Research Article

Toeplitz-Superposition Operators on Analytic Bloch Spaces

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The important purpose of this current work is to study a new class of operators, the so-called Toeplitz-superposition operators as an expansion of the weighted known composition operators, induced by such continuous entire functions mapping on bounded specific sets. Minutely, we have deeply discussed the conditions for boundedness of this new type of operators between certain types of some holomorphic Bloch classes with some specific values of the weighted functions.

1. Introduction

Fundamentals of the needed analytic function spaces as well as the types of concerned operators are briefly introduced. The paper focuses first on the concerned setting of certain classes of function spaces and the new defined operator, which in turn is motivated essentially by some certain classical concepts of known operators such as superposition operators as well as Toeplitz operator. There is an emphasis in the concerned paper on intensive tying together the needed type of analytic function spaces and the concerned operators, to illustrate the roles of the obtained results.

All of the needed information to justify the target of this research is collected in this concerned section. Moreover, here, basic concerned concepts, the Bloch space of analytic-type, certain needed concerned lemmas, and superposition and Toeplitz operators are presented.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in $\mathbb{C}$, and let $\mathcal{H}(\mathbb{D})$ denote the class of all analytic functions in $\mathbb{D}$. Let $dA(z) = dx dy$ denote the concerned Lebesgue measures on $\mathbb{D}$.

Numerous intensive studies on analytic Bloch-type spaces are researched in literature (see [1–5] and others).

Let $h \in \mathcal{H}(\mathbb{D})$ and $0 < b < \infty$, the b -Bloch space $\mathcal{B}^b$ is defined by

$$\mathcal{B}^b = \left\{ f \in \mathcal{H}(\mathbb{D}) : \|h\|_{\mathcal{B}^b} = \sup_{\zeta \in \mathbb{D}} \left(1 - |\zeta|^2\right)^b |f'(\zeta)| < \infty \right\}. \quad (1)$$

The space $\mathcal{B}^1$ is called the Bloch space and denoted by $\mathcal{B}$ (see [3]).

The following interesting needed lemma has been proved in [6].

Lemma 1. For a given $0 < a < \infty$, let the function $h \in \mathcal{B}^a$. Then, we have

$$|h(\zeta)| \lesssim \begin{cases} \|h\|_{\mathcal{B}^a}, & \text{if } 0 < a < 1; \\ \|h\|_{\mathcal{B}^a} \ln \frac{e}{1 - |\zeta|^2}, & \text{if } a = 1; \\ \frac{\|h\|_{\mathcal{B}^a}}{(1 - |\zeta|^2)^{a-1}}, & \text{if } a > 1. \end{cases} \quad (2)$$

The following useful integral estimate is well known and can be found in [7].

Lemma 2. *Let $s > 0$ and $t > -1$. Then*

$$\int_{\mathbb{D}} \frac{(1 - |w|^2)^t dA(w)}{|1 - \bar{z}w|^{2+t+s}} \lesssim \frac{1}{(1 - |z|^2)^s}, \quad \text{for all } z \in \mathbb{D}. \quad (3)$$

For $a > -1$ and $p \in (0, \infty)$, the weighted Bergman spaces $\mathcal{A}_a^p(\mathbb{D})$ is the space of all functions $h \in \mathcal{H}(\mathbb{D})$, for which

$$\|h\|_{\mathcal{A}_a^p}^p = \int_{\mathbb{D}} |h(\zeta)|^p dA_a(\zeta) < \infty, \quad \text{where } dA_a(\zeta) = (1 - |\zeta|^2)^a dA(\zeta). \quad (4)$$

When $a = 0$, we simply write $\mathcal{A}^p(\mathbb{D})$ for $\mathcal{A}_0^p(\mathbb{D})$, and when $p = 2$, $\mathcal{A}_a^2(\mathbb{D})$ is a Hilbert space. It is well known that the Bergman kernel $K_z(w)$ of the Hilbert space $\mathcal{A}_a^2(\mathbb{D})$ is given by $K_z(w) = (1 - \bar{w}z)^{-a-2}$, where $z, w \in \mathbb{D}$. The Bergman projection P_a is the orthogonal projection from $L^2(\mathbb{D}, dA_a)$ onto Hilbert space $\mathcal{A}_a^2(\mathbb{D})$, which given as:

$$P_a h(z) = \int_{\mathbb{D}} K_z(w) h(w) dA_a(w). \quad (5)$$

For $a > -1$ and $h \in \mathcal{H}(\mathbb{D})$, the Toeplitz-type operator T_u^a with symbol $u \in H^\infty(\mathbb{D})$ is defined by

$$T_u^a h(z) = \int_{\mathbb{D}} \frac{u(w) h(w)}{(1 - \bar{w}z)^{a+2}} dA_a(w). \quad (6)$$

