In 1974, Ćirić [1] published the first paper on nonunique fixed point theory. Despite Banach’s theorem, Ćirić [1] focused only on the existence of a fixed point, but not the uniqueness. The motivation of Ćirić [1] was inspired by Banach’s motivation. As it is known, Banach’s fixed point theorem is abstracted from Picard’s paper, in which Picard [2] analyzed both the existence and uniqueness of the solution of the certain differential equation (see [3–5]). On the other hand, not all differential or integral equations have a unique solution. In the differential/integral equations, nonunique solutions are also crucial, for example, periodic solutions. Consequently, Ćirić [1] investigated the corresponding fixed point theorems that would be a tool in finding periodic solutions of the differential/integral equations. In the last five decades, a number of nonunique fixed point results have been reported in two ways: either proposing a new contraction type or changing the structure. The first example for the changing the contraction inequality, in the standard set-up, was given by Achari [6] in 1976 and Pachpatte [7] in 1973. Fifteen years later, Ćirić and Jović [8] proposed a new type of contraction inequalities in the context of complete metric space. This trend was followed by the attractive results [9–13]. On the other hand side, in [14–17], the authors observed several characterizations of the unique fixed point results in the setting of complete b-metric spaces. Indeed, among the several extensions of metric structure, the true extension is the b-metric space. For this reason, observed nonunique fixed theorems in the context of b-metric space is very interesting and important, see also [18–20]. In addition, in [21–23], the characterization of fixed point theorems in partial metric spaces is crucial due to the potential application in the domain theory of computer science. Regarding the applied mathematics, nonunique fixed point results in cone metric spaces have taken attention [24].

In this paper, we consider a nonunique fixed point theorem in the context of the very general frame, partial b-metric spaces. An illustrative example is a set-up to indicate the validity of the main theorem.

Let $M$ be a nonempty set, a real number $s \geq 1$, and $N = \{1, 2, 3, \ldots \}$. In this case, the triplet $(M, p_s, s)$ forms a partial $b$-metric space, on short $p$-ms. Undoubtedly, $b$-metric spaces (and ordinary metric spaces) are closely related to partial $b$-metric spaces. Definitely, a $b$-metric space $(s \geq 1)$ is a partial $b$-metric space with zero self-distance and a partial metric space is a partial $b$-metric space with $s = 1$. Moreover, a partial $b$-metric can define a $b$-metric space. Indeed, for example,
let $p_b$ be a partial $b$-metric on $M$. Then, the functions $b^r, b_p, b_{p,m} : M \rightarrow M$, where

$$b_p(u, v) = \begin{cases} p_b(u, v), & \text{if } u \neq v, \\ 0, & \text{if } u = v, \end{cases}$$  \hspace{1cm} (1)

$$b^r_p(u, y) = 2p_b(u, y) - p_b(u, u) - p_b(y, y),$$  \hspace{1cm} (2)

$$b_{p,m}(u, y) = p_b(u, y) - \min \{p_b(u, u), p_b(y, y)\}$$  \hspace{1cm} (3)

are $b$-metrics on $M$.

**Definition 1.** A function $p_b : M \times M \rightarrow [0, \infty)$ is a partial $b$-metric on $M$ if for all $u, y, w \in M$, it satisfies the following conditions:

$$\begin{align*}
(p_b)_1 & : u = y \iff p_b(u, u) = p_b(u, y) = p_b(y, y) \\
(p_b)_2 & : p_b(u, u) \leq p_b(b(u, y)) \\
(p_b)_3 & : p_b(u, y) \leq p(y, u) \\
(p_b)_4 & : p_b(u, y) \leq sl[p_b(u, w) + p_b(w, u)] - p_b(w, w)
\end{align*}$$

**Example 1.** (see [25]). Let $p_b$ be a partial metric on the set $M$. Then, the functions $p_b : M \times M \rightarrow [0, \infty)$ are given for all $u, y \in M$ by

$$(1) \quad p_b(u, y) = p(y, y) + b(u, y) \text{ is a partial } b\text{-metric on } M$$

$$(2) \quad p_b(u, y) = [p(y, y)]^r \text{ for } r \geq 1, \text{ define a partial } b\text{-metrics on } M \text{ with coefficient } s = 2^{-r}$$

**Remark 2.** From (pb)$_1$ and (pb)$_2$, it follows that if $u, y \in M$ are such that $p_b(u, y) = 0$, then $u = y$.

**Definition 3.** (see [26, 27]). Let $\{u_n\}$ be a sequence on the $p_b$-ms $(M, p_b, s \geq 1)$

$$(1) \quad \{u_n\} \text{ is } p_b\text{-convergent to } u \in M \text{ if } \lim_{n \rightarrow \infty} p_b(u_n, u) = p_b(u, u)$$

$$(2) \quad \{u_n\} \text{ is } p_b\text{-Cauchy if } \lim_{n \rightarrow \infty} p_b(u_n, u) \text{ exists and is finite}$$

$$(3) \quad \{u_n\} \text{ is } 0\text{-}p_b\text{-Cauchy if } \lim_{n \rightarrow \infty} p_b(u_n, u) = 0$$

$$(4) \quad (M, p_b, s \geq 1) \text{ is } p_b\text{-complete if every } p_b\text{-Cauchy sequence in } M \text{ is } p_b\text{-convergent}$$

$$\lim_{n \rightarrow \infty} p_b(u_n, u) = \lim_{n \rightarrow \infty} p_b(u, u) = p_b(u, u)$$  \hspace{1cm} (4)

$$(5) \quad (M, p_b, s \geq 1) \text{ is } 0\text{-}p_b\text{-complete if every } 0\text{-}p_b\text{-Cauchy sequence we can find } u \in M \text{ such that}$$

$$\lim_{n \rightarrow \infty} p_b(u_n, u) = \lim_{n \rightarrow \infty} p_b(u, u) = p_b(u, u) = 0$$  \hspace{1cm} (5)

Moreover, in [26], the following interesting results were proved.

