Hindawi
Journal of Function Spaces
Volume 2021, Article ID 5542843, 6 pages
https://doi.org/10.1155/2021/5542843

Research Article

Penot's Compactness Property in Ultrametric Spaces with an Application

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Received 18 January 2021; Accepted 9 April 2021; Published 19 April 2021

Academic Editor: Shanhe Wu

In this work, we investigate the compactness property in the sense of Penot in ultrametric spaces. Then, we show that spherical completeness is exactly the Penot's compactness property introduced for convexity structures. The spherical completeness property misled some mathematicians to it to hyperconvexity in metric spaces. As an application, we discuss some fixed point results in spherically complete ultrametric spaces.

1. Introduction

Ultrametric spaces are a special type of metric spaces which enjoy a stronger triangular inequality. This stronger triangular inequality implies some amazing properties enjoyed by ultrametric spaces, like any triangle is isosceles. This is why ultrametric spaces are known as isosceles metric spaces. Because of these properties, one may tend to belittle ultrametric spaces and considers them as futile. It happens that these metric spaces are crucial in applications. For example, they appear in a strong manner in logic programming and artificial intelligence [1]. Other areas which involve heavily ultrametric spaces are bioinformatics, distributed networks, microbiology, and learning models [2, 3].

In recent years, many authors followed the work of [4] where a pseudoconnection between ultrametric spaces and hyperconvexity via the 2-Helly intersection property of balls was initiated. In this work, we show that such connection is baseless and not correct. In fact, the correct connection should be made with the work of Penot on convexity structures. Indeed, Penot [5] while investigating an abstract formulation of Kirk's fixed point theorem [6] away from Banach spaces considered an abstract form of convex sets. This abstract formulation captures the weak-compactness beautifully. It is worth mentioning that convexity structures are important in applications [1, 3, 7, 8].

As an application, we discuss the fixed point property in light of some recent papers published on this subject. In particular, we show that many of the known results assumes strong or artificial behavior.

For the interested reader into metric fixed point theory, we recommend the book of Khamsi and Kirk [9].

2. Preliminaries and Basic Results

A metric space (M, d) is said to be ultrametric if and only if

$$d(x, z) \leq \max \{d(x, y) ; d(y, z)\}, \tag{1}$$

for any $x, y, z \in M$. A fundamental property satisfied by ultrametric spaces states that all triangles are isosceles, i.e., $d(x, z) = \max \{d(x, y) ; d(y, z)\}$ whenever $d(x, y) \neq d(y, z)$, for any $x, y, z \in M$. Let x be an element of an ultrametric space (M, d) and $r \geq 0$. Closed balls in ultrametric spaces enjoy interesting properties. First, recall that $B(x, r)$ is a closed ball centered at $x \in M$ with radius $r \geq 0$ provided $B(x, r) = \{y \in M ; d(x, y) \leq r\}$.

Proposition 1. *[2, 9–12] Let (M, d) be an ultrametric space.*

(1) We have $B(y,r) = B(x,r)$ whenever $y \in B(x,r)$, which implies

$$B(x,r) \cap B(y,r) \neq \emptyset \Longrightarrow B(x,r) = B(y,r), \tag{2}$$

for any $x, y \in M$ and $r \geq 0$

(2) If $B(x, r_1)$ and $B(y, r_2)$ are such that $r_1 \leq r_2$, then either $B(x,r_1) \cap B(y,r_2) = \emptyset$ or $B(x,r_1) \subseteq B(y,r_2)$, for any $x_1, x_2 \in M$ and $r_1, r_2 \geq 0$

In other words, for closed balls in ultrametric spaces, the radius is central not the center. Moreover, the inclusion order between balls follows the order on the radiuses. This is amazing if we look at it from the point of view of classical metric and Banach spaces.

The following property is fundamental and gives an insight into the behavior of closed balls in ultrametric spaces.

