Research Article

On Decomposition Formulas Related to the Gaussian Hypergeometric Functions in Three Variables

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In this paper, by using certain inverse pairs of symbolic operators introduced by Choi and Hasanov in 2011, we establish several decomposition formulas associated with the Gaussian triple hypergeometric functions. Some transformation formulas for these functions have also been obtained.

1. Introduction

The use of many mathematical operations goes beyond the class of elementary functions. The calculation of integrals, the summation of series, and solution of algebraic, transcendental, difference, and differential equations and their systems require expanding the class of studied functions. The development of the concept of a function, going in parallel with the development of the concepts of number and space, led to the emergence of new hypergeometric functions of many complex variables.

The great success of the theory of hypergeometric functions in a single variable has stimulated the development of the theory of hypergeometric functions in several variables by the fact that the solutions of partial differential equations arising in many applied problems of mathematical physics are given in terms of such hypergeometric functions (see, e.g., [1–6]). Multiple hypergeometric functions occur in numerous problems in hydrodynamics, control theory, electrical current, heat conduction, and classical and quantum mechanics (see, for details, [7–10], and the references cited therein). In view of theory and applications, a large number of hypergeometric functions have been developed; for example, as many as 205 hypergeometric functions are recorded in

the monograph [11]. In particular, we recall the Gaussian functions $F_{11a}, F_{11c}, F_{15a}, F_{17a}, F_{17b}, F_{17c}, F_{18a}, F_{19a}, F_{20a}$, and F_{23c} in three variables defined by (see [11])

$$F_{11a}(a_1, a_2, a_3, a_4; c_1, c_2; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{n+p}(a_3)_m(a_4)_p}{(c_1)_{m+n}(c_2)_p} \frac{x^m y^n z^p}{m! n! p!}, \quad (1)$$

$$F_{11c}(a_1, a_2, a_3, b; c; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{n+p}(a_3)_p(b)_{m-p}}{(c_1)_{m+n}} \frac{x^m y^n z^p}{m! n! p!}, \quad (2)$$

$$F_{15a}(a_1, a_2, a_3; c_1, c_2; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{n+p}(a_3)_{p+m}}{(c_1)_{m+n}(c_2)_p} \frac{x^m y^n z^p}{m! n! p!}, \quad (3)$$

$$F_{17a}(a_1, a_2, a_3, a_4; c_1, c_2, c_3; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p}(a_2)_m(a_3)_n(a_4)_p}{(c_1)_m(c_2)_n(c_3)_p} \frac{x^m y^n z^p}{m! n! p!}, \quad (4)$$

$$F_{17b}(a_1, a_2, a_3, b; c_1, c_2; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p} (a_2)_n (a_3)_p (b)_{m-n} x^m y^n z^p}{(c_1)_m (c_2)_p m! n! p!}, \quad (5)$$

$$F_{17c}(a_1, a_2, b_1, b_2; c; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p} (a_2)_p (b_1)_{m-n} (b_2)_{n-m} x^m y^n z^p}{(c)_p m! n! p!}, \quad (6)$$

$$F_{18a}(a_1, a_2, a_3, a_4; c_1, c_2; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p} (a_2)_m (a_3)_n (a_4)_p x^m y^n z^p}{(c_1)_{m+n} (c_2)_p m! n! p!}, \quad (7)$$

$$F_{19a}(a_1, a_2, a_3, a_4; c; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p} (a_2)_m (a_3)_n (a_4)_p x^m y^n z^p}{(c)_{m+n+p} m! n! p!}, \quad (8)$$

$$F_{20a}(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p} (a_2)_{m+n} (a_3)_p x^m y^n z^p}{(c_1)_m (c_2)_n (c_3)_p m! n! p!}, \quad (9)$$

$$F_{23c}(a_1, a_2, a_3, b_1, b_2; c; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m (a_2)_m (a_3)_n (b_1)_{n-p} (b_2)_{2p-m} x^m y^n z^p}{(c)_n m! n! p!}. \quad (10)$$

