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Research Article

On the Blowup for the 3D Axisymmetric Incompressible Chemotaxis-Euler Equations

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In this paper, we investigate the 3D incompressible chemotaxis-Euler equations. Taking advantage of the structure of axisymmetric fluids, we establish the blowup criterion of the system using the Fourier localization method.

1. Introduction

The effect of oxygen attraction on the emergence of bioconvective patterns is studied in [1, 2]. Some experiments, such as a colony of Bacillus subtilis suspending in a drop of water, are carried out to identify this phenomenon. From paper [3–6], we can also find the important role of chemotaxis between sperm and eggs. The following model, in [1], is introduced to analyze the above phenomenon:

$$n_t + u \cdot \nabla n = \Delta n + \chi \nabla \cdot \left(n \nabla (\Delta)^{-1} c \right) + \kappa n - \mu n^2, \quad n(x, 0), x \in d. \tag{1}$$

Here, n and u represent the concentration of bacteria and the velocity field of the transported water, respectively. Besides, the vector field u is divergence free and independent of n. The equation describes the evolution of the bacteria transported by the velocity field of the fluid. Moreover, these cells are attracted by the oxygen concentration generated by chemotaxis. For the term $\chi \nabla \cdot (n \nabla (\Delta)^{-1} c)$, $\chi > 0$ is a parameter controlling the influence of the chemotactic effect. In addition, κ is the strength growth rate of the population and μ is a parameter regulating death by overcrowding.

Apart from Equation (1), there are a lot of other models illustrating the procedure of oxygen attraction in biology. An increasing number of mathematicians studied the process in the past years, see [7–15]. Our aim in this paper is to further explore model (1), combined with an oxygen equation and a Navier-Stokes equation, see [16]. Then, we obtain the following model in d, $d = 2, 3$,

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \chi \nabla \cdot (n \nabla c) + \kappa n - \mu n^2, \\ c_t + u \cdot \nabla c = \Delta c - nc, \\ u_t + (u \cdot \nabla) u - \nabla P = \eta \Delta u - n \nabla \Phi, \\ \nabla \cdot u = 0, \\ n(0, x) = n_0(x), c(0, x) = c_0(x), u(0, x) = u_0(x). \end{cases} \tag{2}$$

The unknowns are n, c, u, and P, standing for the bacteria, the oxygen, the velocity field, and the pressure of the fluid separately. The third equation of the above system contained an extra force, buoyancy, which is produced by the density and a given gravitational potential Φ. η is the dissipation coefficient. If $\kappa = \mu = 0$ and $\eta > 0$, the global existence of weak solutions in 2 was shown in [8, 12].

In this paper, we choose $\chi = \kappa = \mu = 1$ and $\eta = 0$, then (2) can be changed into the following one:

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c) + n - n^2, \\ c_t + u \cdot \nabla c = \Delta c - nc, \\ u_t + (u \cdot \nabla) u - \nabla P = -n \nabla \Phi, \\ \nabla \cdot u = 0, \\ n(0,x) = n_0(x), c(0,x) = c_0(x), u(0,x) = u_0(x). \end{cases} \tag{3}$$

The Euler equation is shown as the following form:

$$\begin{cases} u_t + u \cdot \nabla u + \nabla P = 0, \\ \nabla \cdot u = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{4}$$

In three dimensional space, the vorticity equation has the form

$$\omega_t + u \cdot \nabla \omega = \omega \cdot \nabla u. \tag{5}$$

But the chief difficulty is we are lacking information on the vortex-stretching term $\omega \cdot \nabla u$. Although the global existence of classical solutions for the 3D Euler equation is an open problem, some known results are obtained under the circumstances of axisymmetric flows without swirl. That a vector field u is axisymmetric without swirl is defined as follows:

$$u(t,x) = u^r(t,r,z) e_r + u^z(t,r,z) e_z, \quad x = (x_1, x_2, z), r = \left(x_1^2 + x_2^2\right)^{1/2}, \tag{6}$$

where (e_r, e_θ, e_z) is the cylindrical basis of 3 and the components u^r and u^z do not depend on the angular variable. With this structure, vorticity takes the form

$$\omega = (\partial_z u^r - \partial_r u^z) e_\theta := \omega_\theta e_\theta \tag{7}$$

and satisfies

$$\partial_t \omega + u \cdot \nabla \omega = \frac{u^r}{r} \omega. \tag{8}$$

Hence, the quantity $\Gamma := \omega_\theta / r$ obeys to the equation

$$\partial_t \Gamma + u \cdot \nabla \Gamma = 0. \tag{9}$$

The goal of this paper is to build the blowup criterion of smooth solutions for (3) by the Fourier localization technique. Here, we follow ideas introduced in [17–21]. Our result reads as the following:

Theorem 1. *For $s > 3$, suppose the triple $(n_0, c_0, u_0) \in H^s \times H^{s+1} \times H^{s+1}$ and $\Phi \in H^{s+2}$. Let u_0 be an axisymmetric divergence-free vector field and its vorticity satisfies $\omega_0 / r \in L^{3,1}$. Assume that*

$$\begin{gathered} n \in C([0,T]; H^s) \cap L^2([0,T]; H^{s+1}), \\ c \in C([0,T]; H^{s+1}) \cap L^2([0,T]; H^{s+2}), \\ u \in C([0,T]; H^{s+1}) \end{gathered} \tag{10}$$

are the smooth solutions to (3). If the condition

$$\int_0^T \left(\|n(\tau)\|^2_{B^0_{\infty,1}} \right) d\tau < \infty \tag{11}$$

holds true, then the solutions (n, c, u) can be extended beyond $T > 0$.

Remark 2. In paper [20], a regularity criterion in terms of two items is established. But in Theorem 1, we give a different criterion using the only bacteria concentration in 3. The bacteria concentration plays a more important role in this model, and the nonlinear term $-\nabla \cdot (\mathrm{n} \nabla \mathrm{c})$ is difficult to estimate. Hence, using bacteria concentration to show the regularity is natural and physical.

Notation. Throughout the paper, C means a harmless constant and may vary from line to line; C_T denotes a constant C relating to T; $\|\cdot\|_p$ stands for the norm of the Lebesgue space L^p.

2. Preliminaries

In this section, we give the definition of some function spaces and recall some useful lemmas.