This paper is organized as follows: during Section 2, we have defined the Toeplitz-superposition operators on the normed (metric) subspaces. Throughout Section 3, we establish the conditions for the Toeplitz-superposition operators to be bounded from a -Bloch space $\mathcal{B}^a$ into b -Bloch space $\mathcal{B}^b$, in the case $a \in (0, 1)$ and $b > a$ or $b < a$. Section 4 is devoted to a study the boundedness of Toeplitz-superposition operators between weighted Bloch spaces in the case $0 < a \leq b$ or $a = 0, b > 0$.

Remark 3. It is concerned remarkable to say that two concerned quantities N_h and N_h^* , where both depending on the concerned function $h \in \mathcal{H}(\mathbb{D})$, the expression $N_h \leq N_h^*$, can be satisfied when we have a concerned positive constant C_1 , for which $N_h \leq C_1 N_h^*$. When $N_h^* \leq N_h \leq N_h^*$, the expression $N_h \approx N_h^*$ can be written to say that there is an equivalence relation between the concerned quantities N_h and N_h^* . Furthermore, when $N_h \approx N_h^*$, we deduce that $N_h < \infty \Leftrightarrow N_h^* < \infty$.

2. Toeplitz-Superposition Operators

Let $\mathcal{E}(\mathbb{C})$ denote the set of all entire functions on the complex plane $\mathbb{C}$. For a function $\phi \in \mathcal{E}(\mathbb{C})$, the superposition operator $S_\phi : \mathcal{H}(\mathbb{D}) \rightarrow \mathcal{H}(\mathbb{D})$ is defined by $S_\phi(h) = (\phi \circ h)$. Moreover, if $u \in \mathcal{H}(\mathbb{D})$ and $\phi \in \mathcal{E}(\mathbb{C})$, the weighted superposition oper-

ator $S_{\phi,u} : \mathcal{H}(\mathbb{D}) \rightarrow \mathcal{H}(\mathbb{D})$ is defined by $S_{\phi,u}(h)(z) = u(z)\phi(h(z))$, for all $h \in \mathcal{H}(\mathbb{D})$ and $z \in \mathbb{D}$. Note that, if $u(z) = 1$, then $S_{\phi,u} = S_\phi$, for any $z \in \mathbb{D}$.

For any normed subspace $X \subset \mathcal{H}(\mathbb{D})$, we will consider the set $\mathcal{H}(X)$, defined by

$$\mathcal{H}(X) = \{h \in X : \phi \circ h \in X, \quad \text{where } \phi \in \mathcal{E}(\mathbb{C})\}. \quad (7)$$

Now, we define the Toeplitz-superposition operators acting on $\mathcal{H}(\mathbb{D})$.

Definition 4. Let two functions $\phi \in \mathcal{E}(\mathbb{C})$ and $u \in \mathcal{H}(\mathbb{D})$. Then, the Toeplitz-superposition operators $T_u S_\phi$ on the normed (metric) subspace X are given by

$$T_u S_\phi(h) = T_u(\phi \circ h) = P(u \cdot (\phi \circ h)), \quad \text{for all } h \in \mathcal{H}(X). \quad (8)$$

Let α, β be the scalars if ϕ is a fixed entire function and $u, v \in \mathcal{H}(\mathbb{D})$. Then, from the definition of Toeplitz-superposition operators, we have

$$\begin{aligned} T_{\alpha u + \beta v} S_\phi(h) &= T_{\alpha u + \beta v}(\phi \circ h) = P(\alpha u \cdot (\phi \circ h) + \beta v \cdot (\phi \circ h)) \\ &= \alpha P(u \cdot (\phi \circ h)) + \beta P(v \cdot (\phi \circ h)) \\ &= \alpha T_u S_\phi(h) + \beta T_v S_\phi(h), \end{aligned} \quad (9)$$

which holds for all $h \in \mathcal{H}(X)$, and hence, the Toeplitz-Superposition operators are linear on the normed subspace X .

