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Research Article

Blow-Up of Solutions for a Class Quasilinear Wave Equation with Nonlinearity Variable Exponents

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This work deals with the blow-up of solutions for a new class of quasilinear wave equation with variable exponent nonlinearities. To clarify more, we prove in the presence of dispersion term $-\Delta u_{tt}$ a finite-time blow-up result for the solutions with negative initial energy and also for certain solutions with positive energy. Our results are extension of the recent work (Appl Anal. 2017; 96(9): 1509-1515).

1. Introduction

We study in this paper the following nonlinear wave equation:

$$\begin{cases} u_{tt} - \operatorname{div}\left(\left|\nabla u^{s(.)-2}\right|\nabla u\right) - \Delta u_{tt} + \eta u_t |u_t|^{q(.)-2} = \mu u |u|^{p(.)-2}, \text{ in}\Omega \times (0, T), \\ u(x, t) = 0, \text{ on}\Omega \times (0, T), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \text{ in}\Omega. \end{cases} \tag{1}$$

Here, $\Omega \subset \mathbb{R}^n (n \geq 1)$, be a bounded domain with a smooth boundary $\partial\Omega$, and$\eta, \mu > 0$ are constants, and the exponents $q(.)$, $p(.)$, and $s(.)$ are given measurable functions on Ω satisfying

$$2 \leq \max\{q_2, s_2\} < p_1 \leq p(x) \leq p_2 \leq s^*(x), \tag{2}$$

with

$$\begin{aligned} p_1 &:= \operatorname{ess}\inf_{x\in\Omega} p(x), p_2 := \operatorname{ess}\sup_{x\in\Omega} p(x), \\ s_1 &:= \operatorname{ess}\inf_{x\in\Omega} s(x), s_2 := \operatorname{ess}\sup_{x\in\Omega} s(x), \\ q_1 &:= \operatorname{ess}\inf_{x\in\Omega} q(x), q_2 := \operatorname{essp}\sup_{x\in\Omega} q(x), \\ s^*(x) &= \begin{cases} \dfrac{Ns(x)}{\operatorname{ess}\sup_{x\in\Omega}(N - q(x))} & \text{if } s_2 < n, \\ +\infty & \text{if } s_2 \geq n. \end{cases} \end{aligned} \tag{3}$$

Also, we suppose that $q(.)$, $p(.)$, and $s(.)$ satisfy the log-Holder continuity condition:

$$|m(x) - m(y)| \leq -\frac{A}{\log|x - y|} \text{ for a.e.} x, y \in \Omega, \text{with} |x - y| < \delta, \tag{4}$$

$A > 0, 0 < \delta < 1$. In (4), if $x = y$, the inequality is undefined because log 0 is undefined. The inequality is defined for x not equal to y, but the condition that δ is completely greater than zero always makes x not equal to y because $|x - y| < \delta$. The term $\Delta_{s()} u = \text{div}\ (|\nabla u^{s(.)-2}|\nabla u)$ is called $s(.)$-Laplacian.

There are many studies that have studied the problem (P) in the case of constant and variable-exponent nonlinearities.

In the case of constant exponent nonlinearity when $\eta = 0$, $s = 2$, and $\Delta u_{tt} = 0$, the explosion term $\mu u|u|^{p-2}$ forces the negative-energy solutions to explode in finite time ([1, 2]), whereas when $\mu = 0$, $s = 2$, and $\Delta u_{tt} = 0$, the dissipation term $\eta|u_t|^{q-2}u_t$ guarantees the existence of global solutions for any initial data [3].

The problem was first treated by (Levine [2] and Vitillaro [4]) in the case when the both terms are present (the dissipation and source). He debated the case when $q = 2, s = 2$, and determined the result of blow-up for solutions with negative initial energy. To extend Levine's results in [5] considered a different method when $q > 2$ and discussed the cases when $q \geq p$ and $p > q$.

Chen et al. in [6] looked into the nonlinear $p-$ Laplacian wave equation:

$$\partial_{tt} u - \nabla.\left(|\nabla u|^{p-2}\nabla u\right) - \Delta u_t + q(x, u) = f(x), \tag{5}$$

when $2 \leq p < n$ and f, q are given functions. Under suitable conditions on the initial data and the functions f, q, they realized global existence and uniqueness and also discussed the long-time behavior of the solution.