Firstly, we use the dynamic partition of the unity to give the definition of Besov spaces. One may check [22] for exact details. Let $\varphi \in C_0^\infty(d)$ be set in $C = \{\xi \in d, 3/4 \le |\xi| \le 8/3\}$ satisfying

$$\sum_{q \in Z} \varphi(2^{-q}\xi) = 1, \quad \text{for } \xi \neq 0. \tag{12}$$

Let $\chi(\xi) = 1 - \sum_{q \in N} \varphi(2^{-q}\xi)$. For $f \in S'$, Littlewood-Paley operators are defined as follows:

$$\Delta_{-1} f = \chi(D) f; \forall q \in \Delta_q f = \varphi(2^{-q}D) f \text{ and} \forall q \in \dot{\Delta}_q f = \varphi(2^{-q}D) f. \tag{13}$$

The low-frequency cut-offs are denoted:

$$\begin{gathered} S_q f = \sum_{-1 \le q' \le q-1} \Delta_{q'} f, \\ \dot{S}_q f = \sum_{q' \le q-1} \dot{\Delta}_q{}' f. \end{gathered} \tag{14}$$

Now, we introduce the definition of the Besov space. For $s \in \mathbb{R}, 1 \le p, r \le \infty$, the homogenous Besov space $\dot{B}^s_{p,r}$ is defined as the set of tempered distributions of $f \in \mathcal{S}'/\mathcal{P}$ satisfying

$$\|f\|_{\dot{B}^s_{p,r}} := \left(\sum_{q\in Z} 2^{qsr}\left\|\dot{\Delta}_q f\right\|_p^r\right)^{1/r} < \infty, \tag{15}$$

where $\mathcal{P}$ is the polynomial space. The inhomogeneous space $B^s_{p,r}$ is the set of tempered distribution f with the norm

$$\|f\|_{B^s_{p,r}} := \left(\sum_{q\ge -1} 2^{qsr}\left\|\Delta_q f\right\|_p^r\right)^{1/r} < \infty. \tag{16}$$

It is worthwhile to remark that $B^s_{2,2}$ and $B^s_{\infty,\infty}$ coincide with the usual Sobolev spaces H^s and the usual Hölder space C^s for $s \in \mathbb{R} \setminus \mathbb{Z}$, respectively.

In our study, we require the space-time Besov spaces as the following manner: for $T > 0$ and $\rho \ge 1$, we denote by $L^\rho_T B^s_{p,r}$ the set of all tempered distribution f such that

$$\|f\|_{L^\rho_T B^s_{p,r}} \triangleq \left\|\left(\sum_{q\in\mathbb{Z}} 2^{qsr}\left\|\dot{\Delta}_q f\right\|^r_{L^p(\mathbb{R}^d)}\right)^{1/r}\right\|_{L^\rho_T} < \infty. \tag{17}$$

Lemma 3 (see [22]). *Let $1 \le p \le q \le \infty$. Suppose that $f \in L^p$, then there exists a constant C independent of f, j such that*

$$\begin{aligned} &\sup p\widehat{f} \subset \left\{|\xi|\le C2^j\right\} \Rightarrow \|\partial^\alpha f\|_q \le C2^{j|\alpha|+dj(1/p-1/q)}\|f\|_p, \\ &\sup p\widehat{f} \subset \left\{\frac{1}{C}2^j\le|\xi|\le C2^j\right\} \Rightarrow \|f\|_p \le C2^{-j|\alpha|}\sup_{|\beta|=|\alpha|}\|\partial^\beta f\|_p. \end{aligned} \tag{18}$$

Lemma 4 (see [22]). *There exists a constant $C > 0$ such that for $s > 0$, we have*

$$\|uv\|_{H^s} \le C\|u\|_\infty\|v\|_{H^s} + C\|u\|_{H^s}\|v\|_\infty. \tag{19}$$

Lemma 5 (see [23]). *Let u be a solution of the transport equation*

$$\begin{cases} u_t + v\cdot\nabla u = 0, \\ u(x,0) = u_0, \end{cases} \tag{20}$$

and define $R_q := v\cdot\nabla\Delta_q u - \Delta_q(v\cdot\nabla u)$, $1 \le p \le p_1 \le \infty$, $1 \le r \le \infty$, and $s \in$ such that $s > -d \min(1/p_1, 1/p')$ (or $s>-1-d \min(1/p_1, 1/p')$ if $\operatorname{div} v = 0$). There exists a sequence $c_q \in \ell^r(\mathbb{Z})$ such that $\|c_q\|_{\ell^r} = 1$ and a constant C depending only on d, r, s, p, and p_1, which satisfy

$$\forall q \in \mathbb{Z}, 2^{qs}\|R_q\|_p \le Cc_q Z'(t)\|u\|_{B^s_{p,r}}, \tag{21}$$

with

$$Z'(t) := \begin{cases} \|\nabla v\|_{B^{d/p_1}_{p_1,\infty}\cap L^\infty}, \text{ if } s < 1 + \dfrac{d}{p_1}, \\ \|\nabla v\|_{B^{s-1}_{p_1,r}}, \text{ if either } s > 1 + \dfrac{d}{p_1} \text{ or } s = 1 + \dfrac{d}{p_1} \text{ for } r = 1. \end{cases} \tag{22}$$

Lemma 6 (see [24]). *Let $[p, r] \in [1,\infty]^2$, v be a divergence-free vector-field belonging to the space $L^1_{loc}(\mathbb{R}_+; Lip(\mathbb{R}^d)$ and let a be a smooth solution of the following transport equation:*

$$\begin{cases} a_t + v\cdot\nabla a = f, \\ a(x,0) = a_0. \end{cases} \tag{23}$$

If the initial data $a_0 \in B^0_{p,r}$, then we have for all $t \in +\mathbb{R}$

$$\|a\|_{\tilde{L}^\infty_t B^0_{p,r}} \le C\left(\|a_0\|_{B^0_{p,r}} + \|f\|_{\tilde{L}^1_t B^0_{p,r}}\right)\left(1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\right). \tag{24}$$

3. Proof of Theorems

3.1. Local Well-Posedness. We construct the following smoothing system:

$$\begin{cases} n^k_t + u^k\cdot\nabla n^k = \Delta n^k - \nabla\cdot\left(n^k\nabla c^k\right) + n^k - \left(n^k\right)^2, \quad k \in N, \\ c^k_t + u^k\cdot\nabla c^k = \Delta c^k - c^k n^k, \\ u^k_t + \left(u^k\cdot\nabla\right)u^k - \nabla P^k = \Delta u^k + n^k\nabla\Phi, \\ \nabla\cdot u^k = 0, \\ \left(n^k, c^k, u^k\right)\Big|_{t=0} = \left(S_k n^0, S_k c^0, S_k u^0\right). \end{cases} \tag{25}$$

Step 1. Uniform boundedness.