Here, $(a)_m$ denotes the Pochhammer symbol given as

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = a(a+1) \cdots (a+m-1) \quad (m \in \mathbb{N} := \{1, 2, \dots\}), \quad (a)_0 = 1. \quad (11)$$

Burchnall and Chaundy presented the inverse pairs of symbolic operators ∇ and Δ [12, 13] (also see [14]) by means of which they established several decomposition formulas for Appell's double hypergeometric functions in terms of the Gaussian hypergeometric functions in one variable. Recently, Hasanov and Srivastava [15, 16] introduced multivariable analogues of Burchnall-Chaundy's symbolic operators, and with the help of these operators, the authors obtained a number of decomposition formulas associated with multiple Lauricella hypergeometric functions $F_A^{(r)}$, $F_B^{(r)}$, $F_C^{(r)}$, and $F_D^{(r)}$. In [17, 18], the authors gave the following multivariable symbolic operators:

$$H_{x_1, \dots, x_r}(\alpha, \beta) = \frac{\Gamma(\beta) \Gamma(\alpha + \delta_1 + \dots + \delta_r)}{\Gamma(\alpha) \Gamma(\beta + \delta_1 + \dots + \delta_r)} = \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(\beta - \alpha)_{k_1 + \dots + k_r} (-\delta_1)_{k_1} \cdots (-\delta_r)_{k_r}}{(\beta)_{k_1 + \dots + k_r} k_1! \cdots k_r!}, \quad (12)$$

$$\bar{H}_{x_1, \dots, x_r}(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta + \delta_1 + \dots + \delta_r)}{\Gamma(\beta) \Gamma(\alpha + \delta_1 + \dots + \delta_r)} = \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(\beta - \alpha)_{k_1 + \dots + k_r} (-\delta_1)_{k_1} \cdots (-\delta_r)_{k_r}}{(1 - \alpha - \delta_1 - \dots - \delta_r)_{k_1 + \dots + k_r} k_1! \cdots k_r!} \quad (13)$$

$$\left(\delta_j := x_j \frac{\partial}{\partial x_j}, j = 1, \dots, r; r \in \mathbb{N} := \{1, 2, 3, \dots\} \right). \quad (14)$$

Based on the operators (12) and (13), we aim in this work to establish certain decomposition formulas for second-order Gaussian hypergeometric functions in three variables (1), (2), (3), (4), (5), (6), (7), (8), (9), and (10), which are used to derive some interesting transformation formulas for these functions.

2. Symbolic Form

Applying the symbolic operators in (12) and (13), we construct the following set of operator identities involving the classical Gauss hypergeometric function ${}_2F_1$ [19], the Appell functions F_1, F_2, F_4 [20], the Horn functions G_1, H_1, H_7 [21], and the Gaussian triple hypergeometric functions defined in (1), (2), (3), (4), (5), (6), (7), (8), (9), and (10):

$$F_{11a}(a_1, a_2, a_3, a_4; c_1, c_2; x, y, z) = H_{x,y}(a_1, c_1)(1-x)^{-a_3}(1-y)^{-a_2} {}_2F_1\left(a_2, a_4; c_2; \frac{z}{1-y}\right), \quad (15)$$

$$(1-x)^{-a_3}(1-y)^{-a_2} {}_2F_1\left(a_2, a_4; c_2; \frac{z}{1-y}\right) = \bar{H}_{x,y}(a_1, c_1) F_{11a}(a_1, a_2, a_3, a_4; c_1, c_2; x, y, z), \quad (16)$$

$$F_{11c}(a_1, a_2, a_3, b; c; x, y, z) = H_{x,y}(a_1, c)(1-x)^{-b}(1-y)^{-a_2} {}_2F_1\left(a_2, a_3; 1-b; -\frac{(1-x)z}{1-y}\right), \quad (17)$$

$$(1-x)^{-b}(1-y)^{-a_2} {}_2F_1\left(a_2, a_3; 1-b; -\frac{(1-x)z}{1-y}\right) = \bar{H}_{x,y}(a_1, c) F_{11c}(a_1, a_2, a_3, b; c; x, y, z), \quad (18)$$