In [7], Benaissa and Mokeddem considered

$$u_{tt} - \text{div}\ \left(|\nabla u|^{p-2}\nabla u\right) - \sigma(t)\ \text{div}\ \left(|\nabla u_t|^{m-2}\nabla u_t\right) = 0, \tag{6}$$

and they achieved an energy decay estimate for the solutions where $p, m \geq 2$, σ, is a positive function and expanded Yang [8] and Messaoudi [9] results. Recently, Mokeddem and Mansour [10] added some modification in the problem of Benaissa and Mokeddem [10] and established the same decay result.

Messaoudi and Houari [11] studied the nonlinear wave equation:

$$\begin{aligned} u_{tt} - \Delta u_t - c\ \text{div}\ \left(|\nabla u|^{\alpha-2}\nabla u\right) - \text{div}\ \left(|\nabla u_t|^{\beta-2}\nabla u_t\right) \\ + a|u_t|^{m-2}u_t = b|u|^{p-2}u, \end{aligned} \tag{7}$$

where Ω is a bounded domain in $\mathbb{R}^n (n \geq 1)$,$a, b, c > 0$, and $\alpha, \beta, m, p > 2$. They investigated with appropriate conditions imposed on $\alpha, \beta, m, p > 2$, a global nonexistence result for solutions associated with negative initial energy. In [12], Kafini and Messaoudi treated a nonlinear wave equation with delay term and proved, under appropriate hypotheses on the initial data, that the energy of solutions explodes in a finite time. For more results, see the previous studies ([13–35]). In the case of variable-exponent nonlinearity, Antontsev [36] looked into the problem:

$$\partial_{tt} u - \text{div}\ \left(a(x, t)|\nabla u|^{p(x,t)-2}\nabla u\right) - \alpha\Delta u_t = b(x, t)u|u|^{\sigma(x,t)-2}, \tag{8}$$

when α is a nonnegative constant, and a, b, p, σ are given functions. He discussed the case when $\alpha = 0$ and $\alpha > 0$ and demonstrated a blow-up result under particular hypothesis on a, b, p, σ.

Thereafter, Antontsev in [36] considered the same equation and established a local, global existence of weak solutions for specific conditions on a, b, p, σ and realized a blow-up results for solutions with nonpositive initial energy. In [5], Guo and Gao considered the same problem of Antontsev [13], they picked the constant $\sigma(x, t) = r > 2$ and realized a blow-up result in finite time, and also, they alleged without any proof the same blow-up result for $\sigma(x, t) = r(x)$.

Sun et al. in [37] studied the blow-up result for solutions with positive initial energy for the following equation:

$$u_{tt} - \text{div}\ (a(x, t)\nabla u) + c(x, t)u_t|u_t|^{q(x,t)-1} = b(x, t)u|u|^{p(x,t)-1}. \tag{9}$$

They also gave lower and upper bounds for the blow-up time and provided a numerical illustrations for their result.

Lately, Messaoudi and Talahmeh [38] looked into

$$u_{tt} - \text{div}\ \left(|\nabla u|^{m(x)-2}\nabla u\right) + \mu u_t = |u|^{p(x)-2}u, \tag{10}$$

where $\mu \geq 0$. They proved a blow-up result for certain solutions with arbitrary positive initial energy. Korpusov [39] generalized this result and established (10), with m and p are constants.

In this paper, we care to find sufficient conditions on q, p, s and the initial data for which the blowup happens.

In addition to the introduction, our paper is divided into three sections. The second section deals with variable-exponent Lebesgue and Sobolev spaces and some of their characteristics. We also mention the result of existence, but without demonstration, and the second one deals with the result of blow-up for solutions with negative initial energy.

In the fourth one, we present and demonstrate the theorem of blow-up for certain solutions with positive initial energy.

2. Background and Preliminaries

This section contains some essential concepts and definitions about the Lebesgue and Sobolev spaces with variable exponents which will be useful to us later (see Fan and Zhao [40] and Lars et al. [41], Mezouar and Boulaaras [42]).

Let Ω that is a domain of $\mathbb{R}^n, p : \Omega \longrightarrow 1, \infty]$ be a measurable function. We introduce the Lebesgue space with a variable exponent $p(.)$ by

$$\begin{aligned} L^{p(.)}(\Omega) := \Big\{ w : \Omega \longrightarrow \mathbb{R}\ ; \text{measurable in}\ \Omega : \rho_{p(.)}(\lambda w) \\ < +\infty, \text{for}\ \textit{some}\ \lambda > 0 \Big\}, \end{aligned} \tag{11}$$

where

$$\rho_{p(.)}(w) = \int_{\Omega} |w(x)|^{p(x)} dx. \tag{12}$$

Equipped with the Luxembourg-type norm,

$$\|w\|_{p(.)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{w(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}. \tag{13}$$

$L^{p(.)}(\Omega)$ is a Banach space (see Lars et al. [41]).