**Lemma 4.** (see [26]). Every $p_b$-complete $p_b$-ms $(M, p_b, s \geq 1)$ is $0\text{-}p_b\text{-complete}.$

**Lemma 5.** (see [26]). The $p_b$-ms $(M, p_b, s \geq 1)$ is $0\text{-}p_b\text{-complete if and only if the } b\text{-metric space } (M, b_p, s \geq 1) \text{ is complete, where the } b\text{-metric } b_p \text{ was defined in } (3).$

They also showed that the converse affirmation does not hold.

Let $R, S$ to self-mappings on the set $M$. We say that

(i) $S$ commutes with $R$ on $M$ if $RSu = SRu$ for all $u \in M$

(ii) a point $z \in M$ is a point of coincidence of $R$ and $S$ if we can find $u^* \in M$ such that $z = Ru^* = Su^*$

(iii) a point $u^* \in M$ is a common fixed point of $R$ and $S$ if $Ru^* = u^* = Su^*$

We will use the following notations:

$$C_r(R, S)_M = \{u \in M | Ru = Su\}M^* = M \setminus C_r(R, S)_M.$$  \hspace{1cm} (6)

In [28], the notion of $R$-$\beta$-admissible mapping was introduced as follows:

$$(i) \quad \text{Let the function } \beta : M \times M \rightarrow [0, \infty) \text{ and } R, S : M \rightarrow M \text{. The mapping } S \text{ is said to be } R$-$\beta$-admissible if$$

$$\beta(Ru, Ry) \geq 1 \text{ implies } \beta(Su, Sy) \geq 1,$$  \hspace{1cm} (7)

for all $u, y \in M$.

In case that $R = I_M$, the mapping $S$ is said to be $\beta$-admissible.

Let $(M, p_b, s \geq 1)$ be a $p_b$-ms and $\beta : M \times M \rightarrow [0, +\infty)$. The space $M$ is $\beta$-regular if for every sequence $\{z_n\}$ in $M$ such that $z_n \rightarrow z$ and $\beta(z_n, z_{n+1}) \geq 1$, there exists a subsequence $\{z_{n_l}\}$ of $\{z_n\}$ such that

$$\beta(z_{n_l}, z_l) \geq 1,$$  \hspace{1cm} (8)

for all $l \in \mathbb{N}$.

**Lemma 6.** Let $R, S : M \rightarrow M$ such that $S$ is a $R$-$\beta$-admissible. If there exists $u_0 \in M$ such that $\beta(Ru_0, Su_0) \geq 1$, then

$$\beta(Ru_n, Ru_{n+1}) \geq 1,$$  \hspace{1cm} (9)

where the sequence $\{u_n\}$ in $M$ is defined by $Su_n = Ru_{n+1}$, for each $n \in \mathbb{N} \cup \{0\}$.

**Proof.** By the assumption $\beta(Ru_0, Su_0) \geq 1$, since the mapping $S$ is $R$-$\beta$-admissible, we get

$$\beta(Ru_0, Ru_1) = \beta(Ru_0, Su_0) \geq 1 \text{ implies } \beta(Su_1, Su_2) = \beta(Su_0, Su_1) \geq 1.$$  \hspace{1cm} (10)
and by induction, it follows that
$$\beta(Ru_n, Ru_{n+1}) \geq 1,$$
(11)
for \( n \in \mathbb{N} \cup \{0\} \).

## 2. Main Results

Following the idea in [29], we state the following results useful in the sequel.

**Lemma 7.** Let \((M, p_b, s \geq 1)\) be a \( p_b \)-ms. If \( \{u_n\} \) is a sequence in \( M \) such that there exists \( \{z_n\} \) in \( M \), satisfying the inequality
$$p_b(u_n, u_{n+1}) \leq cp_b(u_{n-1}, u_n),$$
(12)
for any \( n \in \mathbb{N} \), then the sequence is \( \{u_n\} \) and is 0-\(p_b\)-Cauchy.