It can be seen that whenever $u \in \mathcal{H}(\mathbb{D})$, then, the operator $T_u S_\phi$ becomes the operator $S_{\phi,u}$. So, Toeplitz-superposition operators can be taken as an extension of weighted superposition operators. The present paper is interested in answering the following interesting questions.

- (i) Can we transform one holomorphic function space into another by what kinds of entire functions?
- (ii) What are the holomorphic spaces that can be transformed one into another by certain weighted classes of entire functions such as specific analytic polynomials of a certain degree and certain entire-type functions of given type and order?
- (iii) When does the holomorphic function ϕ induces a Toeplitz-superposition operators to form one holomorphic function space into another?

As a concerned result, the obtained results will introduce answers of the above mentioned questions by using the class of Toeplitz-superposition operators that are acting between different classes of Bloch functions.

Also, the answers for some of these concerned questions have been introduced by several authors; the following citations can be stated for interesting and intensive studies [8–20].

3. Boundedness in the case $a \in (0, 1)$ and $b > a$ or $b < a$

Several important discussions on boundedness property of the new operator acting on the analytic Bloch spaces are presented in this concerned section. Furthermore, some essential equivalent characterizations for its boundedness are established too.

Now, we will introduce the main results of boundedness.

Theorem 5. For $a \in (0, 1)$ and $b > a$. Suppose that $u \in H^\infty(\mathbb{D})$ and let $\phi \in \mathcal{E}(\mathbb{C})$, with $\phi \neq 0$. Then, the Toeplitz-superposition operator $T_u S_\phi : \mathcal{B}^a \rightarrow \mathcal{B}^b$ is bounded.

Proof. First, assume that $u \in H^\infty(\mathbb{D})$. Let $h \in \mathcal{B}^b$, since

$$\|h\|_{\mathcal{B}^b} \approx \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1} |h(z)|, \quad (10)$$

we have

$$\begin{aligned} \|T_u S_\phi h\|_{\mathcal{B}^b} &\approx \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1} |T_u S_\phi h| \\ &= \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1} \left| \int_{\mathbb{D}} \frac{u(w)(\phi \circ h)(w)}{(1 - \bar{w}z)^2} dA(w) \right| \\ &\leq \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1} \int_{\mathbb{D}} \frac{|u(w)| |\phi(h(w))|}{|1 - \bar{w}z|^2} dA(w) \\ &\leq \|u\|_{H^\infty} \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1} \int_{\mathbb{D}} \frac{|\phi(h(w))|}{|1 - \bar{w}z|^2} dA(w). \end{aligned} \quad (11)$$

Now, let the constant $R > 0$ where $h \in \mathcal{B}^a$ such that $\|h\|_{\mathcal{B}^a} \leq R$, by Lemma 1, we have $|h(z)| \leq R$. Set $R_1 = \max_{|z|=R} |\phi(z)|$, then $|\phi(h(z))| \leq R_1$. Since $b > a$, we have the fact that $\mathcal{B}^a \subset \mathcal{B}^b$, and since $a \in (0, 1)$, we have that $\mathcal{B}^a \subset H^\infty(\mathbb{D})$. Thus,

$$\begin{aligned} \|T_u S_\phi h\|_{\mathcal{B}^b} &\leq R_1 \|u\|_{H^\infty} \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1} \int_{\mathbb{D}} \frac{dA(w)}{|1 - \bar{w}z|^2} \\ &\leq \|u\|_{H^\infty} < \|u\|_{\mathcal{B}^a} < \infty, \end{aligned} \quad (12)$$

where R, R_1 depended only on a, b , and ϕ . This shows that $T_u S_\phi : \mathcal{B}^a \rightarrow \mathcal{B}^b$ is bounded.

Theorem 6. For $0 < b < a < 1$, let $u \in L^1(\mathbb{D})$ be harmonic and let $\phi \in \mathcal{E}(\mathbb{C})$. Then, the Toeplitz-superposition operator $T_u S_\phi : \mathcal{B}^a \rightarrow \mathcal{B}^b$ is bounded if and only if $u \in H^\infty(\mathbb{D})$ and ϕ is a constant entire function.

