Research Article

Ulam-Hyers-Rassias Stability of Stochastic Functional Differential Equations via Fixed Point Methods

Abdellatif Ben Makhlouf ^{1,2}, Lassaad Mchiri ³ and Mohamed Rhaima ^{3,4}

¹Department of Mathematics, College of Science, Jouf University, Saudi Arabia

²Department of Mathematics, Faculty of Sciences of Sfax, University of Sfax, Sfax, Tunisia

³Department of Statistics and Operations Research, College of Sciences, King Saud University, Riyadh 11451, Saudi Arabia

⁴University of Tunis Manar, Faculty of Sciences of Tunis, Department of Mathematics, Tunisia

Correspondence should be addressed to Abdellatif Ben Makhlouf; benmakhloufabdellatif@gmail.com

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The Ulam-Hyers-Rassias stability for stochastic systems has been studied by many researchers using the Gronwall-type inequalities, but there is no research paper on the Ulam-Hyers-Rassias stability of stochastic functional differential equations via fixed point methods. The main goal of this paper is to investigate the Ulam-Hyers Stability (HUS) and Ulam-Hyers-Rassias Stability (HURS) of stochastic functional differential equations (SFDEs). Under the fixed point methods and the stochastic analysis techniques, the stability results for SFDE are investigated. We analyze two illustrative examples to show the validity of the results.

1. Introduction

In recent years, SFDEs play an important role in different areas such as physics, mechanics, population dynamics, ecology, medicine biology, and other areas of sciences. SFDEs have great applications and have been developed very fast, see for example [1–5]. Stability investigation is conducted for stochastic nonlinear differential equations with constant delay. The Lyapunov method is used for stability investigation of different mathematical models such as predator-prey relationships and inverted controlled pendulum.

The HUS problem of functional systems began from a question of S. Ulam, queried in 1940, about the stability of functional differential equations for homomorphism as follows. The question regarding the stability problem of homomorphisms is as follows:

Denote by H_1 the group, and H_2 the metric group with a metric $\tilde{\delta}$ and a constant $\vartheta > 0$. The question is to study if there exists $\lambda > 0$ satisfies for every $h : H_1 \rightarrow H_2$ such that

$$\tilde{\delta}(h(\sigma v), h(\sigma)h(v)) \leq \lambda, \forall \sigma, v \in H_1, \quad (1)$$

there exists a homomorphism $f : H_1 \rightarrow H_2$ satisfies

$$\tilde{\delta}(h(\sigma), f(\sigma)) \leq \vartheta, \forall \sigma \in H_1. \quad (2)$$

In 1941, Hyers [6] presented a partial solution to the question of S. Ulam assuming that D_1, D_2 be two Banach spaces in the case of λ -linear transformations, that is

Let D_1, D_2 be two Banach spaces and set $h : D_1 \rightarrow D_2$ be a linear transformation satisfying

$$\|h(\sigma + v) - h(\sigma) - h(v)\| \leq \lambda, \forall \sigma, v \in D_1, \lambda > 0. \quad (3)$$

There exists a unique linear transformation $\Delta : D_1 \rightarrow D_2$ such that the limit $\Delta(\sigma) = \lim_{n \rightarrow +\infty} h(2^n \sigma)/2^n$ exists for each $\sigma \in D_1$ and $\|h(\sigma) - \Delta(\sigma)\| \leq \lambda$ for all $\sigma \in D_1$, which was the first step towards more answers in this area. Many researchers have analyzed the HUS of various classes of differential systems (see, for instance, [1, 6–21]). In 1978,

Rassias [22] provided a generalized answer to the Ulam question for approximate λ -linear transformations. In [23], Rassias obtained an extension of the Hyers's answer.

In 1994, Gavruta [24] gave a generalization form of Rassias's Theorem for the unbounded Cauchy difference $h(\sigma + v) - h(\sigma) - h(v)$ and introduced the notion of generalized HURS in the sense of Rassias approach.

In the last decades, there is an increasing interest and work on the Ulam stability and the Ulam-Hyers stability of some deterministic systems using the Banach contraction principle and Schaefer's fixed point theorem (see [25, 26]).

In the literature, there are a few papers about the HUS and the HURS of stochastic systems (see [13, 27–30]). The stability of SFDEs has attracted much more attention (see [2, 19] etc.). Consequently, it is interesting to extend the research results on the deterministic functional systems to the stochastic case.

Let us outline the framework of this paper. After some basic notions and assumptions (see Section 2), in Section 3, the HUS and HURS of the solution of the system are proved by using the fixed point methodology. In the last section, two numerical examples are presented to illustrate the main results.