**Proof.** First of all, by (12), we get
$$p_b(u_n, u_{n+1}) \leq c^2 p_b(u_0, u_1),$$
(13)
for all \( n \in \mathbb{N} \). On the other hand, by using (pb)₄, we can derive that
$$p_b(u_n, u_{n+q}) \leq s(p_b(u_n, u_{n+1})$$
$$\quad + p_b(u_{n+1}, u_{n+q}) - p_b(u_{n+1}, u_{n+1})$$
$$\leq s p_b(u_n, u_{n+1})$$
$$\quad + s^2(p_b(u_{n+1}, u_{n+2}) + p_b(u_{n+2}, u_{n+2}, u_{n+q})$$
$$\quad - p_b(u_{n+1}, u_{n+1}) - p_b(u_{n+2}, u_{n+2}) \cdots$$
$$\leq s p_b(u_n, u_{n+1}) + s^2 p_b(u_{n+1}, u_{n+2}) + \cdots$$
$$\quad + s^q p_b(u_{n+q-1}, u_{n+q})$$
$$\leq s^q p_b(u_n, u_{n+1}) + p_b(u_{n+1}, u_{n+2}) \cdots$$
$$\quad + p_b(u_{n+q-1}, u_{n+q})$$
$$\quad - \sum_{i=1}^{q-1} p_b(u_{n+i}, u_{n+1}).$$
(14)

(1) If \( c \in [0, 1/s) \), by (13) and (14), we get
$$p_b(u_n, u_{n+q}) \leq s^q \sum_{i=0}^{q-1} (s^i)^q p_b(u_0, u_1)$$
$$\leq s^q \frac{1 - (s)^q}{1 - s} \longrightarrow 0 \text{ as } n, q \longrightarrow \infty.$$  
(15)
Therefore, \( \{u_n\} \) is a 0-\(p_b\)-Cauchy sequence.

(2) If \( c \in [1/s, 1) \), thus \( c^q \longrightarrow 0 \) (as \( n \rightarrow \infty \)). Moreover, there exists \( l \in \mathbb{N} \) such that \( c^l < 1/s \). This means \( l \geq \log s / \log c \). Again, by (13) together with (14), we have
$$p_b\left(u_{nl}, u_{nl+1}\right) \leq s^l \left[p_b\left(u_{nl}, u_{nl+1}\right) + \cdots + p_b\left(u_{nl+l-1}, u_{nl+l}\right)\right]$$
$$\leq s^l \sum_{i=0}^{l-1} p_b(u_{nl+i}, u_{nl+i})$$
$$\leq s^l \sum_{i=0}^{l-1} c^{i+l} p_b(u_{nl+i}, u_{nl+i})$$
$$\leq s^l \sum_{i=0}^{l-1} c^{i+l} p_b\left(u_{nl+i}, u_{nl+i}\right)$$
$$\leq s^l \sum_{i=0}^{l-1} c^{i+l} p_b\left(u_{nl+i}, u_{nl+i}\right)$$
$$\leq s^l \sum_{i=0}^{l-1} c^{i+l} p_b\left(u_{nl+i}, u_{nl+i}\right)$$
$$\leq s^l \sum_{i=0}^{l-1} c^{i+l} p_b\left(u_{nl+i}, u_{nl+i}\right)$$
$$\leq s^l \sum_{i=0}^{l-1} c^{i+l} p_b\left(u_{nl+i}, u_{nl+i}\right)$$
(16)
Thereby, letting \( \lambda = c^l < 1/s \) by Case (i), we get that the sequence \( \{u_{nl}\} \) is 0-\(p_b\)-Cauchy sequence, which means that
$$\lim_{n,q \rightarrow \infty} p_b\left(u_{nl}, u_{nl}\right) = 0.$$  
(17)
On the other hand,
$$p_b\left(u_{nl+l}, u_n\right) \leq s^l \left[p_b\left(u_{nl+l}, u_{nl+l+1}\right) + \cdots + p_b\left(u_{nl+l}, u_{nl+l+1}\right)\right]$$
$$\leq s^l \left[p_b\left(u_{nl+l}, u_{nl+l+1}\right) + \cdots + p_b\left(u_{nl-l}, u_{nl-1}\right)\right],$$
(18)
and using (13), we have
$$p_b\left(u_{nl+l}, u_n\right) \leq s^l \left[c^{l+l} + \cdots + c^{n-1}\right] p_b\left(u_0, u_1\right)$$
$$\leq s^l \left[c^{l+l} + \cdots + c^{n-1}\right] p_b\left(u_0, u_1\right)$$
$$\leq s^l \left[c^{l+l} + \cdots + c^{n-1}\right] p_b\left(u_0, u_1\right)$$
$$\leq s^l \left[c^{l+l} + \cdots + c^{n-1}\right] p_b\left(u_0, u_1\right)$$
(19)
Finally, combining relations (19) and (17) and keeping in mind (pb)₄, we have
$$p_b\left(u_n, u_{n+q}\right) \leq s^q \sum_{i=0}^{q-1} (s^i)^q p_b(u_0, u_1)$$
$$\leq s^q \frac{1 - (s)^q}{1 - s} \longrightarrow 0 \text{ as } n, q \longrightarrow \infty.$$  
(20)
Thereupon, the sequence \( \{u_n\} \) is 0-\(p_b\)-Cauchy.
Theorem 8. Let \((M, p_0, s \geq 1)\) be a complete \(p_0\)-ms and two mappings \(R, S : M \rightarrow M\). Suppose that there exists \(\kappa \in (0, 1)\) such that
\[
\beta(Ru, Ry) \min \{p_0(Su, Sy), p_0(Sy, Ry)\}
\]
\[
-\min \{b_0(Ru, Ry), b_1(Sy, Ry)\}
\]
\[
\leq \kappa \max \{p_0(Ru, Ry), p_0(Su, Ru)\},
\]
for all \(u, y \in M\), such that \(u \neq y\) when \(u, y \in C_\beta(R, S)_M\). Suppose also that
\[
(a) S(M) \subset R(M) \quad \text{and} \quad (R(M), p_0, s) \text{ is a } 0-p_0\text{-complete } \ 0\text{-ms}
\]
\[
(b) S \text{ is } \beta\text{-admissible, and there exists } u_0 \in M \text{ such that } \beta(Ru_0, Su_0) \geq 1
\]
\[
(c) M \text{ is } \beta\text{-regular}
\]

Then, the mappings \(S\) and \(R\) have a point of coincidence.