Proof. It is trivial that if $u \in H^\infty(\mathbb{D})$ and ϕ is constant, then $T_u S_\phi : \mathcal{B}^a \rightarrow \mathcal{B}^b$ is bounded. If ϕ is constant, not identically 0, and $T_u S_\phi$ maps $\mathcal{B}^a$ into $\mathcal{B}^b$ then it is clear that $u \in H^\infty(\mathbb{D})$. Assume now that $u \neq 0$ and ϕ is not constant, and set $T_u S_\phi$ maps $\mathcal{B}^a$ into $\mathcal{B}^b$. Let h be the constant function defined

by $h(\zeta) = \lambda$, for all $\zeta \in \mathbb{D}$, such that $\phi(\lambda) \neq 0$. Since $h \in \mathcal{B}^a$, it follows that $T_u S_\phi h(\zeta) = T_u \phi(\lambda) \in \mathcal{B}^b$. This implies that $u \in \mathcal{B}^b \subset H^\infty(\mathbb{D})$, since $0 < b < 1$. Finally, since ϕ is not constant, then there is a disk $|w - w_0| < \varepsilon$ and $\delta > 0$, on which $|\phi(w)| > \delta|w|$. Set the test function $h_0(w) = w_0 + r(1 - w)^{1-a} \in \mathcal{B}^a$. Then, for all $w \in \mathbb{D}$, we have

$$\begin{aligned} \|T_u S_\phi h_0\|_{\mathcal{B}^b} &\approx \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1} |T_u S_\phi h_0| \\ &= \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1} \left| \int_{\mathbb{D}} \frac{u(w)(\phi \circ h_0)(w)}{(1 - \bar{w}z)^2} dA(w) \right| \\ &\geq \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1} \int_{\mathbb{D}} \frac{|u(w)|}{|1 - w|^a |1 - \bar{w}z|^2} dA(w). \end{aligned} \quad (13)$$

But, along with the positive radius, we get $|u(w)|/(|1 - w|^a |1 - \bar{w}z|^2) \rightarrow \infty$, as $w \rightarrow 1$. This shows that $T_u S_\phi : \mathcal{B}^a \rightarrow \mathcal{B}^b$ is not bounded.

4. Boundedness in the case $0 < a \leq b$ or $a = 0, b > 0$

Theorem 7. For $a > 0$, let $u \in L^1(\mathbb{D})$ be harmonic and let $\phi \in \mathcal{E}(\mathbb{C})$. Then, $T_u S_\phi$ is bounded on $\mathcal{B}^a$ if and only if $u \in H^\infty(\mathbb{D})$ and ϕ is an affine function (linear function plus a translation).

Proof. First, suppose that $u \in H^\infty(\mathbb{D})$ and ϕ is an affine function. It is easy to explain $T_u S_\phi$ is bounded from $\mathcal{B}^a$ into itself.

On the other hand, assume that $u \in H^\infty(\mathbb{D})$ and $\phi \in \mathcal{E}(\mathbb{C})$ does not linear function. Then, by using the Cauchy estimates for $\phi \in \mathcal{E}(\mathbb{C})$, we can find a sequence $\{w_n\} \subset \mathbb{C}$, for each $n \in \mathbb{N}$ such that $|w_n| \rightarrow \infty$ as $n \rightarrow \infty$ and $|\phi(w_n)| =$

$\max_{|w|=n} |\phi(w)| \geq |w_n|^2$. Also, since the weight $(1 - |\zeta|^2)^a$ is typical, we can find a sequence of points $\{z_n\} \subset \mathbb{D}$ such that $|z_n| \rightarrow 1^-$, with $(0.5 < |z_n| < 1)$ and such that $(1 - |z_n|) |w_n| = 1$, for all $n \in \mathbb{N}$. Now consider the sequence of functions $\{h_n\}$ contained in $\mathcal{B}^a$ satisfies $\|h_n\|_{\mathcal{B}^a} \leq 1$ and $|h_n(z_n)| = |w_n|$. Furthermore, we can suppose that $h_n(z_n) = w_n$. Hence,

$$\begin{aligned} \|T_u S_\phi(h_n)\|_{\mathcal{B}^b} &\approx \sup_{z_n \in \mathbb{D}} (1 - |z_n|^2)^{b-1} |T_u \phi(h_n(z_n))| \\ &\geq \|u\|_{H^\infty} \sup_{z \in \mathbb{D}} (1 - |z_n|^2)^{b-1} \int_{\mathbb{D}} \frac{|w_n|^2}{|1 - \bar{w}z_n|^2} dA(w) \\ &\geq \|u\|_{H^\infty} \sup_{z \in \mathbb{D}} (1 - |z_n|^2)^{b-1} \int_{\mathbb{D}} \frac{|w_n|}{|1 - \bar{w}z|^2} dA(w) \\ &\rightarrow \infty, \text{ as } n \rightarrow \infty. \end{aligned} \quad (14)$$