2. Preliminary

Denote by $\{\Omega, \mathcal{F}, (\mathcal{F}_\sigma)_{\sigma \geq 0}, P\}$ the complete probability space where $\{\mathcal{F}_\sigma\}_{\sigma \geq 0}$ is a filtration satisfying the usual conditions. $W(\sigma)$ is an m -dimensional Brownian motion defined on the probability space. Denote by $\mathcal{L}^2([v, w], \mathbb{R}^b)$ the set of $\mathbb{R}^b$ -valued $\mathcal{F}_\sigma$ -adapted processes $\{\psi(\sigma)\}_{v \leq \sigma \leq w}$ such that $\int_v^w |\psi(\sigma)|^2 d\sigma < \infty$ a.s. and $\mathcal{M}^2([v, w], \mathbb{R}^b)$ the set of processes $\{\psi(\sigma)\}_{v \leq \sigma \leq w}$ in $\mathcal{L}^2([v, w], \mathbb{R}^b)$ satisfies $\mathbb{E} \int_v^w |\psi(\sigma)|^2 d\sigma < \infty$. Let $C([-Q, 0]; \mathbb{R}^b)$ denote the set of functions φ from $[-Q, 0]$ to $\mathbb{R}^b$ that are right-continuous and have limits on the left. $C([-Q, 0]; \mathbb{R}^b)$ is equipped with the norm $\|\varphi\| = \sup_{-Q \leq s \leq 0} |\varphi(s)|$ and $|x| = \sqrt{x^T x}$ for any $x \in \mathbb{R}^b$. Denote by $C_{\mathcal{F}_0}^b([-Q, 0]; \mathbb{R}^b)$ the set of all $\mathcal{F}_0$ -measurable bounded $C([-Q, 0]; \mathbb{R}^b)$ -valued random variables $\chi = \{\chi(\kappa): -Q \leq \kappa \leq 0\}$. Let $L_{\mathcal{F}_t}^2([-Q, 0]; \mathbb{R}^b)$, $t \geq 0$, denote the set of all $\mathcal{F}_t$ -measurable, $C([-Q, 0]; \mathbb{R}^b)$ -valued random variables $\varsigma = \{\varsigma(\kappa): -Q \leq \kappa \leq 0\}$ satisfies $\sup_{-Q \leq \kappa \leq 0} \mathbb{E} |\varsigma(\kappa)|^2 < \infty$.

Consider the following SFDE for $0 \leq \omega_0 < T$ fixed:

$$d\zeta(\omega) = f(\omega, \zeta_\omega) d\omega + g(\omega, \zeta_\omega) dW(\omega), \omega_0 \leq \omega \leq T, \quad (4)$$

with the initial condition

$$\zeta_{\omega_0} = \chi \in L_{\mathcal{F}_{\omega_0}}^2([-Q, 0]; \mathbb{R}^b), \quad (5)$$

and recall that, given $\zeta \in C([\omega_0, T]; \mathbb{R}^b)$, for each $\omega \in [\omega_0, T]$, we denote by $\zeta_\omega(\cdot)$ the function in $C([\omega_0 - Q, 0]; \mathbb{R}^b)$ defined as $\zeta_\omega(p) = \zeta(\omega + p)$, $-Q \leq p \leq 0$. We assume that

$$\begin{aligned} f : [\omega_0, T] \times C([-Q, 0]; \mathbb{R}^b) &\longrightarrow \mathbb{R}^b, g \\ : [\omega_0, T] \times C([-Q, 0]; \mathbb{R}^b) &\longrightarrow \mathbb{R}^{b \times q}. \end{aligned} \quad (6)$$

Using the definition of Itô's stochastic differential and integrating the two sides of equation (4) from ω_0 to ω , we have

$$\zeta(\omega) = \zeta(\omega_0) + \int_{\omega_0}^{\omega} f(v, \zeta_v) dv + \int_{\omega_0}^{\omega} g(v, \zeta_v) dW(v), \omega_0 \leq \omega \leq T. \quad (7)$$

We consider the following assumption:

$\mathcal{A}_1$: (Uniform Lipschitz condition): Suppose that there is a constant $K > 0$ satisfies

$$|f(\omega, v_1) - f(\omega, v_2)|^2 \vee |g(\omega, v_1) - g(\omega, v_2)|^2 \leq K^2 \|v_1 - v_2\|^2, \quad (8)$$

$\forall \omega \in [\omega_0, T]$ and $v_1, v_2 \in C([-Q, 0]; \mathbb{R}^b)$, where $z_1 \vee z_2$ define the maximum of z_1 and z_2 .