Proof. Let \(u_0\) be an arbitrary point in \(M\), such that \(\beta(Ru_0, Su_0) \geq 1\). Thus, since \(S(M) \subset R(M)\), there exists \(u_1 \in M\) such that \(Su_0 = Ru_1\). Thereupon, \(Su_1, Ru_1 \in S(M) \subset R(M)\) and we can find \(u_2 \in M\) such that \(Su_1 = Ru_2\). In this way, we can build a sequence \(\{u_n\} \subseteq M\) as follows:

having defined \(u_n \in M\), we let \(u_{n+1} \in M\) such that \(Su_n = Ru_{n+1}\),

\[
\text{for all } n \in \mathbb{N} \cup \{0\}. \quad \text{Letting } u = u_n \quad \text{and } y = u_{n+1} \text{ in (ref1T1)} \text{ and taking into account Lemma 6, we have}
\]

\[
\min \{p_0(Su_n, Su_{n+1}), p_0(Su_{n+1}, Ru_{n+1})\} \quad \text{-- min } \{b_0(Ru_n, Ru_{n+1}), b_1(Su_{n+1}, Ru_{n+1})\}
\]
\[
\leq \beta(Ru_n, Ru_{n+1}) \min \{p_0(Su_n, Su_{n+1}), p_0(Su_{n+1}, Ru_{n+1})\} \quad \text{-- min } \{b_0(Su_{n+1}, Ru_n), b_1(Su_{n+1}, Ru_n)\}
\]
\[
\leq \kappa \max \{p_0(Ru_n, Ru_{n+1}), p_0(Su_n, Ru_n)\}.
\]

Keeping in mind (22), we get

\[
\min \{p_0(Ru_{n+1}, Ru_{n+2}), p_0(Ru_{n+2}, Ru_{n+1})\} \quad \text{-- min } \{b_0(Ru_{n+1}, Ru_{n+2}), b_1(Ru_{n+2}, Ru_{n+1})\}
\]
\[
\leq \kappa \max \{p_0(Ru_n, Ru_{n+1}), p_0(Ru_{n+1}, Ru_{n})\}
\]
\[
= \kappa p_0(Ru_n, Ru_{n+1}),
\]
which is equivalent with

\[
\min \{p_0(Ru_{n+1}, Ru_{n+2}), p_0(Ru_{n+2}, Ru_{n})\} \quad \text{-- min } \{b_0(Ru_{n+1}, Ru_{n+2}), b_1(Ru_{n+2}, Ru_{n})\}
\]
\[
\leq \kappa p_0(Ru_n, Ru_{n+1}).
\]

Therefore, we get
\[
p_0(Ru_{n+1}, Ru_{n+2}) \leq \kappa p_0(Ru_n, Ru_{n+1}),
\]
for any \(n \in \mathbb{N} \cup \{0\}\). Let now \(\{z_n\}\) be a sequence in \(M\), with \(z_n = Ru_{n+1} = Su_n, n \in \mathbb{N} \cup \{0\}\). First of all, we mention that \(z_n \neq z_{n+1}\) for every \(n \in \mathbb{N}\). Indeed, if we suppose that there exists \(m_0 \in \mathbb{N} \cup \{0\}\) such that \(z_{m_0} = z_{m_0+1}\), thus by (22), we have
\[
Ru_{m+1} = Su_{m_0} = z_{m_0} = z_{m_0+1} = Su_{m_0+1},
\]
so that \(z_{m+1}\) is a point of coincidence. Thus, \(z_n \neq z_{n+1}\) for every \(n \in \mathbb{N} \cup \{0\}\) and (28) can be rewritten as
\[
p_0(z_n, z_{n+1}) \leq \kappa p_0(z_{n-1}, z_n).
\]

Therefore, according to Lemma 7, the sequence \(\{z_n\}\) is \(0\)-\(p_0\)-Cauchy. Since the space is \(0\)-\(p_0\)-complete, it follows that there is \(z \in M\) such that
\[
\lim_{n \to \infty} p_0(z_n, z) = \lim_{n \to \infty} p_0(z_n, z) = p_0(z, z) = 0.
\]

But, on the other hand, since \(z_n = Ru_{n+1}\) and the space \((R(M), p_0, s)\) is \(0\)-\(p_0\)-complete, we can find \(u_\ast \in M\), with \(z = Ru_\ast\). Thus,
\[
\lim_{n \to \infty} p_0(Su_n, Ru_\ast) = \lim_{n \to \infty} p_0(Ru_n, Ru_\ast) = p_0(Ru_\ast, Ru_\ast) = 0.
\]