Now, we introduce the following variable-exponent Sobolev space $W^{1,p(.)}(\Omega)$:

$$W^{1,p(.)}(\Omega) = \left\{ w \in L^{p(.)}(\Omega) \text{ such that} \nabla w \text{ exists and } |\nabla w| \in L^{p(.)}(\Omega) \right\}. \tag{14}$$

This space is a Banach space with respect to the norm $\|w\|_{W^{1,p(.)}(\Omega)} = \|w\|_{p(.)} + \|\nabla w\|_{p(.)}$. Otherwise, we put $W_0^{1,p(.)}(\Omega)$ to be the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(.)}(\Omega)$.

Remark 1. The space $W^{1,p(.)}(\Omega)$ is generally defined in a different way for the variable exponent case.

However, the two definitions are equivalent under condition (4) (see Lars et al. [4]). We define $W^{-1,p'(.)}(\Omega)$ as the dual of $W^{1,p(.)}(\Omega)$, in the same way as the classical Sobolev spaces, where $(1/p(.)) + (1/p'(.)) = 1$.

Lemma 2. *(Poincare inequality [41]).*

Let Ω be a bounded domain and assume that $p(.)$ satisfies (4), then

$$\|w\|_{p(.)} \le C\|\nabla w\|_{p(.)}, \text{for all } w \in W^{1,p(.)}(\Omega), \tag{15}$$

for $C > 0$ that is a constant depends only on Ω and p_1, p_2. Particularly, $\|\nabla w\|_{p()}$ determines an equivalent norm on $W_0^{1,p(.)}(\Omega)$.

Lemma 3. *(Lars et al. [41]).*

If $m : \Omega \longrightarrow [1,\infty)$ is a measurable function and $s(.) \in C(\bar{\Omega})$ such that

$$\begin{aligned} &ess \inf_{x\in\Omega} (s^*(x) - m(x)) > 0 \text{ with } s^*(x) \\ &= \begin{cases} \dfrac{ns(x)}{ess \sup_{x\in\Omega} (n - s(x))} & \text{if } s_2 < n, \\ \infty & \text{if } s_2 \ge n. \end{cases} \end{aligned} \tag{16}$$

Then, the embedding $W_0^{1,s(\bullet)}(\Omega)^{\circ} L^{m()}(\Omega)$ is continuous and compact.

Lemma 4. *(Hölder's inequality [41]).*

Assume that $p, m, r \ge 1$ are measurable functions defined on Ω such that

$$\frac{1}{r(y)} = \frac{1}{p(y)} + \frac{1}{m(y)}, \text{for a.e.} y \in \Omega. \tag{17}$$

If $u \in L^{p(.)}(\Omega)$ and $w \in L^{m(.)}(\Omega)$, then $uw \in L^{r(.)}(\Omega)$, with $\|uw\|_{r(.)} \le 2\|u\|_{p(.)}\|w\|_{m(.)}$.

Lemma 5. *(Unit ball property [41]).*

Assume that p is a measurable function on Ω. Then,

$$\|f\|_{p(.)} \le 1 \text{ if and only if } \rho_{p(.)}(f) \le 1. \tag{18}$$

Lemma 6. *(Lars etal [41]).*

If p is a measurable function on Ω satisfying (4), then for a.e. $x \in \Omega$, we get

$$\min \left\{ \|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2} \right\} \le \rho_{p(.)}(u) \le \max \left\{ \|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2} \right\}, \tag{19}$$

for any $u \in L^{p(.)}(\Omega)$.

Proposition 7. *Let $(u_0, u_1) \in (W_0^{1,s()}(\Omega) \times L^2(\Omega))$ and suppose that the exponents p, q, s satisfy (1) and (2). Then, problem (P) admits a unique weak solution such that*

$$\begin{aligned} &u \in L^{\infty}\left((0,T), W_0^{1,s(\bullet)}(\Omega)\right), \\ &u_t \in L^{\infty}\left((0,T), H_0^1(\Omega)\right), \\ &u_{tt} \in L^{\infty}\left((0,T), W_0^{-1,s'(\bullet)}(\Omega)\right), \end{aligned} \tag{20}$$

where $(1/s()) + (1/s'()) = 1$.