3. Stability Results

In this part, we discuss the HUS and the HURS of equation (4) under the assumption $\mathcal{A}_1$.

Definition 1. Equation (4) is called HUS with respect to (w.r.t) ε if there exists a constant $M > 0$ such that for each $\varepsilon > 0$ and for each solution $\tilde{\zeta} \in \mathcal{M}^2([\omega_0 - Q, T], \mathbb{R}^b)$, with $\tilde{\zeta}_{\omega_0} = \chi$, of the following inequation:

$$\begin{aligned} \mathbb{E} \left| \tilde{\zeta}(\omega) - \tilde{\zeta}(\omega_0) - \int_{\omega_0}^{\omega} f(v, \tilde{\zeta}_v) dv - \int_{\omega_0}^{\omega} g(v, \tilde{\zeta}_v) dW(v) \right|^2 \\ \leq \varepsilon, \forall \omega \in [\omega_0, T], \end{aligned} \quad (9)$$

there exists a solution $\zeta \in \mathcal{M}^2([\omega_0 - Q, T], \mathbb{R}^b)$ of (4), with $\zeta_{\omega_0} = \chi$, such that $\mathbb{E} |\tilde{\zeta}(\omega) - \zeta(\omega)|^2 \leq M\varepsilon$,

$$\forall \omega \in [\omega_0, T]. \quad (10)$$

Definition 2. Equation (4) is called HURS w.r.t $(\varepsilon, \theta(\omega))$, with $\theta(\cdot) \in C([\omega_0, T]; \mathbb{R}_+)$, if there exists a constant $M > 0$ such that for each $\varepsilon > 0$ and for each solution

$\tilde{\zeta} \in \mathcal{M}^2([\omega_0 - Q, T], \mathbb{R}^b)$, with $\tilde{\zeta}_{\omega_0} = \chi$, satisfying

$$\begin{aligned} \mathbb{E} \left| \tilde{\zeta}(\omega) - \tilde{\zeta}(\omega_0) - \int_{\omega_0}^{\omega} f(v, \tilde{\zeta}_v) dv - \int_{\omega_0}^{\omega} g(v, \tilde{\zeta}_v) dW(v) \right|^2 \\ \leq \varepsilon \theta(\omega), \forall \omega \in [\omega_0, T], \end{aligned} \quad (11)$$

there exists a solution $\zeta(\omega) \in \mathcal{M}^2([\omega_0 - \mathbf{Q}, T], \mathbb{R}^b)$ of (4), with $\zeta_{\omega_0} = \chi$, such that $\forall \omega \in [\omega_0, T]$, $E|\tilde{\zeta}(\omega) - \zeta(\omega)|^2 \leq \varepsilon M \theta(\omega)$.

Definition 3. Equation (4) is generalized HURS w.r.t $\theta(\omega)$, with $\theta(\cdot) \in C([\omega_0 - \mathbf{Q}, T]; \mathbb{R}_+)$, if there exists a constant $M > 0$ such that for each solution.

$\tilde{\zeta} \in \mathcal{M}^2([\omega_0 - \mathbf{Q}, T], \mathbb{R}^b)$, with $\tilde{\zeta}_{\omega_0} = \chi$, satisfying

$$\begin{aligned} E \left| \tilde{\zeta}(\omega) - \tilde{\zeta}(\omega_0) - \int_{\omega_0}^{\omega} f(v, \tilde{\zeta}_v) dv - \int_{\omega_0}^{\omega} g(v, \tilde{\zeta}_v) dW(v) \right|^2 \\ \leq \varepsilon \theta(\omega), \forall \omega \in [\omega_0, T], \end{aligned} \quad (12)$$

there exists a solution $\zeta(\omega) \in \mathcal{M}^2([\omega_0 - \mathbf{Q}, T], \mathbb{R}^b)$ of (4), with $\zeta_{\omega_0} = \chi$, such that $\forall \omega \in [\omega_0, T]$, $E|\tilde{\zeta}(\omega) - \zeta(\omega)|^2 \leq M \theta(\omega)$.

Lemma 4 (see [2]). Set $\mathcal{M} = \mathcal{M}^2([\omega_0 - \mathbf{Q}, T], \mathbb{R}^b)$. Let $d : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}_+$ be the function such that

$$\begin{aligned} d^2(\zeta_1, \zeta_2) = \inf \left\{ \Lambda \in [0, +\infty), \frac{E|\zeta_1(\omega) - \zeta_2(\omega)|^2}{h_1(\omega)} \right. \\ \left. \leq \Lambda h_2(\omega), \forall \omega \in [\omega_0 - \mathbf{Q}, T] \right\}, \end{aligned} \quad (13)$$

where $h_1, h_2 \in C([\omega_0 - \mathbf{Q}, T], \mathbb{R}_+^*)$. Then, $(\mathcal{M}, d)$ is a complete metric space.