Taking the operation Δ_q with $q \ge -1$ on the first equation of (25), we obtain

$$\Delta_q n^k_t + \Delta_q\left(u^k\cdot\nabla n^k\right) = \Delta\Delta_q n^k - \nabla\cdot\Delta_q\left(n^k\nabla c^k\right) + \Delta_q n^k - \Delta_q\left(n^k\right)^2. \tag{26}$$

Making the L^2-inner product for (26) with $\Delta_q n^k$ yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta_q n^k\|_2^2+\|\nabla\Delta_q n^k\|_2^2 &= -\int_{\mathbb{R}^d}\Delta_q\left(u^k\cdot\nabla n^k\right)\Delta_q n^k dx\\ &-\int_{\mathbb{R}^d}\nabla\cdot\Delta_q\left(n^k\nabla c^k\right)\Delta_q n^k dx+\int_{\mathbb{R}^d}\Delta_q n^k\Delta_q n^k dx\\ &-\int_{\mathbb{R}^d}\Delta_q\left(n^k\right)^2\Delta_q n^k dx \le \|\Delta_q\left(u^k\cdot\nabla n^k\right)\|_2\|\Delta_q n^k\|_2+\|\Delta_q\\ &\cdot\left(n^k\nabla c^k\right)\|_2\|\nabla\Delta_q n^k\|_2+\|\Delta_q n^k\|_2^2+\|\Delta_q\left(n^k\right)^2\|_2\|\Delta_q n^k\|_2.\end{aligned} \tag{27}$$

Multiplying 2^{2qs} on both sides of the above inequality, then taking the ℓ^1 norm, using Hölder's inequality and Young's inequality together with Lemma 4, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|n^k\|_{H^s}^2+\|n^k\|_{H^{s+1}}^2 &\le \|u^k\cdot\nabla n^k\|_{H^s}\|n^k\|_{H^s}+\|n^k\nabla c^k\|_{H^s}\|n^k\|_{H^{s+1}}\\ &+\|n^k\|_{H^s}^2+\|\left(n^k\right)^2\|_{H^s}\|n^k\|_{H^s}\le C\|u^k\|_{H^s}\|n^k\|_{H^{s+1}}\|n^k\|_{H^s}\\ &+C\|n^k\|_{H^s}\|c^k\|_{H^{s+1}}\|n^k\|_{H^{s+1}}+\|n^k\|_{H^s}^2+C\|n^k\|_{H^s}^2\|n^k\|_{H^s}\\ &\le C\|u^k\|_{H^s}^2\|n^k\|_{H^s}^2+\frac{1}{8}\|n^k\|_{H^{s+1}}^2+C\|n^k\|_{H^s}^2\|c^k\|_{H^{s+1}}^2\\ &+\frac{1}{8}\|n^k\|_{H^{s+1}}^2+\|n^k\|_{H^s}^2+C\left(\|n^k\|_{H^s}^4+\|n^k\|_{H^s}^2\right).\end{aligned} \tag{28}$$

Then, we conclude

$$\frac{d}{dt}\|n^k\|_{H^s}^2+\|n^k\|_{H^{s+1}}^2 \le C\left(\|u^k\|_{H^s}^2\|n^k\|_{H^s}^2+\|n^k\|_{H^s}^2\|c^k\|_{H^{s+1}}^2+\|n^k\|_{H^s}^2+\|n^k\|_{H^s}^4\right). \tag{29}$$

In a similar way to (29), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|c^k\|_{H^{s+1}}^2+\|c^k\|_{H^{s+2}}^2 &\le C\|u^k\|_{H^{s+1}}^2\|c^k\|_{H^{s+1}}^2\\ &+\frac{1}{8}\|c^k\|_{H^{s+2}}^2+C\|c^k\|_{H^{s+1}}^4+\frac{1}{8}\|n^k\|_{H^{s+1}}^2.\end{aligned} \tag{30}$$

Thus, we have

$$\frac{d}{dt}\|c^k\|_{H^{s+1}}^2+\|c^k\|_{H^{s+2}}^2 \le C\left(\|u^k\|_{H^{s+1}}^2\|c^k\|_{H^{s+1}}^2+\|c^k\|_{H^{s+1}}^4\right)+\frac{1}{8}\|n^k\|_{H^{s+1}}^2. \tag{31}$$

Operating Δ_q with $q\ge -1$ to the third equation of (25) implies

$$\begin{aligned}\Delta_q u_t^k+\left(u^k\cdot\nabla\right)\Delta_q u^k-\nabla\Delta_q P^k &= \left(u^k\cdot\nabla\right)\Delta_q u^k\\ &-\Delta_q\left(\left(u^k\cdot\nabla\right)u^k\right)-\Delta_q\left(n^k\nabla\Phi\right).\end{aligned} \tag{32}$$

Taking the L^2-inner product for the above equality with $\Delta_q u^k$ gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta_q u^k\|_2^2 &= \int_{\mathbb{R}^d}\left(\left(u^k\cdot\nabla\right)\Delta_q u^k-\Delta_q\left(\left(u^k\cdot\nabla\right)u^k\right)\right)\Delta_q u^k dx\\ &-\int_{\mathbb{R}^d}\Delta_q\left(n^k\nabla\Phi\right)\Delta_q u^k dx\le \|\left(u^k\cdot\nabla\right)\Delta_q u^k\\ &-\Delta_q\left(\left(u^k\cdot\nabla\right)u^k\right)\|_2\|\Delta_q u^k\|_2+\|\Delta_q\left(n^k\nabla\Phi\right)\|_2\|\Delta_q u^k\|_2.\end{aligned} \tag{33}$$

Multiplying $2^{2q(s+1)}$ on both sides of the above inequality and taking the ℓ^1 norm, we conclude

$$\begin{aligned}\frac{d}{dt}\|u^k\|_{H^{s+1}}^2 &\le \|\nabla u^k\|_\infty\|u^k\|_{H^{s+1}}^2+\frac{1}{8}\|n^k\|_{H^{s+1}}^2+\|u^k\|_{H^{s+1}}^2\\ &\le C\left(\|u^k\|_{H^{s+1}}^4+\|u^k\|_{H^{s+1}}^2\right)+\frac{1}{8}\|n^k\|_{H^{s+1}}^2.\end{aligned} \tag{34}$$