Because $|w_n| \rightarrow \infty$ as $n \rightarrow \infty$. This shows that $T_u S_\phi : \mathcal{B}^a \rightarrow \mathcal{B}^b$ cannot be bounded if $\phi \in \mathcal{E}(\mathbb{C})$ is not a linear function.

Theorem 8. For $0 < a \leq b$, let $u \in L^1(\mathbb{D})$ be harmonic and let $\phi \in \mathcal{E}(\mathbb{C})$ be an increasing and continuous function. Then, $T_u S_\phi : \mathcal{B}^a \rightarrow \mathcal{B}^b$ is bounded if and only if $u \in H^\infty(\mathbb{D})$, and for each $\lambda \in (0, 1)$, there is a positive constant η whenever $|w| > \eta$, such that

$$|\phi(w)| \leq \phi(\lambda |w|). \quad (15)$$

Proof. First, suppose that $u \in H^\infty(\mathbb{D})$ and (15) is true. Now, consider $R_1 > 0$ and let $h \in \mathcal{B}^a$ satisfy $\|h\|_{\mathcal{B}^a} \leq R_1$ and select $\lambda \in (0, 1)$ such that $\lambda R_1 < 1$. Then, there is $\eta > 0$, such that $|\phi(w)| \leq \phi(\lambda |w|)$, whenever $|w| > \eta$. Thus, since $\bar{D}_R = \{w \in \mathbb{C} : |w| \leq R\}$ is a compact set and $\phi \in \mathcal{E}(\mathbb{C})$ is a continuous function, we can assume that $|\phi(w)| \leq 1$, for all $w \in \bar{D}_R$. Hence,

$$\begin{aligned} \|T_u S_\phi h\|_{\mathcal{B}^b} &\approx \sup_{z_n \in \mathbb{D}} (1 - |z_n|^2)^{b-1} |T_u \phi(h(z))| \\ &= \sup_{\{z \in \mathbb{D}, |h(w)| \leq R\}} (1 - |z|^2)^{b-1} |T_u S_\phi h| \\ &\quad + \sup_{\{z \in \mathbb{D}, |h(w)| > R\}} (1 - |z|^2)^{b-1} |T_u \phi(h(z))| \\ &\leq \|u\|_{H^\infty} \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1} \int_{\mathbb{D}} \frac{dA(w)}{|1 - \bar{w}z|^2} \\ &\quad + \|u\|_{H^\infty} \sup_{\{z \in \mathbb{D}, |h(w)| > R\}} (1 - |z|^2)^{b-1} \int_{\mathbb{D}} \phi \\ &\quad \cdot \left(\lambda \left| \frac{\|h\|_{\mathcal{B}^a}}{(1 - |w|^2)^a} \right| \right) \frac{dA(w)}{|1 - \bar{w}z|^2} \\ &\leq \|u\|_{H^\infty} + \|u\|_{H^\infty} \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1} \int_{\mathbb{D}} \phi \\ &\quad \cdot \left(\lambda \left| \frac{1}{(1 - |w|^2)^a} \right| \right) \frac{dA(w)}{|1 - \bar{w}z|^2}. \end{aligned} \quad (16)$$

Using that the function ϕ is increasing and the fact that $\lambda < 1$, we have

$$\begin{aligned} \|T_u S_\phi h\|_{\mathcal{B}^b} &\leq \|u\|_{H^\infty} + \|u\|_{H^\infty} \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1} \int_{\mathbb{D}} \frac{dA(w)}{|1 - \bar{w}z|^2} \\ &\leq \|u\|_{H^\infty}. \end{aligned} \quad (17)$$

This shows that $T_u S_\phi : \mathcal{B}^a \rightarrow \mathcal{B}^b$ is bounded.