$$F_{15a}(a_1, a_2, a_3; c_1, c_2; x, y, z) = H_{x,y}(a_1, c_1)(1-x)^{-a_3}(1-y)^{-a_2} {}_2F_1\left(a_2, a_3; c_2; \frac{z}{(1-x)(1-y)}\right), \quad (19)$$

$$(1-x)^{-a_3}(1-y)^{-a_2} {}_2F_1\left(a_2, a_3; c_2; \frac{z}{(1-x)(1-y)}\right) = \bar{H}_{x,y}(a_1, c_1) F_{15a}(a_1, a_2, a_3; c_1, c_2; x, y, z), \quad (20)$$

$$F_{17a}(a_1, a_2, a_3, a_4; c_1, c_2, c_3; x, y, z) = H_z(a_4, c_3)(1-z)^{-a_1} F_2\left(a_1, a_2, a_3; c_1, c_2; \frac{x}{1-z}, \frac{y}{1-z}\right), \quad (21)$$

$$(1-z)^{-a_1} F_2\left(a_1, a_2, a_3; c_1, c_2; \frac{x}{1-z}, \frac{y}{1-z}\right) = \bar{H}_z(a_4, c_3) F_{17a}(a_1, a_2, a_3, a_4; c_1, c_2, c_3; x, y, z), \quad (22)$$

$$F_{17b}(a_1, a_2, a_3, b; c_1, c_2; x, y, z) = H_z(a_3, c_2)(1-z)^{-a_1} H_1\left(b, a_1, a_2; c_1; \frac{x}{1-z}, \frac{y}{1-z}\right), \quad (23)$$

$$(1-z)^{-a_1} H_1\left(b, a_1, a_2; c_1; \frac{x}{1-z}, \frac{y}{1-z}\right) = \bar{H}_z(a_3, c_2) F_{17b}(a_1, a_2, a_3, b; c_1, c_2; x, y, z), \quad (24)$$

$$F_{17c}(a_1, a_2, b_1, b_2; c; x, y, z) = H_z(a_2, c)(1-z)^{-a_1} G_1\left(a_1, b_2, b_1; \frac{x}{1-z}, \frac{y}{1-z}\right), \quad (25)$$

$$(1-z)^{-a_1} G_1\left(a_1, b_2, b_1; \frac{x}{1-z}, \frac{y}{1-z}\right) = \bar{H}_z(a_2, c) F_{17c}(a_1, a_2, b_1, b_2; c; x, y, z), \quad (26)$$

$$F_{18a}(a_1, a_2, a_3, a_4; c_1, c_2; x, y, z) = H_z(a_4, c_2)(1-z)^{-a_1} F_1\left(a_1, a_2, a_3; c_1; \frac{x}{1-z}, \frac{y}{1-z}\right), \quad (27)$$

$$(1-z)^{-a_1} F_1\left(a_1, a_2, a_3; c_1; \frac{x}{1-z}, \frac{y}{1-z}\right) = \bar{H}_z(a_4, c_2) F_{18a}(a_1, a_2, a_3, a_4; c_1, c_2; x, y, z), \quad (28)$$

$$F_{19a}(a_1, a_2, a_3, a_4; c; x, y, z) = H_{x,y,z}(a_1, c)(1-x)^{-a_2}(1-y)^{-a_3}(1-z)^{-a_4}, \quad (29)$$

$$(1-x)^{-a_2}(1-y)^{-a_3}(1-z)^{-a_4} = \bar{H}_{x,y,z}(a_1, c) F_{19a}(a_1, a_2, a_3, a_4; c; x, y, z), \quad (30)$$

$$F_{20a}(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z) = H_z(a_3, c_3)(1-z)^{-a_1} F_4\left(a_1, a_2; c_1, c_2; \frac{x}{1-z}, \frac{y}{1-z}\right), \quad (31)$$

$$(1-z)^{-a_1} F_4\left(a_1, a_2; c_1, c_2; \frac{x}{1-z}, \frac{y}{1-z}\right) = \bar{H}_z(a_3, c_3) F_{20a}(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z), \quad (32)$$

$$F_{23c}(a_1, a_2, a_3, b_1, b_2; c; x, y, z) = H_y(a_3, c)(1-y)^{-b_1} H_7(b_2, a_1, a_2; 1-b_1; -(1-y)z, x), \quad (33)$$

$$(1-y)^{-b_1} H_7(b_2, a_1, a_2; 1-b_1; -(1-y)z, x) = \bar{H}_y(a_3, c) F_{23c}(a_1, a_2, a_3, b_1, b_2; c; x, y, z). \quad (34)$$