Theorem 5 (see [29]). Suppose (F, d) is a complete metric space and $L : F \rightarrow F$ is a contraction (with $\tau \in [0, 1]$). Suppose that $v \in F$, $\lambda > 0$ and $d(v, L(v)) \leq \lambda$. So, there exists a unique $\beta \in F$ satisfies $\beta = L(\beta)$. Moreover,

$$d(v, \beta) \leq \frac{\lambda}{1 - \tau}. \quad (14)$$

Theorem 6. Suppose that $\mathcal{A}_1$ hold. Let $\tilde{\zeta} \in \mathcal{M}^2([\omega_0 - \rho, T], \mathbb{R}^b)$, with $\tilde{\zeta}_{\omega_0} = \chi$, be a stochastic process satisfies

$$\begin{aligned} E \left| \tilde{\zeta}(\omega) - \tilde{\zeta}(\omega_0) - \int_{\omega_0}^{\omega} f(v, \tilde{\zeta}_v) dv - \int_{\omega_0}^{\omega} g(v, \tilde{\zeta}_v) dW(v) \right|^2 \\ \leq \varepsilon \theta(\omega), \forall \omega \in [\omega_0, T], \end{aligned} \quad (15)$$

where $\varepsilon > 0$ and $\theta(\cdot) \in C([\omega_0, T]; \mathbb{R}_+^*)$ is a nondecreasing function. Then, there is a solution $\zeta \in \mathcal{M}^2([\omega_0 - \rho, T], \mathbb{R}^b)$ of (4), with $\zeta_{\omega_0} = \chi$, such that $\forall \omega \in [\omega_0, T]$,

$$E|\tilde{\zeta}(\omega) - \zeta(\omega)|^2 \leq \frac{1}{(1 - \sqrt{\alpha/\alpha + \delta})^2} \exp((\alpha + \delta)(T - \omega_0)) \varepsilon \theta(\omega), \quad (16)$$

where $\alpha = 2K^2[(T - \omega_0) + 1]$ and δ is any positive constant.

Proof. Consider $d : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}_+$ such that

$$\begin{aligned} d^2(\zeta_1, \zeta_2) = \inf \left\{ \Lambda \in [0, +\infty), \frac{E|\zeta_1(\omega) - \zeta_2(\omega)|^2}{h(\omega)} \right. \\ \left. \leq \Lambda \tilde{\theta}(\omega), \forall \omega \in [\omega_0 - \rho, T] \right\}, \end{aligned} \quad (17)$$

with $h(\omega) = e^{\gamma(\omega - \omega_0)}$ for $\omega \in [\omega_0, T]$ and $h(\omega) = 1$ for $\omega \in [\omega_0 - \mathbf{Q}, \omega_0]$, where $\gamma = \alpha + \delta$, and $\tilde{\theta}(\omega) = \theta(\omega)$ for $\omega \in [\omega_0, T]$ and $\tilde{\theta}(\omega) = \theta(\omega_0)$ for $\omega \in [\omega_0 - \rho, \omega_0]$.

Let the operator $R : \mathcal{M} \rightarrow \mathcal{M}$ such that $(R\zeta)(\omega) = \tilde{\zeta}(\omega)$, for $\omega \in [\omega_0 - \mathbf{Q}, \omega_0]$, and

$$(R\zeta)(\omega) = \tilde{\zeta}(\omega_0) + \int_{\omega_0}^{\omega} f(v, \zeta_v) dv + \int_{\omega_0}^{\omega} g(v, \zeta_v) dW(v), \quad (18)$$

for $\omega \in [\omega_0, T]$.

It is easy to prove that R is well defined.

Let $\zeta_1, \zeta_2 \in \mathcal{M}$, for $\omega \in [\omega_0 - \mathbf{Q}, \omega_0]$, we get $(R\zeta_1)(\omega) - (R\zeta_2)(\omega) = 0$.