On the other hand, assume that $u \in H^\infty(\mathbb{D})$ and $\phi \in \mathcal{E}(\mathbb{C})$ does not satisfy (15). Then, we can find $\lambda_1 \in (0, 1)$ and a sequence $\{w_n\} \subset \mathbb{C}$ such that $|w_n| \rightarrow \infty$ as $n \rightarrow \infty$ and $|\phi(w_n)| \geq \phi(\lambda_1 |w_n|)$, for all $n \in \mathbb{N}$. Since the weight $(1 - |z|^2)^a$ is typical, we can find a sequence of points $\{z_n\} \subset \mathbb{D}$ such that $|z_n| \rightarrow 1^-$ as $n \rightarrow \infty$. Thus, we can consider a sequence of functions $\{h_n\}$ contained in $\mathcal{B}^a$ satisfies $\|h_n\|_{\mathcal{B}^a} \leq 1$ and $|h_n(z_n)| \leq |w_n|$. Now, let $z \in \mathbb{D}$ and set the function $f_n(z) = w_n h_n(z) / h_n(z_n)$ for all $n \in \mathbb{N}$. Then, we have $f_n(z_n) = w_n$ and $\|f_n\|_{\mathcal{B}^a} \leq 1$. For large enough $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \|T_u S_\phi f_n\|_{\mathcal{B}^b} &\approx \sup_{z_n \in \mathbb{D}} (1 - |z_n|^2)^{b-1} |T_u S_\phi f_n(z_n)| \\ &\geq \sup_{z_n \in \mathbb{D}} (1 - |z_n|^2)^{b-1} \int_{\mathbb{D}} \frac{|u(w)| |\phi(w_n)|}{|1 - \bar{w}z_n|^2} dA(w) \\ &\geq \|u\|_{H^\infty} \sup_{z_n \in \mathbb{D}} (1 - |z_n|^2)^{b-1} \int_{\mathbb{D}} \frac{\phi(\lambda_1 |w_n|)}{|1 - \bar{w}z_n|^2} dA(w) \\ &\geq \|u\|_{H^\infty} \sup_{z_n \in \mathbb{D}} (1 - |z_n|^2)^{b-1} \int_{\mathbb{D}} \phi \left(\left| \frac{1}{(1 - |w_n|^2)^a} \right| \right) \\ &\quad \cdot \frac{dA(w)}{|1 - \bar{w}z_n|^2} \rightarrow \infty, \text{ as } n \rightarrow \infty. \end{aligned} \quad (18)$$

Then, we conclude that $T_u S_\phi : \mathcal{B}^a \rightarrow \mathcal{B}^b$ cannot be bounded.

Theorem 9. For $1 < a \leq b$, let $u \in L^1(\mathbb{D})$ be harmonic and let $\phi \in \mathcal{E}(\mathbb{C})$. Then, the following are equivalent:

- (i) $T_u S_\phi$ maps $\mathcal{B}^a$ into $\mathcal{B}^b$
- (ii) $u \in H^\infty(\mathbb{D})$ and ϕ is a polynomial of degree at most $b - 1/a - 1$
- (iii) $T_u S_\phi : \mathcal{B}^a \rightarrow \mathcal{B}^b$ is bounded

Proof. First, suppose that (i) holds, let $\phi = z^n$ be a polynomial with $n \leq (b - 1)/(a - 1)$, then, we have that $u \in H^\infty(\mathbb{D})$.

Now, suppose that the entire function ϕ is a polynomial of degree $m > b - 1/a - 1$. Then, for an integer n and a positive constant δ , there is a sequence $z_n \rightarrow \infty$ such that $|\phi(z_n)| \geq \delta |z_n|^m$. We may assume without losing generality that $|z_n| > 1$ and $|\arg z_n| < \min\{a\pi/4, \pi/2\}$, for an integer n . Now, we let $h_a(w) = (1 - w)^{1-a} \in \mathcal{B}^a$, then, we show that $T_u S_\phi h_a \notin \mathcal{B}^b$. The point $w_n = 1 - (z_n)^{-1/a}$ such that $|1 - w_n| < 1$ and $|\arg(1 - w_n)| < \pi/4$, and satisfies that $|1 - w_n| \leq (1 - w_n)$, for an integer n . Thus, we have