Lemma 2. *Let (M, d) be an ultrametric space. Let $\{B(a_\alpha, r_\alpha)\}_{\alpha\in\Gamma}$ be a collection of closed balls. Assume that $\bigcap_{\alpha\in\Gamma} B(a_\alpha, r_\alpha) \neq \emptyset$. Let $z \in \bigcap_{\alpha\in\Gamma} B(a_\alpha, r_\alpha)$. Then, we have $\bigcap_{\alpha\in\Gamma} B(a_\alpha, r_\alpha) = B(z, \inf_{\alpha\in\Gamma} r_\alpha)$.*

Proof. Since $\inf_{\alpha\in\Gamma} r_\alpha \leq r_\beta$, for any $\beta \in \Gamma$, our previous discussion implies

$$B\left(z, \inf_{\alpha\in\Gamma} r_\alpha\right) \subset \bigcap_{\alpha\in\Gamma} B(a_\alpha, r_\alpha). \tag{3}$$

Conversely, let $x \in \bigcap_{\alpha\in\Gamma} B(a_\alpha, r_\alpha)$; then

$$d(z,x) \leq \max\left\{d\left(x, a_\beta\right); d\left(a_\beta, z\right)\right\} \leq \max\left\{r_\beta; r_\beta\right\} = r_\beta, \tag{4}$$

for any $\beta \in \Gamma$, which implies $d(z,x) \leq \inf_{\alpha\in\Gamma} r_\alpha$, i.e., $x \in B(z, \inf_{\alpha\in\Gamma} r_\alpha)$. This proves our claim.

Let (M, d) be a metric space. An admissible subset is an intersection of closed balls [9]. The collection of admissible subsets of M will be denoted by $\mathscr{A}(M)$. Obviously any closed ball is an admissible subset. Moreover, $\mathscr{A}(M)$ is stable by intersection; i.e., the intersection of admissible subsets is an admissible subset. In ultrametric spaces, nonempty admissible subsets are exactly closed balls according to Lemma 2. This is surprising and may lead mathematicians to a wrong interpretation. This is the case when the authors of [4] connected the ultrametric property to hyperconvexity in metric spaces. Penot [5] is credited as the one who introduced the following definition:

Definition 3. Let (M, d) be a metric space. We say that $\mathscr{A}(M)$ is compact if and only if for any family of closed balls $\{B(a_\alpha, r_\alpha)\}_{\alpha\in\Gamma}$, $\bigcap_{\alpha\in\Gamma} B(a_\alpha, r_\alpha)$ is not empty provided $\bigcap_{\alpha\in\Gamma_f} B(a_\alpha, r_\alpha) \neq \emptyset$, for any finite subset Γ_f of Γ.

Penot introduced this property when he gave the first abstract extension of the classical Kirk's fixed point theorem [6] to metric spaces. His approach allowed him to avoid the linear convexity problem from the underlying Banach space structure. Others stumbled with this issue and specially the weak-compactness argument used heavily by Kirk. For example, Takahashi [13] used the Menger metric convexity and assumes the metric compactness which is very strong and does not capture the weak-compactness in the linear case. In fact, Penot did not work specifically with the family of admissible subsets but used the well-known concept of convexity structures. This concept is very powerful and has applications in areas as vast apart as economics and game theory [1, 3, 7, 8]. Recall that a convexity structure is a family of subsets stable by intersection [5, 8]. A property shared by convex subsets in the linear case. It is clear that $\mathscr{A}(M)$ is the smallest convexity structure which contains the closed balls.

Before we state the main result of this work, we need the following definition:

Definition 4 [12, 14]. A metric space (M, d) is said to be spherically complete if and only if for any sequence $\{B(x_n, r_n)\}_{n\in\mathbb{N}}$ of closed balls such that $r_{n+1} \leq r_n$ and $B(x_{n+1}, r_{n+1}) \subset B(x_n, r_n)$, for any $n \in \mathbb{N}$, we have $\bigcap_{n\in\mathbb{N}} B(x_n, r_n) \neq \emptyset$.