Supposing that \(Ru_\ast \neq Su_\ast\) for \(u = u_n\) and \(y = u_\ast\) and taking into account the \(\beta\)-regularity of the space \(M\), we have
\[
\min \{p_0(Su_n, Su_\ast), p_0(Su_\ast, Ru_\ast)\} \quad - \quad \min \{b_0(Su_n, Ru_\ast), b_1(Su_\ast, Ru_\ast)\}
\]
\[
\leq \beta(z_n, z) \min \{p_0(Su_n, Su_\ast), p_0(Su_\ast, Ru_\ast)\} \quad - \quad \min \{b_0(Su_n, Ru_\ast), b_1(Su_\ast, Ru_\ast)\}
\]
\[
= \beta(Ru_n, Ru_\ast) \min \{p_0(Su_n, Su_\ast), p_0(Su_\ast, Ru_\ast)\} \quad - \quad \min \{b_0(Su_n, Ru_\ast), b_1(Su_\ast, Ru_\ast)\}
\]
\[
\leq \kappa \max \{p_0(Ru_n, Ru_\ast), p_0(Su_n, Ru_\ast)\}.
\]

If \(\min \{p_0(Su_n, Su_\ast), p_0(Su_\ast, Ru_\ast)\} = p_0(Su_\ast, Ru_\ast)\), the above inequality becomes
\[
p_0(Su_\ast, Ru_\ast) - \min \{p_0(Su_n, Su_\ast), p_0(Su_\ast, Ru_\ast)\} \quad - \quad \min \{b_0(Su_n, Ru_\ast), b_1(Su_\ast, Ru_\ast)\}
\]
\[
\leq \kappa \max \{p_0(Ru_n, Ru_\ast), p_0(Su_n, Ru_\ast)\}.
\]

Letting \(l \to \infty\) and taking into account (28) and (30), we get
\[
p_0(Su_\ast, Ru_\ast) = 0,
\]
and by (pb)_1, (pb)_1, we have Su = Ru. If min \{ pb(Su, Su), pb(Su, Ru) \} = pb(Su, Su), we find that \( \lim_{l \to \infty} pb(Su, Su) = 0 \). On the other hand, by (pb)_4,

\[
pb(Su, Ru) \leq s[pb(Su, Su) + pb(Su, Ru)] - pb(Su, Su),
\]

and then, \( pb(Su, Ru) = 0 \), as \( l \to \infty \). This proves that \( z = Su = Ru \), that is, \( z \) is a point of coincidence for \( S \) and \( R \).

**Example 2.** Let \( M = [0, \infty) \) and \( pb : M \times M \to [0, \infty) \) be a partial \( b \)-metric, where \( pb(u, y) = (\max \{ u, y \})^2 \). Let the mappings \( S, R : M \to M \),

\[
Su = \begin{cases} 
\frac{u + 1}{2}, & \text{if } u \in [0, 1], \\
3, & \text{if } u > 1,
\end{cases}
\]

\[
Ru = \begin{cases} 
\frac{u + 2}{4}, & \text{if } u \in [0, 1], \\
\frac{u + 5}{10}, & \text{if } u > 1,
\end{cases}
\]

and the function \( \beta : M \times M \to [0, \infty) \),

\[
\beta(x, y) = \begin{cases} 
2, & \text{for } x = y = 1, \\
3, & \text{for } x = y = 3, \\
1, & \text{for } x, y \geq 3, \\
0, & \text{otherwise}.
\end{cases}
\]

Obviously, since \( x = Ru \geq 4 \) for \( u \geq 35 \) we have

(i) For \( u, y \geq 35 \)

\[
\beta(Ru, Ry) = 1 \iff \beta(Su, Sv) = \beta(3, 3) = 3 > 1,
\]

\[
\beta(\frac{1}{2}, \frac{1}{2}) = \beta(R(0), R(0)) = 2 \iff \beta(S(0), S(0)) = \beta(1, 1) = 2 = 2.
\]

\[
\beta(3, 3) = \beta(R(25), R(25)) = 3 \iff \beta(S(25), S(25)) = \beta(3, 3) = 3.
\]

Moreover,

\[
\beta(Ru, Ry) \min \{ pb(Su, Sy), pb(Sy, Ry) \} \leq \min \{ pb(u, y), pb(Sy, Ru) \} \leq \min \{ pb(3, 3), pb(3, Ry) \}
\]

\[
= 9, \leq \kappa \cdot 16 \leq \kappa \cdot \max \{ pb(Ru, Ry), pb(Su, Ru) \},
\]

for any \( 9/16 < \kappa < 1 \).

(ii) All other cases are uninteresting due to the way the function \( \beta \) was defined

Consequently, by Theorem 8, the mappings \( S, R \) have points of coincidence. These are \( 1/2 = S(0) = R(0) \), respectively, \( 3 = S(25) = R(25) \).

**Corollary 9.** Let \( (M, pb, s \geq 1) \) be a complete \( pb \)-ms and two mappings \( R, S : M \to M \). Suppose that there exists \( \kappa \in (0, 1) \) such that

\[
\min \{ pb(Su, Sy), pb(Sy, Ry) \} \leq \min \{ b_p(Su, Sy), b_p(Sy, Ru) \} \leq \kappa \max \{ pb(Ru, Rv), pb(Su, Ru) \},
\]

for every \( u, y \in M \), such that \( u \neq y \) when \( u, y \in C_s(R, S) \). If \( S (M) \subset R(M) \) and \( (R(M), pb, s) \) is a \( 0-pb \)-complete \( pb \)-ms, then the mappings \( S \) and \( R \) have a point of coincidence.