Remark 8. We can achieve the proof of the previous proposition by using the Galerkin method as in [13].

3. Blowing Up for Negative Initial Energy

To introduce and demonstrate our results, we first define our energy as follows:

$$\begin{aligned} E(t) := &\frac{1}{2}\int_{\Omega} u_t^2 dx + \int_{\Omega} \frac{1}{s(x)} |\nabla u|^{s(x)} dx + \frac{1}{2}\int_{\Omega} |\nabla u_t|^2 dx \\ &- \mu \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx. \end{aligned} \tag{21}$$

Theorem 9. *Assume that the assumptions of Proposition1 hold true and suppose that*

$$E(0) < 0. \tag{22}$$

Then, the solution of problem (P) blows up in finite time.

To state and demonstrate our result, In order to prove our result, we give the following Lemma

Lemma 10. *Suppose the conditions of Lemma 3 hold. Then, we have*

$$\rho_{p(.)}^{r/p_1}(u) \le C\left(\|\nabla u\|_{s_{(.)}}^{s_1} + \rho_{p(.)}(u)\right), s_1 \le r \le p_1, \tag{23}$$

for any $u \in W_0^{1,s()}(\Omega)$, where $C > 1$ is a positive constant which depends on Ω only.

Corollary 11. *Let the assumptions of Lemma 10 hold. Then, for any $u \in W_0^{1,s()}$, we get*

$$\|u\|_{p_1}^r \le C\left(\|\nabla u\|_{s_{(.)}}^{s_1} + \|u\|_{p_1}^{p_1}\right), \tag{24}$$

where $s_1 \le r \le p_1$ and C are a positive constant.

Now, we set

$$H(t) := -E(t) \tag{25}$$

and use, throughout this paper, C to denote a generic positive constant depending on Ω only. From the result of (21) and (23), we give the following Corollary

Corollary 12. *Let the assumptions of Lemma 10 hold. Then, we have*

$$\rho_{p(.)}^{r/p_1}(u) \le C\left(|H(t)| + \|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \rho_{p(.)}(u)\right), \tag{26}$$

for any $u \in u \in W_0^{1,s()}$ and $s_1 \le r \le p_1$.

Corollary 13. *Let the assumptions of Lemma 10 hold. Then, we have*

$$\|u\|_{p_1}^r \le C\left(|H(t)| + \|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|u\|_{p_1}^{p_1}\right), \tag{27}$$

for any $u \in W_0^{1,s()}$ and $s_1 \le r \le p_1$.

Lemma 14. *Assume that (2) and (4) hold and $E(0) < 0$. Then, the solution of (P) satisfies, for some $c > 0$,*

$$\rho_{p(.)}(u) \ge c\|u\|_{p_1}^{p_1}. \tag{28}$$

Lemma 15. *Let u be the solution of problem (P) and assume that (2) holds. Then,*

$$\int_\Omega |u|^{q(x)} dx \le C\left(\left(\rho_{p(.)}(u)\right)^{q_1/p_1} + \left(\rho_{p(.)}(u)\right)^{q_2/p_1}\right). \tag{29}$$

Remark 16. We can achieve the proof of the previous Lemmas and Corollaries as in the paper of Messaoudi and Talahmeh [39].

Lemma 17. *Let u be the solution of (P). Then, there exists a constant $c_1 > 0$ such that*

$$\|\nabla u(.,t)\|_{s(.)} \ge c_1, \forall t \ge 0. \tag{30}$$

Proof. Assume, by contradiction, there exists a sequence t_j such that

$$\left\|\nabla u\left(.,t_j\right)\right\|_{s(.)} \longrightarrow 0 \text{ as } j \longrightarrow \infty. \tag{31}$$

Then, Lemmas 3 and 6 give us

$$\rho_{p(.)}\left(u\left(.,t_j\right)\right) \longrightarrow 0 \text{ as } j \longrightarrow \infty. \tag{32}$$

This yields

$$\lim_{j\longrightarrow\infty} E\left(t_j\right) \ge 0, \tag{33}$$

that contrasts with the fact that $E(t) < 0, \forall t \ge 0$.

Proof. Proof of Theorem 9.