For $\omega \in [\omega_0, T]$, we have

$$\begin{aligned} |(R\zeta_1)(\omega) - (R\zeta_2)(\omega)|^2 \leq 2 \left| \int_{\omega_0}^{\omega} [f(v, \zeta_{1v}) - f(v, \zeta_{2v})] dv \right|^2 \\ + 2 \left| \int_{\omega_0}^{\omega} [g(v, \zeta_{1v}) - g(v, \zeta_{2v})] dW(v) \right|^2. \end{aligned} \quad (19)$$

Taking the expectation on both sides and using assumption $\mathcal{A}_1$, we have

$$\begin{aligned} E|(R\zeta_1)(\omega) - (R\zeta_2)(\omega)|^2 \\ \leq 2K^2 \left[(T - \omega_0) \int_{\omega_0}^{\omega} E\|\zeta_{1v} - \zeta_{2v}\|^2 dv + \int_{\omega_0}^{\omega} E\|\zeta_{1v} - \zeta_{2v}\|^2 dv \right]. \end{aligned} \quad (20)$$

Then,

$$E|(R\zeta_1)(\omega) - (R\zeta_2)(\omega)|^2 \leq \alpha \int_{\omega_0}^{\omega} E\|\zeta_{1v} - \zeta_{2v}\|^2 dv. \quad (21)$$

For $v \in [\omega_0, \omega]$, we have $E\|\zeta_{1v} - \zeta_{2v}\|^2 = E|\zeta_1(v + \sigma) - \zeta_2(v + \sigma)|^2$ where $\sigma \in [-\mathbf{Q}, 0]$. Then,

$$\begin{aligned} E\|\zeta_{1v} - \zeta_{2v}\|^2 &= \frac{E|\zeta_1(v+\sigma) - \zeta_2(v+\sigma)|^2}{h(v+\sigma)\tilde{\theta}(v+\sigma)} h(v+\sigma)\tilde{\theta}(v+\sigma) \\ &\leq d^2(\zeta_1, \zeta_2)h(v)\tilde{\theta}(v). \end{aligned} \quad (22)$$

Therefore,

$$\begin{aligned} E|(R\zeta_1)(\omega) - (R\zeta_2)(\omega)|^2 &\leq \alpha d^2(\zeta_1, \zeta_2)\theta(\omega) \int_{\omega_0}^{\omega} h(v)dv \\ &\leq \frac{\alpha}{\gamma} d^2(\zeta_1, \zeta_2)\theta(\omega)h(\omega). \end{aligned} \quad (23)$$

Then, $d^2(R\zeta_1, R\zeta_2) \leq (\alpha/\gamma)d^2(\zeta_1, \zeta_2)$. Thus, $d(R\zeta_1, R\zeta_2) \leq \sqrt{\alpha/\gamma}d(\zeta_1, \zeta_2)$. Therefore, R is strictly contractive for $c = \sqrt{\alpha/\gamma} + \delta$.

For $\omega \in [\omega_0 - \mathbf{Q}, \omega_0]$, we get $(R\tilde{\zeta})(\omega) - \tilde{\zeta}(\omega) = 0$.

From (15), we get

$$\begin{aligned} \frac{E|\tilde{\zeta}(\omega) - R\tilde{\zeta}(\omega)|^2}{\theta(\omega)h(\omega)} &= \frac{E|\tilde{\zeta}(\omega) - \tilde{\zeta}(\omega_0) - \int_{\omega_0}^{\omega} f(v, \tilde{\zeta}_v)dv - \int_{\omega_0}^{\omega} g(v, \tilde{\zeta}_v)dW(v)|^2}{\theta(\omega)h(\omega)} \leq \varepsilon, \end{aligned} \quad (24)$$

for all $\omega \in [\omega_0, T]$.

Then, $d^2(R\tilde{\zeta}, \tilde{\zeta}) \leq \varepsilon$. Therefore, $d(R\tilde{\zeta}, \tilde{\zeta}) \leq \sqrt{\varepsilon}$. Using the fixed point theorem, there is a solution ζ^* of (4) such that $d(\tilde{\zeta}, \zeta^*) \leq (1/(1-c))\sqrt{\varepsilon}$. Then,

$$\frac{E|\tilde{\zeta}(\omega) - \zeta^*(\omega)|^2}{\theta(\omega)h(\omega)} \leq \frac{1}{(1-c)^2} \varepsilon, \forall \omega \in [\omega_0, T]. \quad (25)$$

Therefore,

$$E|\tilde{\zeta}(\omega) - \zeta^*(\omega)|^2 \leq \frac{e^{(\alpha+\delta)(T-\omega_0)}}{(1-c)^2} \varepsilon \theta(\omega), \forall \omega \in [\omega_0, T], \quad (26)$$

as desired.

Remark 7. In our analysis of the HURS, we do not suppose any condition on K unlike the case of the Theorem 6 in [29].