**Proof.** It is enough to choose \( \beta(u, y) = 1 \) in Theorem 8.

**Theorem 10.** Let \( (M, pb, s \geq 1) \) be a complete \( pb \)-ms and a mapping \( S : M \to M \). Suppose that there exists \( \kappa \in (0, 1) \) such that

\[
\beta(u, y) \min \{ pb(Su, Sy), pb(Sy, y) \} \leq \min \{ b_p(Su, y), b_p(Sy, u) \} \leq \kappa \max \{ pb(Ru, Rv), pb(Su, Ru) \},
\]

for every \( u, y \in M \), such that \( u \neq y \). Suppose also that

(a) \( S \) is \( \beta \)-admissible, and there exists \( u_0 \in M \) such that \( \beta(u_0, Su_0) \geq 1 \)

(b) \( M \) is \( \beta \)-regular

Then, the mapping \( S \) has a fixed point.

**Proof.** Put \( R = I_M \) in Theorem 8.

**Corollary 11.** Let \( (M, pb, s \geq 1) \) be a complete \( pb \)-ms and a mapping \( S : M \to M \). Suppose that there exists \( \kappa \in (0, 1) \) such that

\[
\min \{ pb(Su, Sy), pb(Sy, y) \} \leq \min \{ b_p(Su, y), b_p(Sy, u) \} \leq \kappa \max \{ pb(u, y), pb(Su, u) \},
\]

for every \( u, y \in M, u \neq y \). Then, the mapping \( S \) has a fixed point.

**Proof.** It is enough to choose \( \beta(u, y) = 1 \) in Theorem 10.

**Theorem 12.** Let \( (M, p_b, s \geq 1) \) be a complete \( p_b \)-ms and two mappings \( R, S : M \to M \). Suppose that there exist \( \kappa \in (0, 1) \)
and $a > 0$ such that
\[
\beta(Ru, Ry)M^1_{S,R}(u, y) - a \cdot N_{S,R}^1(u, y) \leq \kappa p_b(Su, Ru) p_b(Sy, Ry),
\]
where
\[
M^1_{S,R}(u, y) = \min \{ |p_b(Su_n, Su_{n+1})|^2, |p_b(Su_{n+1}, Ru_{n+1})|^2 \},
\]
\[
N_{S,R}^1(u, y) = \min \{ |b_p(Su_n, Ry)|, p_b(Su_n, Ru_n)p_b(Su_{n+1}, Su_{n+1}), p_b(Su_{n+1}, Ru_{n+1}) \}.
\]

for every $u, y \in M$, such that $u \neq y$ when $u, y \in C_f(R, S)_M$.

Suppose also that:

(a) $S(M) \subset R(M)$ and $(R(M), p_b, s)$ is a 0-$p_b$-complete $p_b$-ms

\[
M^1_{S,R}(u_n, u_{n+1}) = \min \{ |p_b(Su_n, Su_{n+1})|^2, |p_b(Su_{n+1}, Ru_{n+1})|^2 \}
\]
\[
= \min \{ |p_b(z_n, z_{n+1})|^2, |p_b(z_{n+1}, z_n)|^2 \}
\]
\[
= |p_b(z_{n+1}, z_n)|^2,
\]
\[
N_{S,R}^1(u_n, u_{n+1}) = \min \{ b_p(Su_n, Ru_n)b_p(Su_{n+1}, Ru_n), p_b(Su_n, Ru_n)p_b(Su_{n+1}, Su_{n+1}), p_b(Su_{n+1}, Ru_{n+1}) \}
\]
\[
= \min \{ b_p(z_n, z_n)b_p(z_{n+1}, z_{n-1}), p_b(z_n, z_{n+1})p_b(z_{n+1}, z_n), p_b(z_{n+1}, u_n)p_b(u_n, z_n) \} = 0,
\]

Taking into account (46), the above inequality turns into
\[
|p_b(z_n, z_{n+1})|^2 \leq \kappa p_b(z_n, z_{n-1}) p_b(z_{n+1}, z_n),
\]
or equivalent (since $z_n \neq z_{n+1}$)
\[
p_b(z_n, z_{n+1}) \leq \kappa p_b(z_n, z_{n-1}).
\]

Accordingly, from Lemma 7, it follows that the sequence $\{z_n\}$ is 0-$p_b$-Cauchy and due to the completeness of the space, there exists $z \in M$ such that $\lim n \rightarrow \infty p_b(z_n, z) = p_b(z, z) = 0$. Following the corresponding lines in Theorem 8, we can find $u_0 \in M$ such that $Ru_0 = z$. Supposing that $Ru_0 \neq Su_0$ for $u = u_n$ and $y = u_{n+1}$ and taking into account the assumption (c),
\[
M^1_{S,R}(u_n, u_{n+1}) \leq \beta(Ru_n, Ru_{n+1})M^1_{S,R}(u_n, u_{n+1}) - a \cdot N_{S,R}^1(u_n, u_{n+1})
\]
\[
\leq \kappa p_b(Su_n, Ru_n) \cdot p_b(Su_{n+1}, Ru_{n+1}),
\]
\[
M^1_{S,R}(u_n, u_{n+1}) \leq \beta(Ru_n, Ru_{n+1})M^1_{S,R}(u_n, u_{n+1}) - a \cdot N_{S,R}^1(u_n, u_{n+1})
\]
\[
\leq \kappa p_b(Su_n, Ru_n) \cdot p_b(Su_{n+1}, Ru_{n+1}).
\]

