 - \frac{1}{2}c_1 c_3\right). \quad (34)$$

For a_2, implementing (22) in (31), we obtain

$$|a_2| \leq \frac{1}{2}. \quad (35)$$

For a_3, reordering (32), we get

$$a_3 = \frac{1}{6}\left(c_2 - \frac{1}{2}c_1 c_1\right), \quad (36)$$

and using (21), we have

$$|a_3| \leq \frac{1}{3}. \quad (37)$$

For a_4, we can rewrite (33) as

$$|a_4| = \frac{1}{4}\left|\frac{5}{48}c_1^3 - \frac{1}{2}c_1 c_2 + \frac{1}{2}c_3\right|. \quad (38)$$

Application of triangle inequality plus (23), we get

$$|a_4| \leq \frac{1}{4}\left[2\left|\frac{5}{48}\right| + 2\left|\frac{1}{2} - 2\left(\frac{5}{48}\right)\right| + 2\left|\frac{5}{48} - \frac{1}{2} + \frac{1}{2}\right|\right]. \quad (39)$$

By simple calculations, we obtain

$$|a_4| \leq \frac{1}{4}. \quad (40)$$

These outcomes are sharp. For this, we consider a function

$$f_n'(z) = 1 + \sinh^{-1}(z^n), \text{for } n = 1, 2, 3. \quad (41)$$

Thus, we have

$$f_1(z) = \int_0^z \left(1 + \sinh^{-1}(t)\right)dt = z + \frac{1}{2}z^2 - \frac{1}{24}z^4 + \cdots,$$
$$f_2(z) = \int_0^z \left(1 + \sinh^{-1}(t^2)\right)dt = z + \frac{1}{3}z^3 - \frac{1}{42}z^7 + \cdots,$$
$$f_3(z) = \int_0^z \left(1 + \sinh^{-1}(t^3)\right)dt = z + \frac{1}{4}z^4 - \frac{1}{60}z^{10} + \cdots. \quad (42)$$

Now, we discussed about the Hankel determinant problem, which is explicitly related to the Fekete-Szegö functional which is an extraordinary instance of the Hankel determinant.

Theorem 5. *If f of the form ((1)) belongs to $\mathscr{BT}_s$, then*

$$\left|a_3 - \gamma a_2^2\right| \leq \max\left\{1, \frac{3|\gamma|}{4}\right\}, \text{for } \gamma \in \mathbb{C}. \quad (43)$$

This inequality is sharp.

Proof. Employing (31) and (32), we may write

$$\left|a_3-\gamma a_2^2\right|=\left|\frac{c_2}{6}-\frac{c_1^2}{12}-\gamma\frac{c_1^2}{16}\right|. \quad (44)$$

By rearranging, it yields

$$\left|a_3-\gamma a_2^2\right|=\frac{1}{6}\left|\left(c_2-\left(\frac{3\gamma+4}{8}\right)c_2\right)\right|. \quad (45)$$

Application of (21) leads us to

$$\left|a_3-\gamma a_2^2\right|\leq\frac{1}{6}\max\left\{2,2\left|2\frac{3\gamma+4}{8}-1\right|\right\}. \quad (46)$$

After the simplification, we obtain

$$\left|a_3-\gamma a_2^2\right|\leq\frac{1}{3}\max\left\{1,\frac{3|\gamma|}{4}\right\}. \quad (47)$$

The required result is sharp and is determined by

$$f_2(z)=\int_0^z\left(1+\sinh^{-1}\left(t^2\right)\right)dt=z+\frac{1}{3}z^3-\frac{1}{42}z^7+\cdots. \quad (48)$$

Theorem 6. *If f has the form ((1)) belongs to $\mathscr{BT}_s$, then*

$$\left|a_2a_3-a_4\right|\leq\frac{1}{4}. \quad (49)$$

This inequality is sharp.