Each of the operator identities (15), (16), (17), (18), (19), (20), (21), (22), (23), (24), (25), (26), (27), (28), (29), (30), (31), (32), (33), and (34) can be proved by means of Mellin and inverse Mellin transformation (see, for example, [11, 20, 22]). The proofs of the operator identities are omitted here.

3. Decomposition Formulas

In [23] (p. 93), it is proved that, for every analytic function $f(\xi)$, the following formulas hold true:

$$(-\delta)_n \{f(\xi)\} = (-1)^n \xi^n \frac{d^n}{d\xi^n} \{f(\xi)\}, \quad (35)$$

$$(\alpha + \delta)_n \{f(\xi)\} = \xi^{1-\alpha} \frac{d^n}{d\xi^n} \left\{ \xi^{\alpha+n-1} f(\xi) \right\}, \quad (36)$$

where

$$\delta := \xi \frac{d}{d\xi}; \alpha \in \mathbb{C}; n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}. \quad (37)$$

In view of formulas (35) and (36), and taking into account the differentiation formula for hypergeometric functions, from operator identities (15), (16), (17), (18), (19), (20), (21), (22), (23), (24), (25), (26), (27), (28), (29), (30), (31), (32), (33), and (34), we have

$$F_{11a}(a_1, a_2, a_3, a_4; c_1, c_2; x, y, z) = (1-x)^{-a_3}(1-y)^{-a_2} \times \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j} (a_3)_i (a_2)_j (c_1 - a_1)_{i+j}}{(c_1)_{i+j} i! j!} \cdot \left(\frac{x}{1-x}\right)^i \left(\frac{y}{1-y}\right)^j \times {}_2F_1\left(a_2 + j, a_4; c_2; \frac{z}{1-y}\right), \quad (38)$$

$$(1-x)^{-a_3}(1-y)^{-a_2} {}_2F_1\left(a_2, a_4; c_2; \frac{z}{1-y}\right) = \sum_{i,j=0}^{\infty} \frac{(a_3)_i (a_2)_j (c_1 - a_1)_{i+j}}{(c_1)_{i+j} i! j!} \times x^i y^j F_{11a}(a_1, a_2 + j, a_3 + i, a_4; c_1 + i + j, c_2; x, y, z), \quad (39)$$

$$F_{11c}(a_1, a_2, a_3, b; c; x, y, z) = (1-x)^{-b}(1-y)^{-a_2} \times \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j} (b)_i (a_2)_j (c - a_1)_{i+j}}{(c)_{i+j} i! j!} \left(\frac{x}{1-x}\right)^i \left(\frac{y}{1-y}\right)^j \times {}_2F_1\left(a_2 + j, a_3; 1-b-i; \frac{(1-x)z}{1-y}\right), \quad (40)$$

$$\begin{aligned}
& (1-x)^{-b}(1-y)^{-a_2} {}_2F_1\left(a_2, a_3; 1-b; -\frac{(1-x)z}{1-y}\right) \\
&= \sum_{i,j=0}^{\infty} \frac{(b)_i (a_2)_j (c-a_1)_{i+j}}{(c)_{i+j} i! j!} x^i y^j \\
&\quad \times F_{11c}(a_1, a_2+j, a_3, b+i; c+i+j; x, y, z),
\end{aligned} \tag{41}$$

$$\begin{aligned}
& F_{15a}(a_1, a_2, a_3; c_1, c_2; x, y, z) = (1-x)^{-a_3}(1-y)^{-a_2} \\
&\quad \times \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j} (a_3)_i (a_2)_j (c_1-a_1)_{i+j}}{(c_1)_{i+j} i! j!} \left(\frac{x}{1-x}\right)^i \left(\frac{y}{1-y}\right)^j \\
&\quad \times {}_2F_1\left(a_2+j, a_3+i; c_2; \frac{z}{(1-x)(1-y)}\right),
\end{aligned} \tag{42}$$