Collecting (29)–(34), we have

$$\begin{aligned}&\frac{d}{dt}\left(\|n^k\|_{H^s}^2+\|c^k\|_{H^{s+1}}^2+\|u^k\|_{H^{s+1}}^2\right)+\|n^k\|_{H^{s+1}}^2+\|c^k\|_{H^{s+2}}^2\\ &\le C\left(\|n^k\|_{H^s}^2+\|c^k\|_{H^{s+1}}^2+\|u^k\|_{H^{s+1}}^2\right)\left(1+\|n^k\|_{H^s}^2+\|c^k\|_{H^{s+1}}^2+\|u^k\|_{H^{s+1}}^2\right)\\ &\le \left(1+\|n^k\|_{H^s}^2+\|c^k\|_{H^{s+1}}^2+\|u^k\|_{H^{s+1}}^2\right)^2.\end{aligned} \tag{35}$$

We obtain from the Gronwall inequality that

$$1+\|n^k\|_{H^s}^2+\|c^k\|_{H^{s+1}}^2+\|u^k\|_{H^{s+1}}^2 \le \frac{1+\|n_0^k\|_{H^s}^2+\|c_0^k\|_{H^{s+1}}^2+\|u_0^k\|_{H^{s+1}}^2}{1-C\left(1+\|n_0^k\|_{H^s}^2+\|c_0^k\|_{H^{s+1}}^2+\|u_0^k\|_{H^{s+1}}^2\right)t}. \tag{36}$$

Let

$$T=\frac{1}{2C\left(1+\|n_0^k\|_{H^s}^2+\|c_0^k\|_{H^{s+1}}^2+\|u_0^k\|_{H^{s+1}}^2\right)}>0, \tag{37}$$

then we obtain

$$\begin{aligned}&\sup_{t\in[0,T]}\left(\|n^k(t)\|_{H^s}^2+\|c^k(t)\|_{H^{s+1}}^2+\|u^k(t)\|_{H^{s+1}}^2\right)\\ &+\int_0^t\left(\|n^k\|_{H^{s+1}}^2+\|c^k\|_{H^{s+2}}^2\right)(\tau)d\tau\\ &\le 2\left(1+\|n_0^k\|_{H^s}^2+\|c_0^k\|_{H^{s+1}}^2+\|u_0^k\|_{H^{s+1}}^2\right).\end{aligned} \tag{38}$$

Step 2. Extracting sequences.

According to (38), we get

$$\begin{aligned}n^k &\in L^\infty([0,T],H^s)\cap L^2\left([0,T],H^{s+1}\right),\\ c^k &\in L^\infty\left([0,T],H^{s+1}\right)\cap L^2\left([0,T],H^{s+2}\right),\\ u^k &\in L^\infty\left([0,T],H^{s+1}\right).\end{aligned} \tag{39}$$

In order to prove the convergence, we require uniform boundedness for $\partial_t n^k$, $\partial_t c^k$, and $\partial_t u^k$. By the first equation of (25), we infer

$$\begin{aligned}\|\partial_t n^k\|_{L_t^\infty H^{-1}} \le{}& \|\Delta n^k\|_{L_t^\infty H^{-1}}+\|u^k\cdot\nabla n^k\|_{L_t^\infty H^{-1}}+\|\nabla\cdot\left(n^k\nabla c^k\right)\|_{L_t^\infty H^{-1}}+\|n^k\|_{L_t^\infty H^{-1}}+\|\left(n^k\right)^2\|_{L_t^\infty H^{-1}}\\ \le{}& \|n^k\|_{L_t^\infty H^s}+\|u^k\|_{L_t^\infty H^{s+1}}\|n^k\|_{L_t^\infty H^s}+\|n^k\|_{L_t^\infty H^s}\|c^k\|_{L_t^\infty H^{s+1}}\\ &+\|n^k\|_{L_t^\infty H^s}+\|n^k\|^2_{L_t^\infty H^s}\le C.\end{aligned} \tag{40}$$

In a similar process, we have

$$\begin{aligned}\|\partial_t c^k\|_{L_t^\infty H^{-1}} \le{}& \|\Delta c^k\|_{L_t^\infty H^{-1}}+\|u^k\cdot\nabla c^k\|_{L_t^\infty H^{-1}}+\|c^k n^k\|_{L_t^\infty H^{-1}}\\ \le{}& \|c^k\|_{L_t^\infty H^{s+1}}+\|u^k\|_{L_t^\infty H^{s+1}}\|c^k\|_{L_t^\infty H^{s+1}}\\ &+\|c^k\|_{L_t^\infty H^{s+1}}\|n^k\|_{L_t^\infty H^s}\le C.\\ \|\partial_t u^k\|_{L_t^\infty H^{-1}} \le{}& \|\left(u^k\cdot\nabla\right)u^k\|_{L_t^\infty H^{-1}}+\|n^k\nabla\Phi\|_{L_t^\infty H^{-1}}\\ \le{}& \|u^k\|^2_{L_t^\infty H^{s+1}}+\|n^k\|_{L_t^\infty H^s}\le C.\end{aligned} \tag{41}$$

Since L^2 is locally compactly embedded in H^{-1}, we can apply the Aubin-Lions Lemma to deduce that, extracting a subsequence, the approximate solution sequence (n^k, c^k, u^k) strongly converges in $L^\infty([0,T];H^{-1})$ to some function (n, c, u) such that

$$\begin{aligned} n^k &\in L^\infty([0,T];H^s)\cap L^2\left([0,T],H^{s+1}\right),\\ c^k &\in L^\infty\left([0,T];H^{s+1}\right)\cap L^2\left([0,T],H^{s+2}\right),\\ u^k &\in L^\infty\left([0,T];H^{s+1}\right).\end{aligned} \tag{42}$$

By the above estimates, we can easily have the limit in the approximate system (25) and (n, c, u) solve (3) in the sense of distribution. Using a classical method [12], we have $n\in C([0,T];H^s)$, $c\in C([0,T];H^{s+1})$, and $u\in C([0,T];H^{s+1})$.

Step 3. Uniqueness.