As usual, multiplying by u_t and integrating over Ω in (P) to get

$$E'(t) = -\eta\int_\Omega |u_t(x,t)|^{q(x)} dx \le 0, \tag{34}$$

for almost every t in $[0, T)$ since $E(t)$ is absolutely continuous; hence, $H'(t) \ge 0$ and

$$0 < H(0) \le H(t) \le \frac{\mu}{p_1}\rho_{p(.)}(u), \tag{35}$$

for every t in $[0, T)$, by remembering the condition that $E(0) < 0$. We then introduce

$$L(t) := H^{1-\alpha}(t) + \varepsilon\int_\Omega uu_t(x,t)dx, \tag{36}$$

for ε small to be chosen later and

$$0 < \alpha \le \min\left\{\frac{p_1 - 2}{2p_1}, \frac{p_1 - q_2}{p_1(q_2 - 1)}\right\}. \tag{37}$$

By taking the derivative of (35) and using (1), we obtain

$$L'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int_\Omega u_t^2(x,t)dx + \varepsilon\int_\Omega uu_{tt}^2(x,t)dx, \tag{38}$$

so

$$\begin{aligned}&\left\{L'(t)+\frac{d}{dt}\left(\varepsilon\int_\Omega\{\nabla u_t\nabla u\}\right)\right.\\&\quad=(1-\alpha)H^{-\alpha}(t)H'(t)+\varepsilon\int_\Omega\left[u_t^2-|\nabla u|^{s(x)}+|\nabla u_t|^2\right]\\&\qquad+\varepsilon\mu\int_\Omega|u|^{p(x)}-\eta\varepsilon\int_\Omega uu_t|u_t|^{q(x)-2}.\end{aligned}\tag{39}$$

Adding and subtracting the term $\varepsilon(1-\xi)p_1H(t)$, for $0<\xi<1$, from the right side of (37), we get

$$\begin{aligned}&L'(t)+\frac{d}{dt}\left(\varepsilon\int_\Omega\{\nabla u_t\nabla u\}\right)\\&\quad\geq(1-\alpha)H^{-\alpha}(t)H'(t)+\varepsilon(1-\xi)p_1H(t)\\&\qquad+\varepsilon\mu\xi\int_\Omega|u|^{p(x)}+\varepsilon\left(\frac{(1-\xi)p_1}{2}+1\right)\|u_t\|_2^2\\&\qquad+\varepsilon\left(\frac{(1-\xi)p_1}{s_2}-1\right)\int_\Omega|\nabla u|^{s(x)}\varepsilon\left(\frac{(1-\xi)p_1}{2}+1\right)\\&\qquad\cdot\int_\Omega|\nabla u_t|^2-\eta\varepsilon\int_\Omega uu_t|u_t|^{q(x)-2}dx.\end{aligned}\tag{40}$$

So, for ξ small enough, we obtain

$$\begin{aligned}&L'(t)+\frac{d}{dt}\left(\varepsilon\int_\Omega\{\nabla u_t\nabla u\}\right)\\&\quad\geq\varepsilon\beta\left[H(t)+\|u_t\|_2^2+\|\nabla u_t\|_2^2+\rho_{s(.)}(\nabla u)+\rho_{p(.)}(u)\right]\\&\qquad+(1-\alpha)H^{-\alpha}(t)H'(t)-\eta\varepsilon\int_\Omega uu_t|u_t|^{q(x)-2}dx.\end{aligned}\tag{41}$$

where

$$\beta=\min\left\{(1-\xi)p_1,\mu\xi,\frac{(1-\xi)p_1}{2}+1,\frac{(1-\xi)p_1}{s_2}-1\right\}>0.\tag{42}$$

By using Young's inequality, the last term in (40) yields

$$\begin{aligned}\int_\Omega|u_t|^{q(x)-1}|u|dx\leq\frac{1}{q_1}\int_\Omega\delta^{q(x)}|u|^{q(x)}+\frac{q_2-1}{q_2}\\\cdot\int_\Omega\delta^{-q(x)/q(x)-1}|u_t|^{q(x)}dx\forall\delta>0.\end{aligned}\tag{43}$$

Thus, by picking δ such that

$$\delta^{-q(x)/q(x)-1}=kH^{-\alpha}(t),\tag{44}$$

for a large constant k to be given later, and replacing in (41), we reach to

$$\begin{aligned}\int_\Omega|u_t|^{q(x)-1}|u|dx\leq\frac{1}{q_1}\int_\Omega k^{1-q(x)}|u|^{q(x)}H^{\alpha(q(x)-1)}(t)\\+\frac{q_2-1}{q_2}kH^{-\alpha}(t)H'(t)\forall\delta>0.\end{aligned}\tag{45}$$

