Research Article

Exact Analytical Solutions of Generalized Fifth-Order KdV Equation by the Extended Complex Method

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The recently introduced technique, namely, the extended complex method, is used to explore exact solutions for the generalized fifth-order KdV equation. Appropriately, the rational, periodic, and elliptic function solutions are obtained by this technique. The 3D graphs explain the different physical phenomena to the exact solutions of this equation. This idea specifies that the extended complex method can acquire exact solutions of several differential equations in engineering. These results reveal that the extended complex method can be directly and easily used to solve further higher-order nonlinear partial differential equations (NLPDEs). All computer simulations are constructed by maple packages.

1. Introduction

In the 20 century, nonlinear science (NLS) plays a significant role in special inventions, for example, the invention of the radio, the discovery of DNA structure for biology, the development of quantum theory for theoretical physics and chemistry, and the invention of transistor for computer engineering. It is well known that NLS belongs to the NLPDEs which are introduced in several areas such as fluid thermodynamics, plasma diffusion, biology, physics, geometry, and population dynamics.

Lots of studies are focused on the differential equations [1–10], and many effective techniques are used to acquire analytical and numerical solutions for NLPDEs such as sine-cosine method [11], extended sinh-Gorden equation expansion method [12], variation iteration algorithm [13], homotopy perturbation method [14], F-expansion method [15], Exp-function expansion method [16], first integral method [17], Ansatz method [18], generalized Kudryashov method [19], (G'/G) -expansion method [20], projective Riccati equation method [21], tanh method [22], nonpolynomial spline method [23], B-spline method [24], B-spline collocation

method [25], Weierstrass elliptic function method [26], Laplace decomposition method [27], extended direct algebraic method [28, 29], Sub-ODE method [30], Darboux transformation [31], and extended tanh-coth method [32, 33]. The generalized fifth-order KdV equation [34] is represented by

$$w_t + sww_x + fw^2w_x + ew_{xxx} + \mu w_{xxxx} = 0, \quad (1)$$

where s , f , e , and μ are the arbitrary constants. This equation is a nonlinear model in many long wave physical phenomena. It is used in the shallow water wave with surface tension and magnetoacoustic wave in plasma. Several researchers have explored the analytical solutions of generalized fifth-order KdV equation such as Hedli and Kadem have attained a new analytical solution for the fifth-order KdV equation by the exponential expansion method [35]. Dinarvand et al. have found approximate analytical solutions of the sawada-kotera and Lax's fifth-order KdV equations by homotopy analysis technique [36]. Salas and Lugo have introduced extended tanh method to obtain the exact solutions of the general fifth-order KdV equation [37]. Alam and Xin et al. have attained new exact

solutions by (G'/G) -expansion method of modified KdV-Zakharov-Kuznetsov equation [38]. Ganji and Abdollahzadeh have introduced the sech method and rational expansion method to find the exact traveling wave solutions of the Lax's seventh-order KdV equation [39].

In the present work, our main purpose is to calculate the generalized fifth-order KdV equation by the extended complex method based on the concept of Yuan et al. [40–46]. It is a remarkable approach to attain exact analytical solutions. Our technique would be potentially applied to various processes of the engineering field. This article is organized as mentioned as follows. In Section 2, methods and materials are described. In Section 3, the application of the introduced method is determined. Section 4 deals with physical phenomena of important results. The comparison and conclusions are explained in Section 5.

2. Methods and Materials

Let us consider the general form of NLPDE

$$l(w, w_t, w_x, w_z, w_{tt}, w_{xx}, \dots), \quad (2)$$

where the unknown function is $w = w(x, t)$ and l is a polynomial in $w = w(x, t)$ and its derivatives.

Step 1. A transformation $T : w(x, t) \rightarrow W(z)$ is introduced, and (x, t) can be introduced in different standard; hence, we have used the transformation such as

$$w(x, t) = W(z), z = k(x - \omega t). \quad (3)$$

Step 2. The $w(x, t) = W(z), z = k(x - \omega t)$ transform Eq. (2) into nonlinear ODE:

$$T(W, W', W'', W''', \dots) = 0, \quad (4)$$

in Eq. (4), where W primes are the derivatives w.r.t z . This equation is reduced by further integration.