$$\begin{aligned}
& (1-x)^{-a_3}(1-y)^{-a_2} {}_2F_1\left(a_2, a_3; c_2; \frac{z}{(1-x)(1-y)}\right) \\
&= \sum_{i,j=0}^{\infty} \frac{(a_3)_i (a_2)_j (c_1-a_1)_{i+j}}{(c_1)_{i+j} i! j!} x^i y^j \\
&\quad \times F_{15a}(a_1, a_2+j, a_3+i; c_1+i+j, c_2; x, y, z),
\end{aligned} \tag{43}$$

$$\begin{aligned}
& F_{17a}(a_1, a_2, a_3, a_4; c_1, c_2, c_3; x, y, z) = (1-z)^{-a_1} \\
&\quad \times \sum_{i=0}^{\infty} \frac{(-1)^i (a_1)_i (c_3-a_4)_i}{(c_3)_i i!} \left(\frac{z}{1-z}\right)^i F_2 \\
&\quad \cdot \left(a_1+i, a_2, a_3; c_1, c_2; \frac{x}{1-z}, \frac{y}{1-z}\right),
\end{aligned} \tag{44}$$

$$\begin{aligned}
& (1-z)^{-a_1} F_2\left(a_1, a_2, a_3; c_1, c_2; \frac{x}{1-z}, \frac{y}{1-z}\right) \\
&= \sum_{i=0}^{\infty} \frac{(a_1)_i (c_3-a_4)_i}{(c_3)_i i!} z^i F_{17a} \\
&\quad \cdot (a_1+i, a_2, a_3, a_4; c_1, c_2, c_3+i; x, y, z),
\end{aligned} \tag{45}$$

$$\begin{aligned}
& F_{17b}(a_1, a_2, a_3, b; c_1, c_2; x, y, z) \\
&= (1-z)^{-a_1} \times \sum_{i=0}^{\infty} \frac{(-1)^i (a_1)_i (c_2-a_3)_i}{(c_2)_i i!} \left(\frac{z}{1-z}\right)^i H_1 \\
&\quad \cdot \left(b, a_1+i, a_2; c_1; \frac{x}{1-z}, \frac{y}{1-z}\right),
\end{aligned} \tag{46}$$

$$\begin{aligned}
& (1-z)^{-a_1} H_1\left(b, a_1, a_2; c_1; \frac{x}{1-z}, \frac{y}{1-z}\right) \\
&= \sum_{i=0}^{\infty} \frac{(a_1)_i (c_2-a_3)_i}{(c_2)_i i!} z^i F_{17b}(a_1+i, a_2, a_3, b; c_1, c_2+i; x, y, z),
\end{aligned} \tag{47}$$

$$\begin{aligned}
& F_{17c}(a_1, a_2, b_1, b_2; c; x, y, z) = (1-z)^{-a_1} \\
&\quad \times \sum_{i=0}^{\infty} \frac{(-1)^i (a_1)_i (c-a_2)_i}{(c)_i i!} \left(\frac{z}{1-z}\right)^i G_1 \\
&\quad \cdot \left(a_1+i, b_2, b_1; \frac{x}{1-z}, \frac{y}{1-z}\right),
\end{aligned} \tag{48}$$

$$\begin{aligned}
& (1-z)^{-a_1} G_1\left(a_1, b_2, b_1; \frac{x}{1-z}, \frac{y}{1-z}\right) \\
&= \sum_{i=0}^{\infty} \frac{(a_1)_i (c-a_2)_i}{(c)_i i!} z^i F_{17c}(a_1+i, a_2, b_1, b_2; c+i; x, y, z),
\end{aligned} \tag{49}$$