Combining (40) and (43) yields

$$\begin{aligned}&L'(t)+\frac{d}{dt}\left(\varepsilon\int_\Omega\{\nabla u_t\nabla u\}\right)\\&\quad\geq\varepsilon\beta\left[H(t)+\|u_t\|_2^2+\|\nabla u_t\|_2^2+\rho_{s(.)}(\nabla u)+\rho_{p(.)}(u)\right]\\&\qquad+\left[(1-\alpha)-\varepsilon\frac{q_2-1}{q_2}k\right]H^{-\alpha}(t)H'(t)\\&\qquad-\eta\varepsilon\frac{k^{1-q_1}}{q_1}C_1H^{\alpha(q_2-1)}(t)\int_\Omega|u|^{q(x)}dx.\end{aligned}\tag{46}$$

Exploiting Lemma 15 and (34) to get

$$\begin{aligned}&H^{\alpha(q_2-1)}(t)\int_\Omega|u|^{q(x)}dx\\&\quad\leq C\left[(\rho(u))^{q_1/p_1+\alpha(q_2-1)}+(\rho(u))^{q_2/p_1+\alpha(q_2-1)}\right].\end{aligned}\tag{47}$$

Now, we employ Lemma 10 and (36), and we get

$$r=q_2+\alpha p_1(q_2-1)\leq p_1\text{ and }r=q_1+\alpha p_1(q_2-1)\leq p_1,\tag{48}$$

And it is easy to see from (46) that

$$H^{\alpha(q_2-1)}(t)\int_\Omega|u|^{q(x)}dx\leq C\left(\|\nabla u\|_{s(.)}^{s_1}+\rho_{p(.)}(u)\right).\tag{49}$$

So, by using Lemmas 6 and 17, we obtain

$$\rho_{s(.)}(\nabla u)\geq c_2\|\nabla u\|_{s(.)}^{s_1}.\tag{50}$$

Collecting of (45), (47), and (49), we get

$$\begin{aligned}&L'(t)+\frac{d}{dt}\left(\varepsilon\int_\Omega\{\nabla u_t\nabla u\}\right)\\&\quad\geq\left[(1-\alpha)-\frac{q_2-1}{q_2}\varepsilon k\right]H^{-\alpha}(t)H'(t)\\&\qquad+\varepsilon\left(\beta-\eta\frac{k^{1-q_1}}{q_1}C\right)\left[H(t)+\|u_t\|_2^2\right.\\&\qquad\left.+\|\nabla u_t\|_2^2+\|\nabla u\|_{s(.)}^{s_1}+\rho_{p(.)}(u)\right].\end{aligned}\tag{51}$$

In this step, we choose k so large that the coefficient

$$\gamma=\beta-\eta\frac{k^{1-q_1}}{q_1}C>0.\tag{52}$$

Once k is fixed (thus γ), we put sufficiently small ε so that

$$(1-\alpha)-\frac{q_2-1}{q_2}\varepsilon k \geq 0 \text{ and } L(0)=H^{1-\alpha}(0) + \varepsilon\int_\Omega u_0 u_1(x)dx > 0. \tag{53}$$

Subsequently, (50) becomes

$$\begin{aligned}\left\{L'(t)+\frac{d}{dt}\left(\varepsilon\int_\Omega\{\nabla u_t \nabla u\}\right)\right. &\geq \varepsilon\gamma\left[H(t)+\|u_t\|_2^2+\|\nabla u_t\|_2^2+\|\nabla u\|_{s(.)}^{s_1}+\rho_{p(.)}(u)\right] \\ &\geq \varepsilon\gamma\left[H(t)+\|u_t\|_2^2+\|u\|_{p_1}^{p_1}\right],\end{aligned} \tag{54}$$

by virtue of (28). Therefore, we get

$$L(t) \geq L(0) > 0, \text{ for } all\ t \geq 0. \tag{55}$$

Next, we are in the position to obtain an inequality of the form

$$L'(t)+\frac{d}{dt}\left(\varepsilon\int_\Omega\{\nabla u_t \nabla u\}\right) \geq \Gamma L^{1/1-\alpha}(t), \text{ for all } t \geq 0. \tag{56}$$

Here, Γ is a positive constant that depends on $\varepsilon\gamma$, C (the constant of Corollary 1).