Proof. Using (31), (32), and (33), we have

$$\left|a_2a_3-a_4\right|=\frac{1}{8}\left|c_3-2\left(\frac{2}{3}\right)c_1c_2+\frac{7}{24}c_1^3\right|. \quad (50)$$

From (24), we have

$$0\leq B=\frac{2}{3}\leq 1, B=\frac{2}{3}\geq D=\frac{7}{24}, \quad (51)$$

and also satisfy

$$B(2B-1)=\frac{2}{3}\left(\frac{4}{3}-1\right)\leq D=\frac{7}{24}. \quad (52)$$

Thus, by using (24), we have

$$\left|a_2a_3-a_4\right|\leq\frac{1}{4}. \quad (53)$$

Equality is achieved by using

$$f_3(z)=\int_0^z\left(1+\sinh^{-1}\left(t^3\right)\right)dt=z+\frac{1}{4}z^4-\frac{1}{60}z^{10}+\cdots. \quad (54)$$

Next, we will determine the second-order Hankel determinant $\mathscr{HD}_{2,2}(f)$ for $f\in\mathscr{BT}_s$.

Theorem 7. *If f belongs to $\mathscr{BT}_s$, then the second Hankel determinant*

$$\left|\mathscr{HD}_{2,2}(f)\right|=\left|a_2a_4-a_3^2\right|\leq\frac{1}{9}. \quad (55)$$

This result is the best possible.

Proof. From (31), (32), and (33), we have

$$\mathscr{HD}_{2,2}(f)=-\frac{1}{2304}c_1^4-\frac{1}{288}c_1^2c_2+\frac{1}{32}c_1c_3-\frac{1}{36}c_2^2. \quad (56)$$

Using (18) and (19) to express c_2 and c_3 in terms of c_1 and noting that without loss in generality we can write $c_1=c$, with $0\leq c\leq 2$, we obtain

$$\left|\mathscr{HD}_{2,2}(f)\right|=\left|-\frac{1}{768}c^4-\frac{1}{128}c^2\left(4-c^2\right)x^2-\frac{1}{144}\left(4-c^2\right)^2x^2+\frac{1}{64}c\left(4-c^2\right)\left(1-|x|^2\right)z\right|, \quad (57)$$

with the aid of the triangle inequality and replacing $|z|\leq 1$, $|x|=b$, with $b\leq 1$. So,

$$\left|\mathscr{HD}_{2,2}(f)\right|\leq\frac{1}{768}c^4+\frac{1}{128}c^2\left(4-c^2\right)b^2+\frac{1}{144}\left(4-c^2\right)^2b^2+\frac{1}{64}b\left(4-c^2\right)\left(1-b^2\right)\coloneqq\phi(c,b). \quad (58)$$

It is a simple exercise to show that $\phi'(c,b)\geq 0$ on $[0,1]$, so that $\phi(c,b)\leq\phi(c,1)$. Putting $b=1$ gives

$$\left|\mathscr{HD}_{2,2}(f)\right|\leq\frac{1}{768}c^4+\frac{1}{128}c^2\left(4-c^2\right)+\frac{1}{144}\left(4-c^2\right)^2\coloneqq\phi(c,1). \quad (59)$$

Also, $\phi'(c,1)<0$, and so $\phi(c,1)$ is a decreasing function. Thus, the maximum value at $c=0$ is

$$\left|\mathscr{HD}_{2,2}(f)\right|\leq\frac{16}{144}=\frac{1}{9}. \quad (60)$$

The required second Hankel determinant is sharp and is obtained by

$$f_2(z)=\int_0^z\left(1+\sinh^{-1}\left(t^2\right)\right)dt=z+\frac{1}{3}z^3-\frac{1}{42}z^7+\cdots. \quad (61)$$

4. Third-Order Hankel Determinant

We will now determine the third-order Hankel determinant $\mathscr{HD}_{3,1}(f)$ for $f\in\mathscr{BT}_s$.

Theorem 8. *If f belongs to $\mathscr{BT}_s$, then the third Hankel determinant*

$$\left|\mathscr{HD}_{3,1}(f)\right| \le \frac{1}{16}. \tag{62}$$

This result is sharp.