$$\begin{aligned} &(1 - |z_n|^2)^{b-1} |T_u S_\phi h_a(z_n)| \\ &= (1 - |z_n|^2)^{b-1} \left| \int_{\mathbb{D}} \frac{u(w) \phi(h_a(w))}{(1 - \bar{w}z_n)^2} dA(w) \right| \\ &\geq (1 - |z_n|^2)^{b-1} \int_{\mathbb{D}} \frac{\delta |u(w)|}{|1 - z_n|^{ma} |1 - \bar{w}z_n|^2} dA(w) \\ &\geq \int_{\mathbb{D}} \frac{\delta |u(w)|}{|1 - z_n|^{ma-m-b+1} |1 - w|^a |1 - \bar{w}z_n|^2} dA(w). \end{aligned} \quad (19)$$

Since $ma - m - b + 1 > 0$, then, we obtain

$$(1 - |z_n|^2)^{b-1} |T_u S_\phi h_a(z_n)| \rightarrow \infty, \text{ as } n \rightarrow \infty. \quad (20)$$

This implies that $T_u S_\phi h_a \notin \mathcal{B}^b$. Based on the above it is clear that (i) $\Rightarrow$ (ii).

Second, assume that $u \in H^\infty(\mathbb{D})$ and $\phi = z^n$ are a polynomial of degree $n \leq (b-1)/(a-1)$. For all $h \in \mathcal{B}^a$ and $b-1-na-n \geq 0$, by Lemma 1 and 2, where n is bounded, we have

$$\begin{aligned} \|T_u S_\phi h\|_{\mathcal{B}^b} &\approx \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1} |T_u S_\phi h(z)| \\ &= \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1} \left| \int_{\mathbb{D}} \frac{u(w)(\phi \circ h)(w)}{(1 - \bar{w}z)^2} dA(w) \right| \\ &\leq \|u\|_{H^\infty} \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1} \int_{\mathbb{D}} \frac{|h(w)|^n}{|1 - \bar{w}z|^2} dA(w) \\ &\leq \|u\|_{H^\infty} \|h\|_{\mathcal{B}^a}^n \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1-na-n} \int_{\mathbb{D}} \frac{dA(w)}{|1 - \bar{w}z|^2} \\ &\leq \|u\|_{H^\infty} \|h\|_{\mathcal{B}^a}^n. \end{aligned} \quad (21)$$

This shows that (ii) $\Rightarrow$ (iii). Thus, the proof has been completed.

Theorem 10. For $b > -1$, let $u \in L^1(\mathbb{D})$ be harmonic and let $\phi \in \mathcal{E}(\mathbb{C})$, with order ρ and type τ . Then, the following are equivalent:

- (i) $T_u S_\phi$ maps $\mathcal{B}$ into $\mathcal{B}^b$
- (ii) $u \in H^\infty(\mathbb{D})$ and $\phi \in \mathcal{E}(\mathbb{C})$ with $\rho < 1$ or (with $\rho = 1$ and $\tau = 0$);
- (iii) $T_u S_\phi : \mathcal{B} \rightarrow \mathcal{B}^b$ is bounded

Proof. First, assume that $T_u S_\phi$ maps $\mathcal{B}$ into $\mathcal{B}^b$. Now, we assume on the antithesis that (ii) does not hold. Then, the function $\phi \in \mathcal{E}(\mathbb{C})$ with $\rho > 1$ or (with $\rho = 1$ and $\tau > 0$). Thus, there is a positive constant λ and a sequence $\{w_n\}$ of complex numbers such that $|w_n| \rightarrow \infty$ and

$$|\phi(w_n)| \geq \exp(\lambda |w_n|), \quad \text{for any } n \in \mathbb{N}. \quad (22)$$