Theorem 5. *Let (M, d) be an ultrametric space. The following are equivalent:*

(1) $\mathscr{A}(M)$ is compact (in the sense of Penot)

(2) (M, d) is spherically complete

Proof. Obviously we only need to prove that if (M, d) is spherically complete, then $\mathscr{A}(M)$ is compact. Let $\{B(a_\alpha, r_\alpha)\}_{\alpha\in\Gamma}$ be a family of closed balls such that $\bigcap_{\alpha\in\Gamma_f} B(a_\alpha, r_\alpha) \neq \emptyset$, for any finite subset Γ_f of Γ. Let us prove that $\bigcap_{\alpha\in\Gamma} B(a_\alpha, r_\alpha)$ is not empty. Set $R = \inf_{\alpha\in\Gamma} r_\alpha$. There exists $\{\alpha_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\longrightarrow\infty} r_{\alpha_n} = R$. Set $R_n = \inf\{r_{\alpha_0}, r_{\alpha_2}, \cdots, r_{\alpha_n}\}$, for $n \in \mathbb{N}$. Then, $\{R_n\}$ is decreasing and converges to R. Moreover, by assumption, we have $\bigcap_{i\leq n} B(a_{\alpha_i}, r_{\alpha_i})$ is not empty and is a closed ball with radius R_n, i.e.; there exists $x_n \in M$ such that

$$\bigcap_{i\leq n} B\left(a_{\alpha_i}, r_{\alpha_i}\right) = B(x_n, R_n), \tag{5}$$

for any $n \in \mathbb{N}$. By construction, we have $B(x_{n+1}, R_{n+1}) \subset B(x_n, R_n)$, for any $n \in \mathbb{N}$. Since (M, d) is spherically complete, we conclude that $\bigcap_{n\in\mathbb{N}} B(x_n, R_n)$ is not empty. We claim that

$$\bigcap_{\alpha\in\Gamma} B(a_\alpha, r_\alpha) = \bigcap_{n\in\mathbb{N}} B(x_n, R_n) = \bigcap_{n\in\mathbb{N}} \left(\bigcap_{i\leq n} B\left(a_{\alpha_i}, r_{\alpha_i}\right)\right). \tag{6}$$

In order to see this is true, we only need to prove

$$\bigcap_{n\in\mathbb{N}}\left(\bigcap_{i\le n}B\left(a_{\alpha_i}, r_{\alpha_i}\right)\right)\subset\bigcap_{\alpha\in\Gamma}B\left(a_\alpha, r_\alpha\right). \qquad (7)$$

Let $z\in\bigcap_{n\in\mathbb{N}}B(x_n, R_n)=\bigcap_{n\in\mathbb{N}}(\bigcap_{i\le n}B(a_{\alpha_i}, r_{\alpha_i}))$. Fix $\beta\in\Gamma$ and $n\in\mathbb{N}$. Our assumption on the family $\{B(a_\alpha, r_\alpha)\}_{\alpha\in\Gamma}$ implies $B(x_n, R_n)\cap B(a_\beta, r_\beta)$ is not empty. Hence, $d(x_n, a_\beta)\le\max\{R_n, r_\beta\}$ which implies

$$d\left(a_\beta, z\right)\le\max\left\{d(x_n, z)\,;d\left(x_n, a_\beta\right)\right\}\le\max\left\{R_n, r_\beta\right\}. \quad (8)$$

If we let $n\longrightarrow\infty$, we get $d(a_\beta, z)\le\max\{R, r_\beta\}=r_\beta$, i.e., $z\in B(a_\beta, r_\beta)$. Since β was chosen arbitrarily in Γ, we conclude that $z\in\bigcap_{\alpha\in\Gamma}B(a_\alpha, r_\alpha)$, which completes the proof.

Some mathematicians confused the spherical complete property in ultrametric spaces with the hyperconvex property (see for example [4]). Theorem 5 clarifies this connection by showing that in fact spherical completeness in ultrametric spaces coincides with Penot's compactness of convexity structures in metric spaces [5]. Recall that Penot's compactness in Banach spaces was captured in a wonderful way the compactness of the weak-topology and does not connect to the hyperconvexity in any way. For the sake of being complete, let us recall the definition of hyperconvexity in metric spaces.