Let us consider the two solutions (n_1, c_1, u_1) and (n_2, c_2, u_2) associated with the same initial data and satisfy (3). We use the notation $\delta n = n_1 - n_2$, $\delta c = c_1 - c_2$, and $\delta u = u_1 - u_2$. Then, we have

$$\begin{cases}\partial_t\delta n+\delta u\cdot\nabla n_1+u_2\cdot\nabla\delta n=\Delta\delta n-\nabla\cdot(\delta n\nabla c_1)-\nabla\cdot(n_2\nabla\delta c)+\delta n-n_1\delta n-n_2\delta n,\\ \partial_t\delta c+\delta u\cdot\nabla c_1+u_2\cdot\nabla\delta c=\Delta\delta c-n_1\delta c-c_2\delta n,\\ \partial_t\delta u+(\delta u\cdot\nabla)u_1+(u_2\cdot\nabla)\delta u-\nabla(P_1-P_2)=\Delta\delta u+\delta n\nabla\Phi.\end{cases} \tag{43}$$

Multiplying the first equation of (43) by δn and integrating in spaces, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\delta n(t)\|_2^2+\|\nabla\delta n(t)\|_2^2={}&-\int_{\mathbb{R}^d}(\delta u\cdot\nabla n_1)\delta n dx\\ &-\int_{\mathbb{R}^d}\nabla\cdot(\delta u\nabla c_1)\delta n dx-\int_{\mathbb{R}^d}\nabla\cdot(n_2\nabla\delta c)\delta n dx\\ &+\int_{\mathbb{R}^d}\delta n\delta n dx-\int_{\mathbb{R}^d}n_1\delta n\delta n dx-\int_{\mathbb{R}^d}n_2\delta n\delta n dx\\ \le{}& C\left(\|\delta u\|_2^2+\|\delta n\|_2^2\|n_1\|_{H^s}^2\right)+C\|\delta n\|_2^2\|c_1\|_{H^{s+1}}^2\\ &+\frac{1}{8}\|\nabla\delta n\|_2^2+C\|n_2\|_{H^s}^2\|\nabla\delta c\|_2^2+\frac{1}{8}\|\nabla\delta n\|_2^2+\|\delta n\|_2^2\\ &+\|n_1\|_{H^s}\|\delta n\|_2^2+\|n_2\|_{H^s}\|\delta n\|_2^2,\end{aligned} \tag{44}$$

from which we conclude

$$\begin{aligned}\frac{d}{dt}\|\delta n(t)\|_2^2+\|\nabla\delta n(t)\|_2^2\le{}& C\left(\|\delta u\|_2^2+\|\delta n\|_2^2\|n_1\|_{H^s}^2\right.\\ &+\|\delta n\|_2^2\|c_1\|_{H^{s+1}}^2+\|n_2\|_{H^s}^2\|\nabla\delta c\|_2^2+\|\delta n\|_2^2\\ &\left.+\|n_1\|_{H^s}\|\delta n\|_2^2+\|n_2\|_{H^s}\|\delta n\|_2^2\right).\end{aligned} \tag{45}$$

Then, multiplying the second equation of (43) by δc and integrating in spaces, we know

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\delta c(t)\|_2^2+\|\nabla\delta c(t)\|_2^2={}&-\int_{\mathbb{R}^d}(\delta u\cdot\nabla c_1)\delta c dx\\ &-\int_{\mathbb{R}^d}\delta c n_1\delta c dx-\int_{\mathbb{R}^d}\delta n c_2\delta c dx\\ \le{}& C\left(\|\delta u\|_2^2+\|\delta c\|_2^2\|c_1\|_{H^{s+1}}^2+\|\delta c\|_2^2\|n_1\|_{H^s}\right.\\ &\left.+\|\delta n\|_2^2+\|\delta c\|_2^2\|c_2\|_{H^{s+1}}^2\right).\end{aligned} \tag{46}$$

Hence, we get

$$\begin{aligned}\frac{d}{dt}\|\delta c(t)\|_2^2+\|\nabla\delta c(t)\|_2^2\le{}& C\left(\|\delta u\|_2^2+\|\delta c\|_2^2\|c_1\|_{H^{s+1}}^2+\|\delta c\|_2^2\|n_1\|_{H^s}\right.\\ &\left.+\|\delta n\|_2^2+\|\delta c\|_2^2\|c_2\|_{H^{s+1}}^2\right).\end{aligned} \tag{47}$$

Applying ∂_i on both sides of the second equation of Equation (43) gives

$$\begin{aligned}\partial_t\partial_i\delta c+u_2\cdot\nabla\partial_i\nabla c-\Delta\partial_i\delta c={}&-\partial_i(\delta u\cdot\nabla c_1)-\partial_i u_2\cdot\nabla\delta c\\ &-\partial_i(n_1\delta c)-\partial_i(c_2\delta n).\end{aligned} \tag{48}$$

Taking the L^2-inner product for the above equation with

$\partial_i \delta c$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\delta c(t)\|_2^2+\|\Delta\delta c(t)\|_2^2 &= -\sum_i\int_{\mathbb{R}^d}\partial_i(\delta u\cdot\nabla c_1)\partial_i\delta c dx \\ &- \sum_i\int_{\mathbb{R}^d}\partial_i u_2\cdot\nabla\delta c\partial_i\delta c dx \\ &- \sum_i\int_{\mathbb{R}^d}\partial_i(n_1\delta c)\partial_i\delta c dx \\ &- \sum_i\int_{\mathbb{R}^d}\partial_i(c_2\delta n)\partial_i\delta c dx = \int_{\mathbb{R}^d}(\delta u\cdot\nabla c_1)\Delta\delta c dx \\ &- \int_{\mathbb{R}^d}(\nabla\delta c\cdot\nabla)u_2\cdot\nabla\delta c dx + \int_{\mathbb{R}^d} n_1\delta c\Delta\delta c dx \\ &+ \int_{\mathbb{R}^d} c_2\delta n\Delta\delta c dx \le C\|\delta u\|_2^2\|c_1\|_{H^{s+1}}^2 + \frac{1}{8}\|\Delta\delta c\|_2^2 \\ &+ C\|\nabla\delta c\|_2^2\|u_2\|_{H^{s+1}} + C\|\delta c\|_2^2\|n_1\|_{H^s}^2 + \frac{1}{8}\|\Delta\delta c\|_2^2 \\ &+ C\|\delta n\|_2^2|c_2\|_{H^{s+1}}^2 + \frac{1}{8}\|\Delta\delta c\|_2^2.\end{aligned} \tag{49}$$

Hence, we have

$$\begin{aligned}\frac{d}{dt}\|\nabla\delta c(t)\|_2^2+\|\Delta\delta c(t)\|_2^2 &\le C\big(\|\delta u\|_2^2\|c_1\|_{H^{s+1}}^2+\|\nabla\delta c\|_2^2\|u_2\|_{H^{s+1}} \\ &+\|\delta c\|_2^2\|n_1\|_{H^s}^2+\|\delta n\|_2^2 \mid c_2\|_{H^{s+1}}^2\big).\end{aligned} \tag{50}$$