Theorem 8. Assume that $\mathcal{A}_1$ hold. Let $\tilde{\zeta} \in \mathcal{M}^2([\omega_0 - \mathbf{Q}, T], \mathbb{R}^b)$, with $\tilde{\zeta}_{\omega_0} = \chi$, be a stochastic process satisfies

$$\begin{aligned} E\left|\tilde{\zeta}(\omega) - \tilde{\zeta}(\omega_0) - \int_{\omega_0}^{\omega} f(v, \tilde{\zeta}_v)dv - \int_{\omega_0}^{\omega} g(v, \tilde{\zeta}_v)dW(v)\right|^2 \\ \leq \varepsilon, \forall \omega \in [\omega_0, T], \end{aligned} \quad (27)$$

where $\varepsilon > 0$. Then, there is a solution $\zeta \in \mathcal{M}^2([\omega_0 - \rho, T], \mathbb{R}^b)$ of (4), with $\zeta_{\omega_0} = \chi$, such that $\forall \omega \in [\omega_0, T]$,

$$E|\tilde{\zeta}(\omega) - \zeta(\omega)|^2 \leq \frac{1}{(1 - \sqrt{\alpha/\alpha + \delta})^2} \exp((\alpha + \delta)(T - \omega_0))\varepsilon, \quad (28)$$

where $\alpha = 2K^2[(T - \omega_0) + 1]$ and δ is a positive constant.

Proof. The proof of this theorem is similar to Theorem 6.

Theorem 9. Assume that $\mathcal{A}_1$ hold. Let $\tilde{\zeta} \in \mathcal{M}^2([\omega_0 - \mathbf{Q}, T], \mathbb{R}^b)$, with $\tilde{\zeta}_{\omega_0} = \chi$, be a stochastic process satisfies

$$\begin{aligned} E\left|\tilde{\zeta}(\omega) - \tilde{\zeta}(\omega_0) - \int_{\omega_0}^{\omega} f(v, \tilde{\zeta}_v)dv - \int_{\omega_0}^{\omega} g(v, \tilde{\zeta}_v)dW(v)\right|^2 \\ \leq \theta(\omega), \forall \omega \in [\omega_0, T], \end{aligned} \quad (29)$$

where $\theta(\cdot) \in C([\omega_0, T]; \mathbb{R}_+^*)$ is a nondecreasing function. Then, there is a solution $\zeta \in \mathcal{M}^2([\omega_0 - \rho, T], \mathbb{R}^b)$ of (4), with $\zeta_{\omega_0} = \chi$, such that $\forall \omega \in [\omega_0, T]$,

$$E|\tilde{\zeta}(\omega) - \zeta(\omega)|^2 \leq \frac{1}{(1 - \sqrt{\alpha/\alpha + \delta})^2} \exp((\alpha + \delta)(T - \omega_0))\theta(\omega), \quad (30)$$

where $\alpha = 2K^2[(T - \omega_0) + 1]$ and $\delta > 0$.

Proof. The proof of this theorem is similar to Theorem 6.

4. Examples

Two examples are studied to show the interest of the main results.

Example 10. Consider the following SFDE for each $\varepsilon > 0$

$$\begin{cases} d\zeta(\omega) = f(\omega, \zeta_\omega)d\omega + g(\omega, \zeta_\omega)dW(\omega), \\ E\left|\zeta(\omega) - \zeta(\omega_0) - \int_{\omega_0}^{\omega} f(v, \zeta_v)dv - \int_{\omega_0}^{\omega} g(v, \zeta_v)dW(v)\right|^2 \leq \varepsilon(\omega + 1), \end{cases} \quad (31)$$

where

$$\chi \in L^2_{\mathcal{F}_{\omega_0}}([-\rho, 0]; \mathbb{R}), \zeta(\omega) \in \mathcal{M}^2([\omega_0 - \mathbf{Q}, T], \mathbb{R}),$$

$$f(\omega, \xi_1) = \sin(\omega)\xi_1(0) + \cos(\omega)\xi_1(-\mathbf{Q}), \xi_1 \in C([-\mathbf{Q}, 0]; \mathbb{R}),$$

$$g(\omega, \xi_1) = \frac{1}{\sqrt{1 + \omega^2}}\xi_1(0) + \frac{e^{-\omega}}{\sqrt{1 + \omega^2}}\xi_1(-\mathbf{Q}), \xi_1 \in C([-\mathbf{Q}, 0]; \mathbb{R}), \quad (32)$$

with $Q > 0$. Then, replacing now ξ_1 by the segment of a solution ζ_ω , we get

$$g(\omega, \zeta_\omega) = \frac{1}{\sqrt{1+\omega^2}} \zeta(\omega) + \frac{e^{-\omega}}{\sqrt{1+\omega^2}} \zeta(\omega - Q), \quad (33)$$

$$f(\omega, \zeta_\omega) = \sin(\omega) \zeta(\omega) + \cos(\omega) \zeta(\omega - Q).$$