Definition 6. The metric space (M, d) is said to be hyperconvex if for any family $\{B(x_\alpha, r_\alpha)\}_{\alpha\in\Gamma}$ of closed balls in M such that $d(x_\alpha, x_\beta)\le r_\alpha+r_\beta$, for any $\alpha, \beta\in\Gamma$, we have

$$\bigcap_{\alpha\in\Gamma}B(x_\alpha, r_\alpha)\neq\varnothing. \qquad (9)$$

For more details more on hyperconvex metric spaces, we refer the reader to [15, 16]. Note that hyperconvex metric spaces are convex in the sense of Menger, i.e., for any $x, y\in M$ and $t\in[0,1]$; there exists $z_t\in M$ such that $d(x, z_t)=t\,d(x, y)$ and $d(y, z_t)=(1-t)\,d(x, y)$. Taking into account this property, it is clear that ultrametric spaces can not be convex in the sense of Menger. Therefore, ultrametric spaces can not be hyperconvex.

Remark 7. Note that if an ultrametric space (M, d) is spherically complete, then, it is complete.

3. An Application

As we discussed in the previous section, the spherical completeness should be understood within the compactness framework introduced by Penot. His work was initiated to extend Kirk's fixed point theorem to abstract metric spaces away from the linear convexity.

Definition 8. Let (M, d) be a metric space. A mapping $T: M\longrightarrow M$ is said to be Lipschitzian if there exists $K\ge 0$ such that

$$d(T(x), T(y))\le K\,d(x, y), \qquad (10)$$

for any $x, y\in M$. The smallest K will be denoted by $\mathrm{Lip}(T)$. T is said to be nonexpansive if $\mathrm{Lip}(T)\le 1$, i.e.,

$$d(T(x), T(y))\le d(x, y), \qquad (11)$$

for any $x, y\in M$. A point $x\in M$ is said to be a fixed point of T provided $T(x)=x$. The set of fixed points of T is denoted by $\mathrm{Fix}(T)$.

The following result is known and is amazing and has no equivalence in general metric spaces. We will give its proof here.

Theorem 9 [9, 11, 17]. *Let (M, d) be an ultrametric space which is spherically complete. Let $T: M\longrightarrow M$ be nonexpansive.*

(1) The closed ball $B(x, d(x, T(x)))$ is T-invariant, i.e., $T(B(x, d(x, T(x))))\subset B(x, d(x, T(x)))$, for any $x\in M$

(2) Let $B(x, r)$ be a minimal closed ball invariant by T. Then, for any $y\in B(x, r)$, we have $d(x, T(x))=d(y, T(y))$; i.e., the displacement function $z\longrightarrow d(z, T(z))$ is constant when restricted to $B(x, r)$

Proof. Proof of (1) is easy. Indeed, we have $B(T(x), d(x, T(x)))=B(x, d(x, T(x)))$, since $T(x)\in B(x, d(x, T(x)))$. Let $y\in B(x, d(x, T(x)))$. Since T is nonexpansive, we have

$$d(T(x), T(y))\le d(x, y)\le d(x, T(x)), \qquad (12)$$

which implies $T(y)\in B(T(x), d(x, T(x)))=B(x, d(x, T(x)))$, which proves our claim.

As for the proof of (2), note first that since (M, d) is spherically complete, the family $\mathscr{A}(M)$ is compact by Theorem 5. Fix $x\in M$. Consider the subfamily

$$\mathscr{F}_x=\{A\in\mathscr{A}(M)\,;T(A)\subset A\,\text{and}\,x\in A\}. \qquad (13)$$