Proof. The third Hankel determinant can be written as

$$\mathscr{HD}_{3,1}(f) = \left(a_2a_4 - a_3^2\right)a_3 - \left(a_1a_4 - a_2a_3\right)a_4 + \left(a_1a_3 - a_2^2\right)a_5. \tag{63}$$

After simplification of the above equation, we have

$$\mathscr{HD}_{3,1}(f) = 2a_2a_3a_4 - a_3^3 - a_4^2 + a_3a_5 - a_2^2a_5. \tag{64}$$

Let $c_1 = c$ and putting the estimations of a_i's from (31), (32), (33), and (34), we get

$$\begin{aligned}\mathscr{HD}_{3,1}(f) = \frac{1}{552960}&\left(-151c^6 + 144c^4c_2 + 1584c^3c_3 - 768c^2c_2^2\right.\\ &\left.- 8064c^2c_4 + 13824cc_2c_3 - 7168c_2^3 + 9216c_2c_4 - 8640c_3^2\right).\end{aligned} \tag{65}$$

To simplify computation, let $t = 4 - c^2$ in (18), (19), and (20). Now, using the simplified form of these formulae, we obtain

$$\begin{aligned}144c^4c_2 &= 72\left(c^6 + c^4tx\right),\\ 1584c^3c_3 &= 396c^6 + 792c^4tx - 396c^4tx^2 + 792c^3t\left(1 - |x|^2\right)\sigma,\\ 768c^2c_2^2 &= 192c^6 + 384c^4tx + 192c^2t^2x^2,\\ 8064c^2c_4 &= 1008c^4tx^3 - 4032c^3tx\left(1 - |x|^2\right)\sigma\\ &\quad - 4032c^2t\bar{x}\left(1 - |x|^2\right)\sigma^2 - 3024c^4tx^2\\ &\quad + 4032c^2t\left(1 - |x|^2\right)\left(1 - |\sigma|^2\right)\rho + 4032c^3t\left(1 - |x|^2\right)\sigma\\ &\quad + 3024c^4tx + 1008c^6 + 4032c^2tx^2,\end{aligned}$$

$$\begin{aligned}13824cc_2c_3 &= -1728c^2t^2x^3 - 1728c^4tx^2 + 3456ct^2x\left(1 - |x|^2\right)\sigma\\ &\quad + 3456c^2t^2x^2 + 3456c^3t\left(1 - |x|^2\right)\sigma + 5184c^4tx\\ &\quad + 1728c^6,\end{aligned}$$

$$\begin{aligned}7168c_2^3 &= 896t^3x^3 + 2688c^2t^2x^2 + 2688c^4tx + 896c^6,\\ 9216c_2c_4 &= 2304c^2tx^2 + 2304t^2x^3 + 576c^6 + 2304c^4tx\\ &\quad + 2304c^3t\left(1 - |x|^2\right)\sigma + 2304c^2t\left(1 - |x|^2\right)\left(1 - |\sigma|^2\right)\rho\\ &\quad + 1728c^2t^2x^2 + 2304ct^2x\left(1 - |x|^2\right)\sigma\\ &\quad + 2304t^2x\left(1 - |x|^2\right)\left(1 - |\sigma|^2\right)\rho - 1728c^4tx^2\\ &\quad - 2304c^2t\bar{x}\left(1 - |x|^2\right)\sigma^2 - 2304c^3tx\left(1 - |x|^2\right)\sigma\\ &\quad - 1728c^2t^2x^3 - 2304t^2x\bar{x}\left(1 - |x|^2\right)\sigma^2 + 576c^4tx^3\\ &\quad + 576c^2t^2x^4 - 2304ct^2x^2\left(1 - |x|^2\right)\sigma,\end{aligned}$$

$$\begin{aligned}8640c_3^2 &= 540c^2t^2x^4 - 2160ct^2x^2\left(1 - |x|^2\right)\sigma - 2160c^2t^2x^3\\ &\quad - 1080c^4tx^2 + 2160t^2\left(1 - |x|^2\right)^2\sigma^2\\ &\quad + 4320ct^2x\left(1 - |x|^2\right)\sigma + 2160c^2t^2x^2 + 2160c^3t\left(1 - |x|^2\right)\sigma\\ &\quad + 2160c^4tx + 540c^6.\end{aligned} \tag{66}$$