To achieve (54), we estimate the term$|\int_\Omega uu_t(x,t)dx| \leq \|u\|_2\|u_t\|_2 \leq C(\|u\|_{p_1}\|u_t\|_2)$. (32)Hence,

$$\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\alpha)} \leq C\|u\|_{p_1}^{1/(1-\alpha)}\|u_t\|_2^{1/(1-\alpha)}. \tag{57}$$

From Young's inequality that yields the following estimate

$$\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\alpha)} \leq C\left[\|u\|_{p_1}^{\omega/(1-\alpha)}+\|u_t\|_2^{\chi/(1-\alpha)}\right], \tag{58}$$

where $1/\omega+1/\chi=1$. Putting $\chi=2/(1-\alpha)$, we find $\omega/(1-\alpha) = 2/(1-2\alpha) \leq p_1$ by (37). Thus, (56) becomes

$$\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\alpha)} \leq C\left[\|u\|_{p_1}^{r}+\|u_t\|_2^2\right], \tag{59}$$

with $r=2/(1-2\alpha) \leq p_1$. We obtain after using Corollary 3

$$\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\alpha)} \leq C\left[H(t)+\|u_t\|_2^2+\|\nabla u_t\|_2^2+\|u\|_{p_1}^{p_1}\right], \quad \text{for all } t \geq 0. \tag{60}$$

In the end, by noting that

$$\begin{aligned}L^{1/(1-\alpha)}(t) &= \left[H^{1-\alpha}(t)+\varepsilon\int_\Omega uu_t(x,t)dx\right]^{1/(1-\alpha)} \\ &\leq 2^{1/(1-\alpha)}\left[H(t)+\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\alpha)}\right],\end{aligned} \tag{61}$$

and combining it with (51) and (58)), the inequality (54) is achieved.

The proof is completed.

4. Blowing Up for Positive Initial Energy

Now, we are in the position to present and prove one of the main results of this section which is the blow-up for certain solutions with positive energy. For this goal, let A be the best constant of the Sobolev embedding $W_0^{1,s()}(\Omega)^\circ L^{p()}(\Omega)$ and set

$$A_1=\max\left\{1, A, \left(\frac{1}{\mu}\right)^{1/s_2}\right\}, \alpha_1=\left(\left(\frac{1}{\mu A_1^{p_1}}\right)^{s_2/(p_1-s_2)}\right), \tag{62}$$

$$\alpha_0=\|\nabla u_0\|_{s(.)}^{s_2}, E_1=\left(\frac{1}{s_2}-\frac{1}{p_1}\right)\alpha_1, \tag{63}$$

$$H(t)=E_1-E(t), \tag{64}$$

$$K(t)=H^{1-\lambda}(t)+\varepsilon\int_\Omega uu_t(x,t)dx, \tag{65}$$

for $0<\lambda<1, \varepsilon>0$ that are to be specified later.

We state here the following theorem which will be our main result.

Theorem 18. *Assume that the conditions of Proposition 7 hold true and suppose that*

$$E(0)<E_1, \alpha_1<\alpha_0 \leq A_1^{-s_2}. \tag{66}$$

Then, the solution of (P) blows up in a finite time.

To demonstrate our theorem, we refer the following two lemmas.

Lemma 19. *Let the assumptions in Theorem 18 be fulfilled, and then there exists a constant $\alpha_2>\alpha_1$ such that*

$$\|\nabla u(.,t)\|_{s(.)}^{s_2} \geq \alpha_2, \forall t \geq 0. \tag{67}$$

Proof. . Exploiting (21), we get

$$\begin{aligned} E(t) &\geq \frac{1}{s_2}\rho_{s(.)}(\nabla u) - \frac{\mu}{p_1}\rho_{p(.)}(u) \\ &\geq \frac{1}{s_2}\min\left\{\|\nabla u\|_{s(.)}^{s_1}, \|\nabla u\|_{s(.)}^{s_2}\right\} \\ &\quad - \frac{\mu}{p_1}\max\left\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right\} \\ &\geq \frac{1}{s_2}\min\left\{\|\nabla u\|_{s(.)}^{s_1}, \|\nabla u\|_{s(.)}^{s_2}\right\} - \frac{\mu}{p_1}\max \\ &\quad \cdot\left\{\left(A_1\|\nabla u\|_{s(.)}\right)^{p_1}, \left(A_1\|\nabla u\|_{s(.)}\right)^{p_2}\right\} \\ &= \frac{1}{s_2}\min\left\{\alpha^{s_1/s_2}, \alpha\right\} - \frac{\mu}{p_1}\max\left\{\left(A_1^{s_2}\alpha\right)^{p_1/s_2}, \left(A_1^{s_2}\alpha\right)^{p_2/s_2}\right\} \\ &:= h(\alpha), \forall\alpha \in [0,\infty), \end{aligned} \tag{68}$$