$\mathscr{F}_x$ is not empty since $B(x, d(x, T(x))\in\mathscr{F}_x$ from (1). Using the compactness of $\mathscr{A}(M)$, Zorn's lemma implies the existence of minimal elements in $\mathscr{F}_x$. Let $A\in\mathscr{F}_x$ be such a minimal element. Lemma 2 implies that A is a closed ball. Therefore, minimal closed balls invariant by T do exist. Let $B(x, r)$ be a minimal closed ball invariant by T. Then $B(x, d(x, T(x)))=B(x, r)$. Indeed, from (1), we know that $B(x, d(x, T(x)))$ is invariant by T. Moreover, we have $d(x, T(x))\le r$ which implies $B(x, d(x, T(x)))\subset B(x, r)$. The minimality implies $B(x, d(x, T(x)))=B(x, r)$ as claimed. Next, we prove that $d(x, T(x))=d(y, T(y))$, for any $y\in B(x, d(x, T(x)))$. Without loss of generality, we assume $d(x, T(x))>0$. Note that since $y\in B(x, d(x, T(x)))$ and T is nonexpansive, we have

$$d(T(x), T(y)) \leq d(x, y) \leq d(x, T(x)). \quad (14)$$

Hence, $T(y) \in B(T(x), d(x, T(x))) = B(x, d(x, T(x))) = B(y, d(x, T(x)))$, which implies $d(y, T(y)) \leq d(x, T(x))$. By the symmetric role played by x and y, we get $d(x, T(x)) = d(y, T(y))$ as claimed.

The conclusion (2) of Theorem 9 was first proved by Petalas and Vidalis in [18] and was extended by many authors (see for instance [11, 19]). Therefore, given a nonexpansive mapping defined in a spherically complete metric space, it has a minimal closed ball with two options: either the radius of the minimal ball is $r = 0$ or it is $r > 0$. If $r = 0$, the minimal ball is reduced to one point which is a fixed point of the map. Otherwise, T fails to have a fixed point in that minimal ball.

Theorem 10 [11, 18, 19]. *Let (M, d) be a spherically complete ultrametric space. Let $T : M \longrightarrow M$ be nonexpansive. Fix $x \in M$. Then, the closed ball $B(x, d(x, T(x))$ contains either a fixed point of T or a minimal invariant closed ball B such that*

$$d(y, T(y)) = d(z, T(z)) > 0, \quad (15)$$

for any $y, z \in B$.

From the conclusion of Theorem 10, it is clear that a nonexpanive mapping defined in a spherically complete ultrametric space has a fixed point provided; we violate the condition

$$d(y, T(y)) = d(z, T(z)) > 0, \quad (16)$$

for any y and z in a minimal closed ball invariant by the mapping. First, we note that a fixed point may not exist in general.

Example 11 [18]. Let K be a field with a discrete non-Archimedean valuation $|.|$, and let $c_0(K)$ the set of all sequences $(x_j)_{j\in\mathbb{N}}$ in K such that

$$\lim_{j\longrightarrow\infty} |x_j| = 0. \quad (17)$$

It is known that $c_0(K)$ endowed with the non-Archimedean norm

$$\left\| (x_j)_{j\in\mathbb{N}} \right\| = \sup_{j\in\mathbb{N}} |x_j| \quad (18)$$

is a spherically complete non-Archimedean vector space. Let $a \in K$ such that $0 < |a| < 1$. Define $T : c_0(K) \longrightarrow c_0(K)$ by

$$T(x_1, x_2, x_3, \cdots) = (a, x_1, x_2, \cdots). \quad (19)$$

T is a nonexpansive mapping in $c_0(K)$ with no fix point.

The above example suggests that more assumptions are needed in order to secure the existence of a fixed point for nonexpansive mappings defined in spherically complete ultrametric space. The first type of assumptions which secure a fixed point is the one introduced by Priess-Crampe and Ribenboim [20].

Definition 12 [20]. Let (M, d) be a metric space. A map $T : M \longrightarrow M$ is strictly contracting on orbits if and only if $T(x) \neq x$ implies

$$d\left(T^2(x), T(x)\right) < d(T(x), x), \quad (20)$$

for each $x \in M$.

Most of the mathematicians who came afterward used the same or similar assumption and obtained the following:

Theorem 13 [11, 18–20]. *Let (M, d) be a spherically complete ultrametric space. Let $T : M \longrightarrow M$ be nonexpansive. Assume T is strictly contracting on orbits. Then, T has a fixed point in any closed ball $B(x, d(x, T(x))$, for any $x \in M$.*

Proof. The proof follows easily from Theorem 10.