$$\begin{aligned}
& F_{18a}(a_1, a_2, a_3, a_4; c_1, c_2; x, y, z) = (1-z)^{-a_1} \\
&\quad \times \sum_{i=0}^{\infty} \frac{(-1)^i (a_1)_i (c_2-a_4)_i}{(c_2)_i i!} \left(\frac{z}{1-z}\right)^i F_1 \\
&\quad \cdot \left(a_1+i, a_2, a_3; c_1; \frac{x}{1-z}, \frac{y}{1-z}\right),
\end{aligned} \tag{50}$$

$$\begin{aligned}
& (1-z)^{-a_1} F_1\left(a_1, a_2, a_3; c_1; \frac{x}{1-z}, \frac{y}{1-z}\right) \\
&= \sum_{i=0}^{\infty} \frac{(a_1)_i (c_2-a_4)_i}{(c_2)_i i!} z^i F_{18a}(a_1+i, a_2, a_3, a_4; c_1, c_2+i; x, y, z),
\end{aligned} \tag{51}$$

$$\begin{aligned}
& F_{19a}(a_1, a_2, a_3, a_4; c; x, y, z) = (1-x)^{-a_2}(1-y)^{-a_3}(1-z)^{-a_4} \\
&\quad \times F_D^{(3)}\left(c-a_1, a_2, a_3, a_4; c; \frac{x}{x-1}, \frac{y}{y-1}, \frac{z}{z-1}\right),
\end{aligned} \tag{52}$$

$$\begin{aligned}
& (1-x)^{-a_2}(1-y)^{-a_3}(1-z)^{-a_4} \\
&= \sum_{i,j,k=0}^{\infty} \frac{(a_2)_i (a_3)_j (a_4)_k (c-a_1)_{i+j+k}}{(c)_{i+j+k} i! j! k!} x^i y^j z^k \\
&\quad \times F_{19a}(a_1, a_2+i, a_3+j, a_4+k; c+i+j+k; x, y, z),
\end{aligned} \tag{53}$$

$$\begin{aligned}
& F_{20a}(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z) \\
&= (1-z)^{-a_1} \times \sum_{i=0}^{\infty} \frac{(-1)^i (a_1)_i (c_3-a_3)_i}{(c_3)_i i!} \left(\frac{z}{1-z}\right)^i F_4 \\
&\quad \cdot \left(a_1+i, a_2; c_1, c_2; \frac{x}{1-z}, \frac{y}{1-z}\right),
\end{aligned} \tag{54}$$

$$\begin{aligned}
& (1-z)^{-a_1} F_4\left(a_1, a_2; c_1, c_2; \frac{x}{1-z}, \frac{y}{1-z}\right) \\
&= \sum_{i=0}^{\infty} \frac{(a_1)_i (c_3-a_3)_i}{(c_3)_i i!} z^i F_{20a}(a_1+i, a_2, a_3; c_1, c_2, c_3+i; x, y, z),
\end{aligned} \tag{55}$$

$$\begin{aligned}
& F_{23c}(a_1, a_2, a_3, b_1, b_2; c; x, y, z) \\
&= (1-y)^{-b_1} \times \sum_{i=0}^{\infty} \frac{(-1)^i (b_1)_i (c-a_3)_i}{(c)_i i!} \\
&\quad \cdot \left(\frac{y}{1-y}\right)^i H_7(b_2, a_1, a_2; 1-b_1-i; -(1-y)z, x),
\end{aligned} \tag{56}$$

$$\begin{aligned}
& (1-y)^{-b_1} H_7(b_2, a_1, a_2; 1-b_1; -(1-y)z, x) \\
&= \sum_{i=0}^{\infty} \frac{(b_1)_i (c-a_3)_i}{(c)_i i!} y^i F_{23c}(a_1, a_2, a_3, b_1+i, b_2; c+i; x, y, z).
\end{aligned} \tag{57}$$