where
\[
M^1_{S,R}(u_n, u_{n+1}) = \min \{ |p_b(Su_n, Su_{n+1})|^2, |p_b(Su_{n+1}, Ru_{n+1})|^2 \},
\]
\[
N_{S,R}^1(u_n, u_{n+1}) = \min \{ |b_p(Su_n, Ru_n)|, p_b(Su_n, Ru_n)p_b(Su_{n+1}, Su_{n+1}), p_b(Su_{n+1}, Ru_{n+1})p_b(Ru_n, Ru_{n+1}) \}.
\]
Since \( \lim_{l \to \infty} N_{S,R}^{1}(u_n, u_S) = 0 \) and \( \lim_{l \to \infty} p_b(Su_n, Ru_n) \), \( p_b(Su_n, Ru_n) = 0 \) (by) letting \( l \to \infty \) in (50), we have

\[
(1) \text{ If } [p_b(Su_n, Ru_n)]^2 = 0, \text{ it follows that } Su_n = Ru_n.
\]

\[
(2) \text{ If } \lim_{l \to \infty} [p_b(Su_n, Su_n)]^2 = 0, \text{ by (pb)_4}
\]

so \( p_b(Su_n, Su_n) = 0 \).

Thereupon, \( Ru_n = Su_n = z \) and \( z \) is a point of coincidence of \( R \) and \( S \).

**Example 3.** Let \( M = \{a_1, a_2, a_3, a_4, a_5\} \) and the partial \( b \)-metric \( p_b : M \times M \to [0, +\infty) \) defined as follows (Table 1).

\[
M_{S,R}^{1}(a_2, a_5) = \min \{ [p_b(Sa_2, Sa_5)]^2, [p_b(Sa_5, Ra_2)]^2 \} = \min \{ [p_b(a_3, a_2)]^2, [p_b(a_2, a_3)]^2 \} = 9,
\]

\[
N_{S,R}^{1}(a_2, a_5) = \min \{ b_p(Sa_2, Ra_2)b_p(Sa_5, Ra_2), \ldots \} = \min \{ b_p(a_3, a_2)b_p(a_2, a_3), \ldots \} = 0,
\]

\[
p_b(Sa_2, Ra_2)p_b(Sa_5, Ra_5) = p_b(a_3, a_2)p_b(a_2, a_3) = 22 \cdot 3 = 66.
\]

\[
(2) \text{ (u, y)} = (a_3, a_4)
\]

\[
M_{S,R}^{1}(a_3, a_4) = \min \{ [p_b(Sa_3, Sa_4)]^2, [p_b(Sa_4, Ra_3)]^2 \} = \min \{ [p_b(a_4, a_3)]^2, [p_b(a_3, a_2)]^2 \} = 1,
\]

\[
N_{S,R}^{1}(a_3, a_4) = \min \{ b_p(Sa_3, Ra_4)b_p(Sa_4, Ra_3), \ldots \} = \min \{ b_p(a_4, a_3)b_p(a_3, a_2), \ldots \} = 0,
\]

\[
p_b(Sa_3, Ra_4)p_b(Sa_4, Ra_3) = p_b(a_4, a_3)p_b(a_2, a_5) = 3 \cdot 1 = 3.
\]

So, for any \( \kappa \in (0, 1) \), the inequality (42) holds. Therefore, the mappings \( S, R \) have a point of coincidence, which is \( z = a_2 \).

**Corollary 13.** Let \( (M, p_b, s \geq 1) \) be a complete \( p_b \)-ms and two mappings \( R, S : M \to M \). Suppose that there exist \( \kappa \in (0, 1) \) and \( a > 0 \) such that

\[
M_{S,R}^{1}(u, y) - a \cdot N_{S,R}^{1}(u, y) \leq \kappa p_b(Su, Ru)p_b(Sy, Ry),
\]

where

\[
M_{S,R}^{1}(u, y) = \min \{ [p_b(Su, Sy)]^2, [p_b(Sy, Ry)]^2 \},
\]

\[
N_{S,R}^{1}(u, y) = \min \{ b_p(Su, Ry)b_p(Sy, Ru)p_b(Su, Ry)p_b(Su, Sy), p_b(Sy, Ru)p_b(Ru, Ry) \}.
\]
for every \( u, y \in M \), such that \( u \neq y \) when \( u, y \in C_{\mathcal{C}}(R, S) \).

Then, the mappings \( S \) and \( R \) have a point of coincidence providing that \( S(M) \subset R(M) \) and \( (R(M), p_b, s) \) is a \( 0 \)-\( \beta \)-complete \( p_b \)-ms.

**Proof.** Put \( \beta(u, y) = 1 \) in Theorem 12.

**Theorem 14.** Let \( (M, p_b, s \geq 1) \) be a complete \( p_b \)-ms a mapping \( S : M \rightarrow M \). Suppose that there exist \( \kappa \in (0, 1) \) and \( a > 0 \) such that

\[
\beta(u, y)M^2_{S,u,y}(u, y) - a \cdot N^1_{S,u,y}(u, y) \leq \kappa p_b(Su, u)p_b(Sy, y),
\]

where

\[
M^2_{S,u,y}(u, y) = \min \left\{ [p_b(Su, Sy)]^2, [p_b(Sy, y)]^2 \right\},
\]

\[
N^1_{S,u,y}(u, y) = \min \left\{ b_p(Su, y)b_p(Sy, u), p_b(Su, y)p_b(Sy, u), p_b(Su, u)p_b(y, y) \right\},
\]

for every \( u, y \in M, u \neq y \). Suppose also that

(a) \( S \) is \( \beta \)-admissible, and there exists \( u_0 \in M \) such that \( \beta(u_0, Su_0) \geq 1 \)

(b) \( M \) is \( \beta \)-regular

Then, the mapping \( S \) possesses a fixed point.