Multiplying the third equation of system (43) by δu and integrating in spaces, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\delta u(t)\|_2^2 &= -\int_{\mathbb{R}^d}((\delta u\cdot\nabla)u_1)\cdot\delta u dx + \int_{\mathbb{R}^d}\delta n\nabla\Phi\cdot\delta u dx \\ &\le C\|\delta u\|_2^2\|u_1\|_{H^{s+1}} + C\big(\|\delta n\|_2^2+\|\delta u\|_2^2\big).\end{aligned} \tag{51}$$

Thus,

$$\frac{d}{dt}\|\delta u(t)\|_2^2 \le C\big(\|\delta u\|_2^2\|u_1\|_{H^{s+1}}+\|\delta n\|_2^2+\|\delta u\|_2^2\big). \tag{52}$$

From (45)–(52), we obtain

$$\begin{aligned}&\frac{d}{dt}\big(\|\delta n(t)\|_2^2+\|\delta c(t)\|_2^2+\|\nabla\delta c(t)\|_2^2+\|\delta u(t)\|_2^2\big)+\|\nabla\delta n\|_2^2 \\ &+\|\nabla\delta c\|_2^2+\|\Delta\delta c\|_2^2 \le C\big(\|\delta u\|_2^2+\|\delta n\|_2^2\|n_1\|_{H^s}^2+\|\delta n\|_2^2\|c_1\|_{H^{s+1}}^2 \\ &+\|n_2\|_{H^s}^2\|\nabla\delta c\|_2^2+\|\delta n\|_2^2+\|n_1\|_{H^s}\|\delta n\|_2^2+\|n_2\|_{H^s}\|\delta n\|_2^2+\|\delta u\|_2^2 \\ &+\|\delta c\|_2^2\|c_1\|_{H^{s+1}}^2+\|\delta c\|_2^2\|n_1\|_{H^s}+\|\delta n\|_2^2+\|\delta c\|_2^2\|c_2\|_{H^{s+1}}^2 \\ &+\|\delta u\|_2^2\|c_1\|_{H^{s+1}}^2+\|\nabla\delta c\|_2^2\|u_2\|_{H^{s+1}}+\|\delta c\|_2^2\|n_1\|_{H^s}^2 \\ &+\|\delta n\|_2^2 \mid c_2\|_{H^{s+1}}^2+\|\delta u\|_2^2\|u_1\|_{H^{s+1}}+\|\delta n\|_2^2+\|\delta u\|_2^2\big).\end{aligned} \tag{53}$$

Then, we have

$$\begin{aligned}&\frac{d}{dt}\big(\|\delta n(t)\|_2^2+\|\delta c(t)\|_2^2+\|\nabla\delta c(t)\|_2^2+\|\delta u(t)\|_2^2\big) \\ &\quad\le CF(t)\big(\|\delta n\|_2^2+\|\delta c\|_2^2+\|\nabla\delta c\|_2^2+\|\delta u\|_2^2\big),\end{aligned} \tag{54}$$

where

$$\begin{aligned}F(t) &= 1+\|n_1\|_{H^s}+\|n_2\|_{H^s}+\|u_1\|_{H^{s+1}}+\|u_2\|_{H^{s+1}}+\|n_1\|_{H^s}^2+\|n_2\|_{H^s}^2 \\ &+\|c_1\|_{H^{s+1}}^2+\|c_2\|_{H^{s+1}}^2.\end{aligned} \tag{55}$$

From (3), we infer that $F(t)$ is integrable. Using the Gronwall inequality gives the uniqueness.

3.2. Blowup Criterion. Operating Δ_q with $q \ge -1$ to the first equation of (3) gives

$$\begin{aligned}\Delta_q n_t + u\cdot\nabla\Delta_q n &= \Delta\Delta_q n + u\cdot\nabla\Delta_q n - \Delta_q(u\cdot\nabla n) - \nabla\cdot\Delta_q(n\nabla c) \\ &+ \Delta_q n - \Delta_q n^2.\end{aligned} \tag{56}$$

Taking the L^2-inner product for the above equation with $\Delta_q n$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta_q n\|_2^2+\|\nabla\Delta_q n\|_2^2 &= \int_d\big(u\cdot\nabla\Delta_q n - \Delta_q(u\cdot\nabla n)\big)\Delta_q n dx \\ &+ \int_d \Delta_q(n\nabla c)\cdot\nabla\Delta_q n dx + \int_d \Delta_q n\Delta_q n dx \\ &- \int_d \Delta_q n^2\Delta_q n dx \le \|u\cdot\nabla\Delta_q n - \Delta_q(u\cdot\nabla n)\|_2\|\Delta_q n\|_2 \\ &+\|\Delta_q(n\nabla c)\|_2\|\nabla\Delta_q n\|_2+\|\Delta_q n\|_2^2+\|\Delta_q n^2\|_2\|\Delta_q n\|_2.\end{aligned} \tag{57}$$

Multiplying 2^{2qs} on both sides of the above inequality and performing ℓ_1 norm, we have

$$\begin{aligned}\frac{d}{dt}\|n\|_{H^s}^2+\|n\|_{H^{s+1}}^2 &\le \|\nabla u\|_\infty\|n\|_{H^s}^2+\|n\nabla c\|_{H^s}\|n\|_{H^{s+1}}+\|n\|_{H^s}^2 \\ &+\|n^2\|_{H^s}\|n\|_{H^s}.\end{aligned} \tag{58}$$

Using Young's inequality, we conclude

$$\begin{aligned}\frac{d}{dt}\|n\|_{H^s}^2+\|n\|_{H^{s+1}}^2 &\le C\|\nabla u\|_\infty\|n\|_{H^s}^2 + C\|n\|_\infty^2\|\nabla c\|_{H^s}^2 \\ &+ C\|n\|_{H^s}^2\|\nabla c\|_\infty^2 + \frac{1}{8}\|n\|_{H^{s+1}}^2+\|n\|_{H^s}^2 + C\|n\|_\infty^2\|n\|_{H^s}^2 \\ &+ \frac{1}{8}\|n\|_{H^{s+1}}^2.\end{aligned} \tag{59}$$

In terms of the second equation of (3), we know

$$\Delta_q c_t + u\cdot\nabla\Delta_q c = \Delta\Delta_q c + u\cdot\nabla\Delta_q c - \Delta_q(u\cdot\nabla c) - \Delta_q(nc). \tag{60}$$