4. Transformation Formulas

The transformation formulas defined below follow from the expansion formulas (38), (39), (40), (41), (42), (43), (44), (45), (46), (47), (48), (49), (50), (51), (52), (53), (54), (55), (56), and (57):

$$\begin{aligned}
F_{11a}(a_1, a_2, a_3, a_4; c_1, c_2; x, y, z) &= (1-x)^{-a_3} (1-y)^{-a_2} F_{11a} \\
&\cdot \left(c_1 - a_1, a_2, a_3, a_4; c_1, c_2; \frac{x}{x-1}, \frac{y}{y-1}, \frac{z}{1-y} \right),
\end{aligned}$$

$$\begin{aligned}
F_{11c}(a_1, a_2, a_3, b; c; x, y, z) &= (1-x)^{-b} (1-y)^{-a_2} F_{11c} \\
&\cdot \left(c - a_1, a_2, a_3, b; c; \frac{x}{x-1}, \frac{y}{y-1}, \frac{(1-x)z}{1-y} \right),
\end{aligned}$$

$$\begin{aligned}
F_{15a}(a_1, a_2, a_3; c_1, c_2; x, y, z) &= (1-x)^{-a_3} (1-y)^{-a_2} F_{15a} \\
&\cdot \left(c_1 - a_1, a_2, a_3; c_1, c_2; \frac{x}{x-1}, \frac{y}{y-1}, \frac{z}{(1-x)(1-y)} \right),
\end{aligned}$$

$$\begin{aligned}
F_{17a}(a_1, a_2, a_3, a_4; c_1, c_2, c_3; x, y, z) &= (1-z)^{-a_1} F_{17a} \\
&\cdot \left(a_1, a_2, a_3, c_3 - a_4; c_1, c_2, c_3; \frac{x}{1-z}, \frac{y}{1-z}, \frac{z}{z-1} \right),
\end{aligned}$$

$$\begin{aligned}
F_{17b}(a_1, a_2, a_3, b; c_1, c_2; x, y, z) &= (1-z)^{-a_1} F_{17b} \\
&\cdot \left(a_1, a_2, c_2 - a_3, b; c_1, c_2; \frac{x}{1-z}, \frac{y}{1-z}, \frac{z}{z-1} \right),
\end{aligned}$$

$$\begin{aligned}
F_{17c}(a_1, a_2, b_1, b_2; c; x, y, z) &= (1-z)^{-a_1} F_{17c} \\
&\cdot \left(a_1, c - a_2, b_1, b_2; c; \frac{x}{1-z}, \frac{y}{1-z}, \frac{z}{z-1} \right),
\end{aligned}$$

$$\begin{aligned}
F_{18a}(a_1, a_2, a_3, a_4; c_1, c_2; x, y, z) &= (1-z)^{-a_1} F_{18a} \\
&\cdot \left(a_1, a_2, a_3, c_2 - a_4; c_1, c_2; \frac{x}{1-z}, \frac{y}{1-z}, \frac{z}{z-1} \right),
\end{aligned}$$

$$\begin{aligned}
F_{19a}(a_1, a_2, a_3, a_4; c; x, y, z) &= (1-x)^{-a_2} (1-y)^{-a_3} (1-z)^{-a_4} \\
&\times F_D^{(3)} \left(c - a_1, a_2, a_3, a_4; c; \frac{x}{x-1}, \frac{y}{y-1}, \frac{z}{z-1} \right),
\end{aligned}$$

$$\begin{aligned}
F_{20a}(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z) &= (1-z)^{-a_1} F_{20a} \\
&\cdot \left(a_1, a_2, c_3 - a_3; c_1, c_2, c_3; \frac{x}{1-z}, \frac{y}{1-z}, \frac{z}{z-1} \right),
\end{aligned}$$

$$\begin{aligned}
F_{23c}(a_1, a_2, a_3, b_1, b_2; c; x, y, z) &= (1-y)^{-b_1} F_{23c} \\
&\cdot \left(a_1, a_2, c - a_3, b_1, b_2; c; x, \frac{y}{y-1}, (1-y)z \right).
\end{aligned} \tag{58}$$

5. Concluding Remarks

In this present paper, with the help of the inverse pairs of symbolic operators, we established a number of decomposition formulas for some Gaussian triple hypergeometric functions. Also, we investigated certain transformation formulas for these functions. We conclude that mutually inverse operators (12) and (13) can be applied to other multiple hypergeometric functions.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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