Substituting these expressions in (65), by simple but too long computation,

$$\begin{aligned}\mathscr{HD}_{3,1}(f) = \frac{1}{552960}&\left\{-15c^6 + 2304t^2x^3 - 896t^3x^3 - 1728c^2tx^2\right.\\ &- 432c^4tx^3 + 252c^4tx^2 + 96c^4tx + 36c^2t^2x^4\\ &- 1296c^2t^2x^3 + 144c^2t^2x^2 - 2160t^2\left(1 - |x|^2\right)^2\sigma^2\\ &+ 360c^3t\left(1 - |x|^2\right)\sigma + 1728c^3tx\left(1 - |x|^2\right)\sigma\\ &+ 1728c^2t\bar{x}\left(1 - |x|^2\right)\sigma^2 - 1728c^2t\left(1 - |x|^2\right)\left(1 - |\sigma|^2\right)\rho\\ &- 144ct^2x^2\left(1 - |x|^2\right)\sigma - 2304t^2x\bar{x}\left(1 - |x|^2\right)\sigma^2 + 1440ct^2x\left(1 - |x|^2\right)\sigma\\ &\left.+ 2304t^2x\left(1 - |x|^2\right)\left(1 - |\sigma|^2\right)\rho\right\}.\end{aligned} \tag{67}$$

Since $t = 4 - c^2$,

$$\mathscr{HD}_{3,1}(f) = \frac{1}{552960}\left(v_1(c,x) + v_2(c,x)\sigma + v_3(c,x)\sigma^2 + \Psi(c,x,\sigma)\rho\right), \tag{68}$$

where $\rho, x, \sigma \in \bar{\mathbb{D}}$, and

$$\begin{aligned}v_1(c,x) &= -15c^6 + \left(4 - c^2\right)\left[\left(4 - c^2\right)\left(-1280x^3 - 400c^2x^3 + 36c^2x^4\right.\right.\\ &\quad\left.\left. + 144c^2x^2\right) - 1728c^2x^2 - 432c^4x^3 + 252c^4x^2 + 96c^4x\right],\\ v_2(c,x) &= \left(4 - c^2\right)\left(1 - |x|^2\right)\left[\left(4 - c^2\right)\left(1440cx - 144cx^2\right)\right.\\ &\quad\left. + 1728c^3x + 360c^3\right],\\ v_3(c,x) &= \left(4 - c^2\right)\left(1 - |x|^2\right)\left[\left(4 - c^2\right)\left(-144x^2 - 2160\right) + 1728c^2\bar{x}\right],\\ \Psi(c,x,z) &= \left(4 - c^2\right)\left(1 - |x|^2\right)\left(1 - |\sigma|^2\right)\left[-1728c^2 + 2304x\left(4 - c^2\right)\right].\end{aligned} \tag{69}$$

Now, by using $|x| = x, |\sigma| = y$ and utilizing the fact $|\rho| \le 1$, we get

$$\begin{aligned}\left|\mathscr{HD}_{3,1}(f)\right| \le \frac{1}{552960}&\left(|v_1(c,x)| + |v_2(c,x)|y + |v_3(c,x)|y^2\right.\\ &\left.+ |\Psi(c,x,\sigma)|\right),\end{aligned} \tag{70}$$

$$\le \frac{1}{552960}G(c,x,y), \tag{71}$$

where

$$G(c,x,y) = g_1(c,x) + g_2(c,x)y + g_3(c,x)y^2 + g_4(c,x)\left(1 - y^2\right), \tag{72}$$

with

$$\begin{aligned}g_1(c,x) &= 15c^6 + (4-c^2)\left[(4-c^2)\left(1280x^3 + 400c^2x^3 + 36c^2x^4 + 144c^2x^2\right) + 1728c^2x^2 + 432c^4x^3 + 252c^4x^2 + 96c^4x\right],\\ g_2(c,x) &= (4-c^2)(1-x^2)\left[(4-c^2)\left(1440cx + 144cx^2\right) + 1728c^3x + 360c^3\right],\\ g_3(c,x) &= (4-c^2)(1-x^2)\left[(4-c^2)\left(144x^2 + 2160\right) + 1728c^2\bar{x}\right],\\ g_4(c,x) &= (4-c^2)(1-x^2)\left[1728c^2 + 2304x(4-c^2)\right]. \end{aligned} \tag{73}$$