Multiplying the above equality by $\Delta_q c$ and integrating in spaces mean

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta_q c\|_2^2+\|\nabla\Delta_q c\|_2^2 &= \int_{\mathbb{R}^d}\left(u\cdot\nabla\Delta_q c - \Delta_q(u\cdot\nabla c)\right)\Delta_q c dx \\ &+ \int_{\mathbb{R}^d}\Delta_q(nc)\Delta_q c dx \le \|u\cdot\nabla\Delta_q c - \Delta_q(u\cdot\nabla c)\|_2\|\Delta_q c\|_2 \\ &+\|\Delta_q(nc)\|_2\|\Delta_q c\|_2.\end{aligned} \tag{61}$$

Multiplying $2^{2q(s+1)}$ on both sides of the above inequality and taking ℓ_1 norm, we obtain

$$\frac{d}{dt}\|c\|_{H^{s+1}}^2+\|c\|_{H^{s+2}}^2 \le \|\nabla u\|_\infty\|c\|_{H^{s+1}}^2+\|nc\|_{H^{s+1}}\|c\|_{H^{s+1}}. \tag{62}$$

Utilizing Young's inequality, we have

$$\begin{aligned}\frac{d}{dt}\|c\|_{H^{s+1}}^2+\|c\|_{H^{s+2}}^2 &\le C\|\nabla u\|_\infty\|c\|_{H^{s+1}}^2 + C\|n\|_\infty^2\|c\|_{H^{s+1}}^2 \\ &+ C\|c\|_\infty^2\|c\|_{H^{s+1}}^2 + \frac{1}{8}\|n\|_{H^{s+1}}^2.\end{aligned} \tag{63}$$

According to the third equation of (3), we get

$$\Delta_q u_t + (u\cdot\nabla)\Delta_q u - \nabla\Delta_q P = (u\cdot\nabla)\Delta_q u - \Delta_q((u\cdot\nabla)u) - \Delta_q(n\nabla\Phi). \tag{64}$$

Taking the L^2-inner product for the above equality with $\Delta_q u$ implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta_q u\|_2^2 &= \int_{\mathbb{R}^d}\left((u\cdot\nabla)\Delta_q u - \Delta_q((u\cdot\nabla)u)\right)\Delta_q u dx \\ &- \int_{\mathbb{R}^d}\Delta_q(n\nabla\Phi)\Delta_q u dx \le \|(u\cdot\nabla)\Delta_q u \\ &- \Delta_q((u\cdot\nabla)u)\|_2\|\Delta_q u\|_2+\|\Delta_q(n\nabla\Phi)\|_2\|\Delta_q u\|_2.\end{aligned} \tag{65}$$

Multiplying $2^{2q(s+1)}$ on both sides of the above inequality and taking the ℓ_1 norm, we have

$$\frac{d}{dt}\|u\|_{H^{s+1}}^2 \le C\|\nabla u\|_\infty\|u\|_{H^{s+1}}^2 + \frac{1}{8}\|n\|_{H^{s+1}}^2 + C\|u\|_{H^{s+1}}^2. \tag{66}$$

Collecting (59)–(66), we deduce

$$\begin{aligned}&\frac{d}{dt}\left(\|n\|_{H^s}^2+\|c\|_{H^{s+1}}^2+\|u\|_{H^{s+1}}^2\right)+\|n\|_{H^{s+1}}^2+\|c\|_{H^{s+2}}^2 \\ &\quad\le \left(\|n\|_{H^s}^2+\|c\|_{H^{s+1}}^2+\|u\|_{H^{s+1}}^2\right)\left(1+\|\nabla u\|_\infty+\|n\|_\infty^2+\|\nabla c\|_\infty^2\right).\end{aligned} \tag{67}$$

The Gronwall inequality implies

$$\begin{aligned}&\left(\|n(t)\|_{H^s}^2+\|c(t)\|_{H^{s+1}}^2+\|u(t)\|_{H^{s+1}}^2\right) \\ &\quad+\int_0^t\left(\|n(\tau)\|_{H^{s+1}}^2+\|c(\tau)\|_{H^{s+2}}^2\right)d\tau \\ &\quad\le C\exp\left(\int_0^t\left(1+\|\nabla u(\tau)\|_\infty+\|n(\tau)\|_\infty^2+\|\nabla c(\tau)\|_\infty^2\right)d\tau\right).\end{aligned} \tag{68}$$

Next, we turn to prove condition (11). Applying ∇ on both sides of the second equation of (3) means

$$\partial_t\nabla c + u\cdot\nabla^2 c - \Delta\nabla c = -\nabla(nc) - \nabla u\cdot\nabla c. \tag{69}$$

Multiplying the above equality with $|\nabla c|^{p-2}\nabla c$, we obtain

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\nabla c\|_p^p + \int_{\mathbb{R}^d}u\cdot\nabla^2 c|\nabla c|^{p-2}\nabla c dx + \frac{4(p-1)}{p^2}\|\nabla|\nabla c|^{p/2}\|_2^2 = \\ &-\int_{\mathbb{R}^d}\nabla(nc)|\nabla c|^{p-2}\nabla c dx - \int_{\mathbb{R}^d}\nabla u\cdot\nabla c|\nabla c|^{p-2}\nabla c dx \\ &\quad\le (p-1)\int_{\mathbb{R}^d}nc|\nabla c|^{p-2}\Delta c dx+\|\nabla u\|_\infty\|\nabla c\|_p^p \\ &\quad\le \frac{2(p-1)}{p}\|n\|_\infty\|c\|_p\||\nabla c|^{(p-2)/2}\|_{2p/(p-2)}\|\nabla|\nabla c|^{p/2}\|_2+\|\nabla u\|_\infty\|\nabla c\|_p^p \\ &\quad\le \frac{2(p-1)}{p}\left(\frac{p}{2(p-1)\varepsilon}\|n\|_\infty^2\|c\|_p^2\||\nabla c|\|_{p-2}^2 + \frac{2(p-1)\varepsilon}{p}\|\nabla|\nabla c|^{p/2}\|_2^2\right) \\ &+\|\nabla u\|_\infty\|\nabla c\|_p^p.\end{aligned} \tag{70}$$

Because of

$$\int_{\mathbb{R}^d}u\cdot\nabla^2 c|\nabla c|^{p-2}\nabla c dx = \frac{1}{p}\int_{\mathbb{R}^d}u\cdot\nabla|\nabla c|^p dx = 0, \tag{71}$$

we get

$$\frac{d}{dt}\|\nabla c\|_p^2 \le C\|n\|_\infty^2+\|\nabla u\|_\infty\|\nabla c\|_p^2. \tag{72}$$