Let $\xi_1, \xi_2 \in C([-Q, 0]; \mathbb{R})$, then

$$\begin{aligned} |f(\omega, \xi_1) - f(\omega, \xi_2)|^2 &= |\sin(\omega)(\xi_1(0) - \xi_2(0)) + \cos(\omega)(\xi_1(-Q) - \xi_2(-Q))|^2 \\ &\leq 2 \sin^2(\omega) |\xi_1(0) - \xi_2(0)|^2 + 2 \cos^2(\omega) |\xi_1(-Q) - \xi_2(-Q)|^2 \\ &\leq 4 \|\xi_1 - \xi_2\|^2, \end{aligned}$$

$$\begin{aligned} |g(\omega, \xi_1) - g(\omega, \xi_2)|^2 &= \left| \frac{1}{\sqrt{1+\omega^2}} (\xi_1(0) - \xi_2(0)) + \frac{e^{-\omega}}{\sqrt{1+\omega^2}} (\xi_1(-Q) - \xi_2(-Q)) \right|^2 \\ &\leq \frac{2}{1+\omega^2} |\xi_1(0) - \xi_2(0)|^2 + \frac{2e^{-2\omega}}{1+\omega^2} |\xi_1(-Q) - \xi_2(-Q)|^2 \\ &\leq 4 \|\xi_1 - \xi_2\|^2. \end{aligned} \quad (34)$$

Hence, the uniform Lipschitz condition is satisfied.

Therefore, by Theorem 6, there is a solution $\zeta \in \mathcal{M}^2([\omega_0 - Q, T], \mathbb{R})$ of (31), with $\zeta_{\omega_0} = \chi$, such that $\forall \omega \in [\omega_0, T]$,

$$E|\tilde{\zeta}(\omega) - \zeta(\omega)|^2 \leq \frac{1}{(1 - \sqrt{\alpha/\alpha + \delta})^2} \exp((\alpha + \delta)(T - \omega_0)) \varepsilon(\omega + 1), \quad (35)$$

where $\alpha = 32[(T - \omega_0) + 1]$ and $\delta > 0$.

For System (31), we conduct a simulation based on the Euler-Maruyama scheme with step size 10^{-3} , for which we set $\omega_0 = 0$, $Q = 0.5$ and the initial, data χ as a map, namely, $\chi = \omega^2$ for all $-0.5 \leq \omega \leq 0$. In Figure 1, we give a sequence of computer simulations of the exact solution path $\zeta(\omega)$ and the rough solution path $\tilde{\zeta}(\omega)$ for System (31) on the interval $[-0.5, 8]$. Choosing $\varepsilon = 10^{-4}$, $\delta = 15$, $\alpha = 35.2$, and $T = 0.1$ one obtain $M = (1/(1 - \sqrt{\alpha/\alpha + \delta})^2) \exp((\alpha + \delta)(T - \omega_0)) = 5.72 \times 10^3$. We use the time step 10^{-4} of the interval $[0, 0.1]$ and 10000 realizations for this discretisation; we give in Figure 2 the trajectory of $\varepsilon \times M \times (\omega + 1)$ and simulation of the mean square of $|\tilde{\zeta}(\omega) - \zeta(\omega)|$ on the interval $[0, 0.1]$. It is clear that the convergence plot verifies the theoretical findings.

Example 11. Consider the following SFDE for each $\varepsilon > 0$

$$\begin{cases} d\tilde{\zeta}(\omega) = f(\omega, \tilde{\zeta}_\omega) d\omega + g(\omega, \tilde{\zeta}_\omega) dW(\omega), \\ E\left|\tilde{\zeta}(\omega) - \tilde{\zeta}(0) - \int_0^\omega f(v, \tilde{\zeta}_v) dv - \int_0^\omega g(v, \tilde{\zeta}_v) dW(v)\right|^2 \leq \varepsilon, \end{cases} \quad (36)$$

FIGURE 1: Simulation of $\zeta(\omega)$ and $\tilde{\zeta}(\omega)$ are trajectory in System (31) with $Q = 0.5$ and $\chi(\omega) = \omega^2$ for $\omega \in [-0.5, 8]$.