Clearly, in the last four functions, $g_2(c,x)$ and $g_3(c,x)$ are nonnegative in the interval $[0,2]\times[0,1]$. So from (70) along with $y=|\sigma|$ in the interval $[0,1]$, we get

$$G(c,x,y) = G(c,x,1). \tag{74}$$

Therefore,

$$G(c,x,1) = g_1(c,x) + g_2(c,x) + g_3(c,x) + g_4(c,x) = F(c,x). \tag{75}$$

Here, we shall maximize $F(c,x)$ over the interval $[0,2]\times[0,1]$. For this purpose, we consider possible cases:

(i) By taking $x=0$, we have

$$F(c,0) = 15c^6 - 360c^5 + 2160c^4 + 144c^3 - 17280c^2 + 34560 = f_1(c). \tag{76}$$

Since $f_1'(c)<0$ in $[0,2]$, so, $f_1(c)$ is decreasing over $[0,2]$. Thus, $f_1(c)$ has its maxima at $c=0$ which is equal to 34560

(ii) By taking $x=1$, we have

$$F(c,1) = -185c^6 - 1968c^4 + 5952c^2 + 20480 = f_2(c). \tag{77}$$

As $f_2'(c)=0$ has its optimum point at $c_0=0.674788$. Therefore, $f_2(c)$ is an increasing function for $c\le c_0$ and decreasing for $c_0\le c$. Hence, $f_2(c)$ has its maxima at $c=c_0$ that is approximately equal to 22764.68167

(iii) By taking $c=0$, we have

$$F(0,x) = -2304x^4 + 20480x^3 - 32256x^2 + 34560 = f_3(x). \tag{78}$$

As $f_3'(x)<0$ over $[0,1]$, so, $f_3(x)$ is decreasing over $[0,1]$. Thus, $f_3(x)$ has its maxima at $x=0$ which is equal to 34560. Now, by taking $c=2$, we obtain

$$F(2,x) \le 960 \tag{79}$$

(iv) When (c,x) lies in $[0,2]\times[0,1]$. Then, some simple computation shows that there exists real solution for these equations

$$\begin{aligned}\frac{\partial F(c,x)}{\partial x} &= 0,\\ \frac{\partial F(c,x)}{\partial c} &= 0,\end{aligned} \tag{80}$$

lies inside this region $[0,2]\times[0,1]$ at $(c,x)\approx(0,0)$. Consequently, we obtain

$$F(c,x) = 34560. \tag{81}$$

Thus, from all the above cases, we conclude that

$$G(c,x,y) \le 34560 \text{ on } [0,2]\times[0,1]\times[0,1]. \tag{82}$$

From (71), we can write

$$\left|\mathscr{H}\mathscr{D}_{3,1}(f)\right| \le \frac{1}{552960}G(c,x,y) \le \frac{1}{16} \approx 0.0625. \tag{83}$$

If $f\in\mathscr{B}\mathscr{T}_s$, then the equality is obtained from the function

$$f_3(z) = \int_0^z \left(1+\sinh^{-1}\left(t^3\right)\right)dt = z + \frac{1}{4}z^4 - \frac{1}{60}z^{10} + \cdots. \tag{84}$$

5. Conclusion

For the family of bounded turning functions connected with a petal-shaped domain, we studied the problems such as the bounds of the first three coefficients, the estimate of the Fekete-Szegö inequality, and the bounds of Hankel determinants of order three. All the bounds which we investigated are sharp.

Data Availability

We have not used any data.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally in this research paper.

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