Utilizing Gronwall's inequality, we have

$$\begin{aligned}\|\nabla c(t)\|_p^2 &\le C\left(\|\nabla c_0\|_p + \int_0^t\|n(\tau)\|_\infty^2 d\tau\right)\exp\int_0^t\|\nabla u(\tau)\|_\infty d\tau \\ &\le C\left(\|c_0\|_{H^s} + \int_0^t\|n(\tau)\|_\infty^2 d\tau\right)\exp\int_0^t\|\nabla u(\tau)\|_\infty d\tau.\end{aligned} \tag{73}$$

Setting $p \to \infty$, we conclude

$$\|\nabla c(t)\|_\infty^2 \le \left(\|c_0\|_{H^s} + \int_0^t \|n(\tau)\|_\infty^2 d\tau\right) \exp \int_0^t \|\nabla u(\tau)\|_\infty d\tau. \tag{74}$$

Submitting (74) into (68) gives

$$\|n(t)\|_{H^s}^2 + \|c(t)\|_{H^{s+1}}^2 + \|u(t)\|_{H^{s+1}}^2 + \int_0^t \left(\|n(\tau)\|_{H^{s+1}}^2 + \|c(\tau)\|_{H^{s+2}}^2\right) d\tau \le C \exp\exp\left(\int_0^t 1 + \|\nabla u(\tau)\|_\infty + \|n(\tau)\|_\infty^2\right) d\tau. \tag{75}$$

On the other hand, using the inhomogeneous dynamic partition of the unity, we have

$$\|\nabla u\|_{L^\infty} = \|\sum_{j\ge -1} \nabla \Delta_j u\|_{L^\infty} \le \|\nabla \Delta_{-1} u\|_{L^\infty} + \sum_{j=0}^\infty \|\nabla \Delta_j\|_{L^\infty} \le C\left(\|u_0\|_{L^2} + \sum_{j=0}^\infty \|\Delta_j \omega\|_{L^\infty}\right) \le C\left(\|u_0\|_{L^2} + \|\omega\|_{B^0_{\infty,1}}\right). \tag{76}$$

Taking the curl to the third equation of (3) implies

$$\omega_t + u \cdot \nabla \omega = \frac{u^r}{r} \omega - \text{curl } (n \nabla \phi). \tag{77}$$

Using Lemma 6, we obtain

$$\begin{aligned} \|\omega\|_{B^0_{\infty,1}} &\le C\left(\|\omega_0\|_{B^0_{\infty,1}} + \int_0^t \|curl(n\nabla\phi)(\tau)\|_{B^0_{\infty,1}} + \|\frac{u^r}{r}\omega\|_{B^0_{\infty,1}}\right) \\ &\quad \times \left(1 + \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau\right) \\ &\le C\left(\|\omega_0\|_{B^0_{\infty,1}} + \int_0^t \|curl(n\nabla\phi)(\tau)\|_{B^0_{\infty,1}} + \|\frac{u^r}{r}\|_{L^\infty} \|\omega\|_{B^0_{\infty,1}}\right) \\ &\quad \times \left(1 + \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau\right). \end{aligned} \tag{78}$$

For the term $\|\text{curl } (n\nabla\phi)(\tau)\|_{B^0_{\infty,1}}$, using Bony's decomposition, we have

$$\begin{aligned} \|\text{curl } (n\nabla\phi)(\tau)\|_{B^0_{\infty,1}} &= \sum_{|q-q'|\le 4} 2^q \|\Delta_q\left(S_q{}'^{-1} n \Delta_q{}' \nabla\phi\right)\|_{L^\infty} \\ &\quad + \sum_{|q-q'|\le 4} 2^q \|\Delta_q\left(S_q{}'^{-1} \nabla\phi \Delta_q{}' n\right)\|_{L^\infty} \\ &\quad + \sum_{\substack{|q''-q'|\le 1 \\ q'\ge q-3}} 2^q \|\Delta_q\left(\Delta_q{}' n \Delta_q{}'' \nabla\phi\right)\|_{L^\infty} \triangleq I_1 + I_2 + I_3. \end{aligned} \tag{79}$$

For the term I_1, we have

$$I_1 \le C \sum_{|q-q'|\le 4} \sum_{j\le q'-2} 2^q \|\Delta_q{}' \nabla\phi\|_{L^\infty} \|\Delta_j n\|_{L^\infty} \le C\|\nabla\phi\|_{B^1_{\infty,\infty}} \|n\|_{B^0_{\infty,1}}. \tag{80}$$

Similarly,

$$I_2 \le C\|\nabla\phi\|_{B^1_{\infty,\infty}} \|n\|_{B^0_{\infty,1}}. \tag{81}$$

As for I_3,

$$\begin{aligned} I_3 &\le C \sum_{\substack{|q''-q'|\le 1 \\ q''\ge q-3}} 2^{q-q''} 2^{q''} \|\Delta_q{}'' \nabla\phi\|_{L^\infty} \|\Delta_q{}' n\|_{L^\infty} \\ &\le C\|\nabla\phi\|_{B^1_{\infty,\infty}} \|n\|_{B^0_{\infty,1}}. \end{aligned} \tag{82}$$

Plugging (80)–(82) into (79) yields

$$\|\text{curl } (n\nabla\phi)(\tau)\|_{B^0_{\infty,1}} \le C\|\nabla\phi\|_{B^1_{\infty,\infty}} \|n\|_{B^0_{\infty,1}}. \tag{83}$$

Putting (83) into (78), using the fact $\|u^r(t)/r\|_{L^\infty} \le C\|\omega_0/r\|_{L^{3,1}}$ [25] and Gronwall's inequality, we have

$$\|\omega\|_{B^0_{\infty,1}} \le C\left(\|\omega_0\|_{B^0_{\infty,1}} + \int_0^t \|n(\tau)\|_{B^0_{\infty,1}} d\tau\right) e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau}. \tag{84}$$

Substituting (84) into (76) gives

$$\|\nabla u\|_{L^\infty} \le C\left(\|u_0\|_{H^s} + \int_0^t \|n(\tau)\|_{B^0_{\infty,1}} d\tau\right) e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau}. \tag{85}$$

Applying the Gronwall inequality, we get

$$\|\nabla u\|_{L^\infty} \le C e^{\exp\left(C\int_0^t \|n(\tau)\|_{B^0_{\infty,1}}\right)}. \tag{86}$$

Substituting (86) into (75) and using the fact $B^0_{\infty,1} \mathring{} L^\infty$, we obtain the desired result.

This completes the proof of Theorem 1.

Data Availability

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors completed the paper together. All authors read and approved the final manuscript.

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