**Proof.** Choose \( R = I_M \) in Theorem 12.

**Corollary 15.** Let \( (M, p_b, s \geq 1) \) be a complete \( p_b \)-ms a mapping \( S : M \rightarrow M \). Suppose that there exist \( \kappa \in (0, 1) \) and \( a > 0 \) such that

\[
M^2_{S,u,y}(u, y) - a \cdot N^1_{S,u,y}(u, y) \leq \kappa p_b(Su, u)p_b(Sy, y),
\]

where
Thus, the inequality (64) becomes
\[
[p_b(z_{n+1}, z_{n+1})]^2 \leq \kappa p_b(z_{n-1}, z_n) \cdot \max \{p_b(z_{n-1}, z_n), p_b(z_n, z_{n+1})\}.
\]
(67)

Since for the case \(\max \{p_b(z_{n-1}, z_n), p_b(z_n, z_{n+1})\} = p_b(z_n, z_{n+1})\) we get \(p_b(z_{n+1}, z_{n+1}) = p_b(z_n, z_{n+1})\), for any \(n \in \mathbb{N}\). Therefore, by Lemma L2A and using similar arguments as in Theorems 8 and 12, there exists \(u_n \in M\) such that
\[
l_{\rightarrow \infty} p_b(Su_n, Ru_n) = \lim_{n \rightarrow \infty} p_b(Ru_{n+1}, Ru_n) = p_b(Ru_\ast, Ru_\ast) = 0.
\]
(69)

Finally, we claim that \(Su_\ast = Ru_\ast\). From the assumptions (c), there exists a subsequence \(\{u_{n_l}\}\) of \(\{u_n\}\) such that \(\beta(u_{n_l}, u_\ast) \geq 1\). Thus, replacing \(u\) by \(u_{n_l}\) and \(y\) by \(u_\ast\), we get (as \(l \rightarrow \infty\))
\[
l_{\rightarrow \infty} M^{2}_{S,R}(u_{n_l}, u_\ast) = \lim_{n \rightarrow \infty} \left[p_b(Su_{n_l}, Su_{n_l})p_b(Su_{n_l}, Ru_{n_l}) - a \cdot \min \{b_y(Su_{n_l}, Ru_{n_l}), b_y(Su_{n_l}, Ru_{n_l})\}\right]
\]
\[= p_b(Su_{n_l}, Ru_{n_l}) \cdot \lim_{n \rightarrow \infty} \left[p_b(Su_{n_l}, Su_{n_l})\right],
\]
\[l_{\rightarrow \infty} N^{2}_{S,R}(u_{n_l}, u_\ast) = \lim_{n \rightarrow \infty} \left[p_b(Ru_{n_l}, Ru_{n_l}) \cdot \max \left\{p_b(Su_{n_l}, Ru_{n_l}), p_b(Su_{n_l}, Ru_{n_l})\right\} = 0\right].
\]
(70)

Consequently, (64) becomes \(p_b(Su_\ast, Ru_\ast) \cdot \lim_{n \rightarrow \infty} [p_b(Su_{n_l}, Ru_{n_l})]
\]
\[\leq \kappa \cdot N^{2}_{S,R}(u_\ast, y),
\]
(71)

for every \(u, y \in M\), such that \(u \neq y\) when \(u, y \in C_{s}(R, S)\). If \(S(M) \subset R(M)\) and \(R(M), p_{b_0}\) is a \(0-p_{b_0}\)-complete \(\kappa\)-ms, then, the mappings \(S\) and \(R\) have a point of coincidence.

Proof. Let \(\beta(u, y) = 1\) in Theorem 16.

Theorem 18. Let \((M, p_b, s \geq 1)\) be a complete \(p_b\)-ms and a mapping \(S : M \rightarrow M\). Suppose that there exist \(\kappa \in (0, 1)\) and \(a > 0\) such that
\[
\beta(u, y)M^{2}_{S}(u, y) \leq \kappa \cdot N^{2}_{S}(u, y),
\]
(72)

for every \(u, y \in M\). Suppose also that
(i) \(S\) is \(\beta\)-admissible, and there exists \(u_0 \in M\) such that \(\beta(Ru_0, Su_0) \geq 1\);
(ii) \(M\) is \(\beta\)-regular

Then, the mapping \(S\) admits a fixed point.

Proof. Choose \(R = I_M\).

Corollary 19. Let \((M, p_b, s \geq 1)\) be a complete \(p_b\)-ms and two mappings \(R, S : M \rightarrow M\). Suppose that there exist \(\kappa \in (0, 1)\) and \(a > 0\) such that
\[
M^{2}_{S}(u, y) \leq \kappa \cdot N^{2}_{S}(u, y),
\]
(73)

where
for every \( u, y \in M \). Then, the mapping \( S \) has a fixed point.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no competing interests.

**Authors’ Contributions**

Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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