where $\alpha = \|\nabla u\|_{s(.)}^{s_2}$. Let

$$g(\alpha) = \frac{1}{s_2}\alpha - \frac{\mu}{p_1}\left(A_1^{s_2}\alpha\right)^{p_1/r_2}. \tag{69}$$

By noting that $g(\alpha) = h(\alpha)$, for $0 < \alpha \leq A_1^{s_2}$, we can easily verify that the function $g(\alpha)$ is increasing for $0 < \alpha < \alpha_1$ and decreasing for $\alpha_1 < \alpha \leq +\infty$.

Because $E(0) < E_1 = g(\alpha_1)$, there exists a positive constant $\alpha_2 \in (\alpha_1, \infty)$ such that $g(\alpha_2) = E(0)$. So, we get $g(\alpha_0) = h(\alpha_0) \leq E(0) = g(\alpha_2)$. This means that $\alpha_0 \geq \alpha_2$.

To demonstrate (67), we suppose that $\|\nabla u(t_0)\|_{s(.)}^{s_2} < \alpha_2$, for some $t_0 > 0$. Then, there exists $t_1 > 0$ such that $\alpha_1 < \|\nabla u(t_1)\|_{s(.)}^{s_2} < \alpha_2$. Exploiting the monotonicity of $g(\alpha)$ to find

$$E(t_1) \geq g\left(\|\nabla u(t_1)\|_{s(.)}^{s_2}\right) > g(\alpha_2) = E(0), \tag{70}$$

which contradicts $E(t) < E(0)$, for all $t \in (0, T)$. Consequently, (67) is determined.

Lemma 20. *Let the assumptions in Theorem 18 be fulfilled, and so we have*

$$0 < H(0) \leq H(t) \leq \frac{\mu}{p_1}\rho_{p(.)}(u). \tag{71}$$

Proof. Exploiting (21), (30), and (64) to get

$$\begin{aligned} 0 < H(0) \leq H(t) \leq E_1 &- \left[\frac{1}{2}\int_\Omega u_t^2 dx + \int_\Omega \frac{1}{s(x)}|\nabla u|^{s(x)} dx \right. \\ &\left. + \frac{1}{2}\int_\Omega |\nabla u_t|^2 dx\right] + \mu\int_\Omega \frac{1}{p(x)}|u|^{p(x)} dx, \end{aligned} \tag{72}$$

then from (67), we find

$$\begin{aligned} E_1 &- \left[\frac{1}{2}\int_\Omega u_t^2 dx + \int_\Omega \frac{1}{s(x)}|\nabla u|^{s(x)} dx + \frac{1}{2}\int_\Omega |\nabla u_t|^2 dx\right] \\ &\leq E_1 - \int_\Omega \frac{1}{s_2}|\nabla u|^{s(x)} dx \leq E_1 - \frac{1}{s_2}\min\left\{\|\nabla u\|_{s(.)}^{s_1}, \|\nabla u\|_{s(.)}^{s_2}\right\} \\ &\leq E_1 - \frac{1}{s_2}\min\left\{\alpha_2^{s_1/s_2}, \alpha_2\right\} \leq E_1 - \frac{1}{s_2}\min\left\{\alpha_1^{s_1/s_2}, \alpha_1\right\} \\ &= E_1 - \frac{1}{s_2}\alpha_1 = -\frac{\alpha_1}{p_1} < 0, \forall t \geq 0. \end{aligned} \tag{73}$$

Therefore,

$$0 < H(0) \leq H(t) \leq \frac{\mu}{p_1}\rho_{p(.)}(u). \forall t \geq 0. \tag{74}$$

Proof of Theorem 2. It is not hard to determine the proof precisely by repeating the same steps (35) to (58) of the proof of Theorem 1, with the use of Lemma 20.

Data Availability

No data were used to support the study.

Conflicts of Interest

The author(s) declare(s) that they have no conflicts of interest.

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