Hence, as in the proof of Theorem 8, we can consider the sequence $\{w_n\} \subset \mathbb{D}$ and $\{h_n\} \subset \mathcal{B}$ satisfies $\|h_n\|_{\mathcal{B}} \leq 1$ and $|h_n(z_n)| \leq |w_n|$. Now, let $z \in \mathbb{D}$ and set the function $f_n(z) = w_n h_n(z)/h_n(z_n)$ for all $n \in \mathbb{N}$. Then, we have $f_n(z_n) = w_n$ and $\|f_n\|_{\mathcal{B}} \leq 1$. For large enough, $n \in \mathbb{N}$, since $|w_n| \rightarrow \infty$, we obtain

$$\begin{aligned} \|T_u S_\phi f_n\|_{\mathcal{B}^b} &\approx \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1} |T_u S_\phi f_n(z)| \\ &\geq \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1} \int_{\mathbb{D}} \frac{|u(w)| |\phi(w_n)|}{|1 - \bar{w}z|^2} dA(w) \\ &\geq \|u\|_{H^\infty} \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1} \int_{\mathbb{D}} \frac{\exp(\lambda_1 |w_n|)}{|1 - \bar{w}z|^2} dA(w) \\ &\rightarrow \infty, \text{ as } n \rightarrow \infty. \end{aligned} \quad (23)$$

Then, we conclude that $T_u S_\phi : \mathcal{B} \rightarrow \mathcal{B}^b$ cannot be bounded. Based on the above results, it is clear that (i) $\Rightarrow$ (ii).

Second, set $M(t, \phi) = \max_{|w|=t} |\phi(w)|$, where $t \geq 0$, the order ρ of $\phi \in \mathcal{E}(\mathbb{C})$ is

$$\rho = \limsup_{t \rightarrow \infty} \frac{\log \log M(t, \phi)}{\log t}. \quad (24)$$

If $0 < \rho < \infty$, then the type τ of $\phi \in \mathcal{E}(\mathbb{C})$ is

$$\tau = \limsup_{t \rightarrow \infty} \frac{\log M(t, \phi)}{t^\rho}. \quad (25)$$

For given $\lambda = b/R > 0$, the condition (ii) implies that (see for example [18])

$$|\phi(w)| \leq \exp(\lambda |w|), \quad \text{for any } w \in \mathbb{C}. \quad (26)$$

Moreover, for a function $h \in \mathcal{B}$, with $\|h\|_{\mathcal{B}} \leq 1$, we know that

$$|h(w)| \leq \left(1 + \log \frac{1}{1 - |w|}\right), \quad \text{for } w \in \mathbb{D}. \quad (27)$$

Then,

$$|\phi(h(w))| \leq \exp(\lambda |h(w)|) \leq \left(\frac{e}{1 - |w|}\right)^b \leq (2e)^b. \quad (28)$$

Thus, we have

$$\begin{aligned} \|T_u S_\phi h\|_{\mathcal{B}^b} &\approx \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1} |T_u S_\phi h(z)| \\ &\leq (2e)^b \|u\|_{H^\infty} \sup_{z \in \mathbb{D}} (1 - |z|^2)^{b-1} \int_{\mathbb{D}} \frac{dA(w)}{|1 - \bar{w}z|^2} \\ &\leq (2e)^b \|u\|_{H^\infty} < \infty. \end{aligned} \quad (29)$$

This shows that $T_u S_\phi : \mathcal{B} \rightarrow \mathcal{B}^b$ is bounded. So, (ii) $\Rightarrow$ (iii).

5. Conclusion and Future Study

This manuscript deals with a radical study of a concerned class of Toeplitz superposition operators acting between some certain classes of analytic function spaces of Bloch-type. Global discussions of the boundedness property of the new class of operators are presented class of the univalent Bloch functions. All concerned entire functions which transform a class of holomorphic Bloch-type spaces into another using the so-called Toeplitz superposition operators in terms of their order and type or the degree of polynomials are characterized in this paper. Moreover, all the defined Toeplitz-superposition operators induced by concerned entire functions are cleared to be bounded actually. We have cleared that for two spaces of normed-type which belonging

to $\mathcal{H}(\mathbb{D})$, where $X = \mathcal{B}^a$ and $Y = \mathcal{B}^b$, we can find certain concerned functions ϕ and u , with $\phi \in \mathcal{E}(\mathbb{C})$ and $u \in \mathcal{H}(\mathbb{D})$, for which the newly Toeplitz-superposition operators $T_u S_\phi$ can map $\mathcal{B}^a$ into $\mathcal{B}^b$ for some specific values of a and b . Furthermore, the operator $T_u S_\phi : X \rightarrow Y$ is shown to be actually bounded.

Data Availability

The data is not applicable to this concerned article as no concerned data sets were created or used through this concerned study.

Conflicts of Interest

The authors declare that they have no competing interest.

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