It is natural to ask whether a larger class of mappings will lead to a similar conclusion. This is the case of the main fixed point theorem of [11] where the authors weakened the nonexpansiveness assumption. Instead, they considered mappings $T : M \longrightarrow M$ which satisfies

$$(\mathrm{KS}) \begin{cases} z \neq T(z) \\ d(x, T(z)) \leq d\left(T^2(z), T(z)\right) \end{cases} \text{implies } d(x, T(x)) \leq d(z, T(z)), \quad (21)$$

for any $x, z \in M$. The only problem with this class of mappings is the lack of natural examples. Another class is inspired from the work of Browder and Petryshyn [21] dealing with the existence of fixed points of nonexpansive mappings in Banach spaces. They proved a regularity property satisfied by these mappings. Indeed, if $T : C \longrightarrow C$ is a nonexpansive mapping and C a bounded and convex subset in Banach space; then for any $\lambda \in (0, 1)$ the mapping $T_\lambda : C \longrightarrow C$ defined by $T_\lambda(x) = (1 - \lambda)\, x + \lambda\, T(x)$ satisfies the following:

$$\lim_{n\longrightarrow\infty} d\left(T_\lambda^n(x), T_\lambda^{n+1}(x)\right) = 0, \quad (22)$$

for any $x \in C$. Note that T_λ and T have the same fixed points.

Definition 14. Let (M, d) be a metric space and $T : M \longrightarrow M$. We say that T is asymptotically regular at $x \in M$ if and only if

$$\lim_{n\longrightarrow\infty} d\left(T^n(x), T^{n+1}(x)\right) = 0. \quad (23)$$

The next result extends the conclusion of Theorem 13.

Theorem 15. *Suppose (M, d) is a complete ultrametric space and $T : X \longrightarrow X$ a continuous map. Then, if there exists $x \in M$ such that T is asymptotically regular at x, then $Fix(T) \neq \emptyset$.*

Proof. Set $x_n = T^n(x)$, for $n \in \mathbb{N}$. Our assumption implies $\lim_{n\longrightarrow\infty} d(x_n, x_{n+1}) = 0$. Since (M, d) is complete, then we have

$$d(x_n, x_{n+h}) \leq \max \{d(x_n, x_{n+1}); d(x_{n+1}, x_{n+2}); \cdots; d(x_{n+h-1}, x_{n+h})\}, \tag{24}$$

for any $n, h \in \mathbb{N}$. This will imply that $\{x_n\}$ is Cauchy. Since (M, d) is spherically complete, it is also complete. Let $\omega \in M$ be the limit of $\{x_n\}$. Since T is continuous, we have

$$T(\omega) = \lim_{n\longrightarrow\infty} T(x_n) = \lim_{n\longrightarrow\infty} x_{n+1} = \omega, \tag{25}$$

i.e., ω is a fixed point of T.

The asymptotically regular assumption may be relaxed using a weaker condition.

Definition 16. Let (M, d) be a metric space. We will say that M has the weak-regular property if

$$\limsup_{n\longrightarrow\infty} d\left(T^n(x), T^{n+1}(x)\right) < d(x, T(x)), \tag{26}$$

for each x in M such that $x \neq T(x)$.

Note that, if $T : M \longrightarrow M$ is strictly contracting on orbits, then, T has the weak-regular property. Indeed, let $x \in M$ such that $x \neq T(x)$. Assume there exists $n_0 \in \mathbb{N}$ such that $T^{n_0+1}(x) = T^{n_0}(x)$; then we have

$$\limsup_{n\longrightarrow\infty} d\left(T^n(x), T^{n+1}(x)\right) = 0 < d(x, T(x)). \tag{27}$$

Otherwise, we assume $T^n(x) \neq T^{n+1}(x)$, for any $n \in \mathbb{N}$. In this case, the sequence $\{d(T^n(x), T^{n+1}(x))\}$ is decreasing. Hence,

$$\limsup_{n\longrightarrow\infty} d\left(T^n(x), T^{n+1}(x)\right) \leq d\left(T(x), T^2(x)\right) < d(x, T(x)), \tag{28}$$

which completes the proof of our claim.