FIGURE 2: HURS with respect to $(\varepsilon, \theta(\omega))$ of $\zeta(\omega)$ on the interval $[0, 0.1]$ with $\chi(\omega) = \omega^2$.

where

$$\begin{aligned} \chi &\in L^2_{\mathcal{F}_0}([- \rho, 0]; \mathbb{R}), \zeta(\omega) \in \mathcal{M}^2([-Q, 3], \mathbb{R}), \\ f(\omega, \xi_1) &= \omega \xi_1(0) + \omega^2 \xi_1(-Q), \xi_1 \in C([-Q, 0]; \mathbb{R}), \\ g(\omega, \xi_1) &= \omega^2 \xi_1(0) + \omega \xi_1(-Q), \xi_1 \in C([-Q, 0]; \mathbb{R}), \end{aligned} \quad (37)$$

with $\rho > 0$. Then, replacing now ξ_1 by the segment of a solution ζ_ω , we have

$$\begin{aligned} f(\omega, \zeta_\omega) &= \omega \zeta(\omega) + \omega^2 \zeta(\omega - Q), \\ g(\omega, \zeta_\omega) &= \omega^2 \zeta(\omega) + \omega \zeta(\omega - Q). \end{aligned} \quad (38)$$

FIGURE 3: Simulation of $\zeta(\omega)$ and $\tilde{\zeta}(\omega)$ are trajectory in System (36) with $\mathbf{q} = 0.5$ and $\chi(\omega) = -\omega + 1$ for $\omega \in [-0.5, 2]$.

FIGURE 4: HUS with respect to ε of $\zeta(\omega)$ on the interval $[0, 0.06]$ with $\chi(\omega) = -\omega + 1$.

Let $\xi_1, \xi_2 \in C([-q, 0]; \mathbb{R})$, then

$$\begin{aligned} |f(\omega, \xi_1) - f(\omega, \xi_2)|^2 &= |\omega(\xi_1(0) - \xi_2(0)) + \omega^2(\xi_1(-q) - \xi_2(-q))|^2 \\ &\leq 2\omega^2|\xi_1(0) - \xi_2(0)|^2 + 2\omega^4|\xi_1(-q) - \xi_2(-q)|^2 \\ &\leq 180\|\xi_1 - \xi_2\|^2, \end{aligned}$$

$$\begin{aligned} |g(\omega, \xi_1) - g(\omega, \xi_2)|^2 &= |\omega^2(\xi_1(0) - \xi_2(0)) + \omega(\xi_1(-q) - \xi_2(-q))|^2 \\ &\leq 2\omega^4|\xi_1(0) - \xi_2(0)|^2 + 2\omega^2|\xi_1(-q) - \xi_2(-q)|^2 \\ &\leq 180\|\xi_1 - \xi_2\|^2. \end{aligned}$$

(39)

Hence, the uniform Lipschitz condition is satisfied.

Therefore, by Theorem 8, there is a solution $\zeta \in \mathcal{M}^2([-q, 3], \mathbb{R})$ of (36), with $\zeta_0 = \chi$, such that $\forall \omega \in [0, 3]$,

$$E|\tilde{\zeta}(\omega) - \zeta(\omega)|^2 \leq \frac{1}{(1 - \sqrt{\alpha/\alpha + \delta})^2} \exp(3(\alpha + \delta))\varepsilon, \quad (40)$$

where $\alpha = 1440$ and $\delta > 0$.

We use again Euler-Maruyama scheme with step size 10^{-5} to conduct a simulation for System (36). We fix $\rho = 0.5$ and the initial data χ as a linear mapping, namely, $\chi = -\omega + 1$ for all $-0.5 \leq \omega \leq 0$. In Figure 3, we plot the path of the exact solution $\zeta(\omega)$ and the rough solution path $\tilde{\zeta}(\omega)$ for System (36) on the interval $[-0.5, 2]$. Choosing $\varepsilon = 10^{-9}$, $\delta = 10$, $\alpha = 1440$, and $T = 0.06$ one obtain a large value of $M = (1/(1 - \sqrt{\alpha/\alpha + \delta})^2) \exp(3(\alpha + \delta))$. We use the time step 10^{-4} of the interval $[0, 0.06]$ and 10000 realizations for this discretisation; we give in Figure 4 the trajectory of the constant function $\varepsilon \times M$ and simulation of the mean square of $|\tilde{\zeta}(\omega) - \zeta(\omega)|$ on the interval $[0, 0.06]$. It is clear that the convergence plot verifies the theoretical findings.

5. Conclusion

In this paper, we investigate the Ulam-Hyers-Rassias stability of stochastic functional differential equations. To obtain the main results, we used the fixed point theorem and the classical stochastic calculus techniques. Moreover, we extend the Ulam-Hyers-Rassias stability for a generalization version. An example is presented to show the applicability of our results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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