Theorem 17. *Let (M, d) be a spherically complete ultrametric space. Let $T : X \longrightarrow X$ be a nonexpansive map which has the weak-regular property. Then, T has a fixed point in any T-invariant closed ball.*

Proof. Let B be a closed ball such that $T(B) \subset B$. Assume that T does not have a fixed point in B, i.e., $x \neq T(x)$, for any $x \in B$. Theorem 9 implies the existence of a minimal ball $B(x_0, d(x_0, T(x_0)))$ in B such that

$$d(y, T(y)) = d(x_0, T(x_0)) \neq 0, \tag{29}$$

for any $y \in B(x_0, d(x_0, T(x_0)))$. In particular, we have $d(T^n(x_0), T^{n+1}(x_0)) = d(x_0, T(x_0))$, for any $n \in \mathbb{N}$. Hence,

$$d(x_0, T(x_0)) = \limsup_{n\longrightarrow\infty} d\left(T^n(x_0), T^{n+1}(x_0)\right) < d(x_0, T(x_0)). \tag{30}$$

This contradiction completes the proof of Theorem 17.

Next, we give an example of a map and a metric space which satisfy the assumptions of Theorem 17 but will fail to be strictly contracting. Therefore, the results of [11] will not apply.

Example 18. Take $X = \{1, 2, 3, \cdots\}$, and consider the discrete distance $d : X \times X \longrightarrow [0, +\infty)$, i.e.,

$$\begin{cases} d(x, y) = 0, & \text{iff } x = y, \\ d(x, y) = 1, & \text{iff } x \neq y. \end{cases} \tag{31}$$

Then, (X, d) is a spherically complete ultrametric space. Consider the mapping $T : X \longrightarrow X$ defined by

$$\begin{cases} T(2n+1) = 2n+1, \\ T(2n) = n. \end{cases} \tag{32}$$

Then, it is easy to see that $d(T^i(x), T^{i+1}(x)) = 0$ for large i. So if $x \neq T(x)$, we will have

$$\lim_{n\longrightarrow\infty} d\left(T^n(x), T^{n+1}(x)\right) = 0 < d(x, T(x)). \tag{33}$$

In other words, T is weak-regular. Note that T is nonexpansive but fails to be strictly contracting on orbits since $d(8, T(8)) = d(8, 4) = 1$ and $d(T(8), T^2(8)) = d(4, 2) = 1$.

We close this section by giving an example which sheds some light on the above condition (KS) introduced in [11] and show that this condition is not natural and cannot be applied.

Example 19. Take $X = \{\omega, 1, 2, 3, \cdots\}$. Define $d : X \times X \longrightarrow [0, +\infty)$ by

$$\begin{cases} d(x, y) = 0, & \text{iff } x = y, \\ d(\omega, n) = \dfrac{1}{n}, \\ d(n, m) = \max\left\{\dfrac{1}{n}, \dfrac{1}{m}\right\}. \end{cases} \tag{34}$$

Then, (X, d) is a spherically complete ultrametric space. Define $T : X \longrightarrow X$ by

$$\begin{cases} T(\omega) = \omega, \\ T(2n+1) = 2n+1, \\ T(2n) = n. \end{cases} \tag{35}$$

Again, it is easy to see that $d(T^i(x), T^{i+1}(x)) = 0$ for large i. So as in the previous example, we have for $x \neq T(x)$

$$\lim_{n\longrightarrow\infty} d\left(T^n(x), T^{n+1}(x)\right) = 0 < d(x, T(x)). \tag{36}$$

In other words, T is weak-regular. But it fails the condition (KS). Indeed, if we take $x = 4$ and $z = 8$, then, $T(z) \neq z$, $d(x, T(z)) = 0 \leq d(T(z), T^2(z)) = 1/2$ and $d(x, T(x)) = 1/2 > d(z, T(z)) = 1/4$

Data Availability

There is no data used for this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for funding this research group (No. RG-1435-079).

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