Step 3. Let the meromorphic solutions W of Eq. (4) have at least one pole, and let us consider $p, q \in \mathbb{Z}$. For this condition, we substitute the Laurent series

$$W(z) = \sum_{k=-q}^{\infty} B_k z^k, q > 0, B_{-q} \neq 0, \quad (5)$$

into Eq. (4), if we can find p distinct Laurent singular parts:

$$\sum_{k=-q}^{-1} B_k z^k, \quad (6)$$

then the weak $\langle p, q \rangle$ condition of Eq. (4) holds. Weierstrass elliptic function $\wp(z) := \wp(z, g_2, g_3)$ with double periods of the equation is given as below:

$$\left(\wp'(z)\right)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \quad (7)$$

and the addition formula is mentioned as below:

$$\wp(z - z_0) = -\wp(z) + \frac{1}{4} \left[\frac{\wp'(z) + \wp'(z_0)}{\wp(z) - \wp(z_0)} \right]^2 - \wp(z_0). \quad (8)$$

Step 4. Putting the indeterminate forms

$$W(z) = \sum_{i=1}^{y-1} \sum_{j=2}^q \frac{(-1)^j \delta_{-ij}}{(j-1)!} \frac{d^{j-2}}{dz^{j-2}} \left(\frac{1}{4} \left[\frac{\wp'(z) + G_i}{\wp(z) - H_i} \right]^2 - \wp(z) \right) + \sum_{i=1}^{y-1} \frac{\delta_{-i1}}{2} \frac{\wp'(z) + G_i}{\wp(z) - H_i} + \sum_{j=2}^q \frac{(-1)^j \delta_{-yj}}{(j-1)!} \frac{d^{j-2}}{dz^{j-2}} \wp(z) + \delta_0, \quad (9)$$

$$W(z) = \sum_{i=1}^y \sum_{j=1}^q \frac{\delta_{ij}}{(z - z_i)^j} + \delta_0, \quad (10)$$

$$W(e^{\alpha z}) = \sum_{i=1}^y \sum_{j=1}^q \frac{\delta_{ij}}{(e^{\alpha z} - e^{\alpha z_i})^j} + \delta_0, \quad (11)$$

into Eq. (4); hence, the number of equations is computed by adjusting the coefficient to zero. These algebraic equations are calculated by the source of maple. Equation (9) is the elliptic solution W with pole at $z = 0$, where δ_{-ij} are attained by (4), $G_i^2 = 4H_i^3 - g_2H_i - g_3$, $\sum_{i=1}^y \delta_{-i1} = 0$. Equation (10) is the rational function, and Eq. (11) is the exponential function which are denoted as $W(z)$, $W(e^{\alpha z})$ ($\alpha \in \mathbb{C}$), and they have y ($\leq p$) distinct poles of multiplicity q .

Step 5. The meromorphic solutions are got with the arbitrary pole. Substitute inverse transformation T^{-1} into meromorphic solutions; then, we obtain the exact analytical solutions of NLPDEs.

3. Application of the Method

In this section, we would like to find the exact analytical solutions of a generalized fifth-order KdV equation by extended complex approach. Substitute

$$w(x, t) = W(z), z = k(x - \omega t), \quad (12)$$

into Eq. (1), then obtain

$$-\omega W' + s W W' + f W^2 W' + ek^2 W''' + \mu k^4 W'''' = 0, \quad (13)$$

now, we integrate Eq. (13) w.r.t z ; then, we attain new ODE

$$-\omega W + s \frac{W^2}{2} + f \frac{W^3}{3} + ek^2 W'' + \mu k^4 W'''' = 0. \quad (14)$$

Putting (5) into (14) then we have $p=1$ and $q=2$; hence, the weak $\langle 1, 2 \rangle$ condition of (14) holds. By weak $\langle 1, 2 \rangle$ and (10), then rational solutions with pole at $z=0$ are

$$W_r(z) = \frac{\delta_{12}}{(z-1)^2} + \frac{\delta_{11}}{z-1} + \delta_{10}, \quad (15)$$

substituting the $W_r(z)$ into Eq. (14); then, we have

$$\frac{1}{6} \sum_{i=1}^7 c_{1i} z^{(7-i)} (z-1)^{-6} = 0, \quad (16)$$

where

$$c_{11} = 2\delta_{10}^3 f + 3\delta_{10}^2 s - 6\delta_{10} \omega,$$

$$c_{12} = 6\delta_{11} \delta_{10}^2 f - 12\delta_{10}^3 f + 6\delta_{11} \delta_{10} s - 18\delta_{10}^2 s - 6\delta_{11} \omega + 36\delta_{10} \omega,$$

$$c_{13} = 6\delta_{12} \delta_{10}^2 f + 6\delta_{11}^2 \delta_{10} f - 30\delta_{11} \delta_{10}^2 f + 30\delta_{10}^3 f + 6\delta_{12} \delta_{10} s + 3\delta_{11}^2 s - 30\delta_{11} \delta_{10} s + 45\delta_{10}^2 s - 6\delta_{12} \omega + 30\delta_{11} \omega - 90\delta_{10} \omega,$$

$$c_{14} = 12\delta_{12} \delta_{11} \delta_{10} f - 24\delta_{12} \delta_{10}^2 f + 2\delta_{11}^3 f - 24\delta_{11}^2 \delta_{10} f + 60\delta_{11} \delta_{10}^2 f + 12\delta_{11}^2 e k^2 - 40\delta_{10}^3 f + 6\delta_{12} \delta_{11} s - 24\delta_{12} \delta_{10} s - 12\delta_{11}^2 s + 60\delta_{11} \delta_{10} s - 60\delta_{12}^2 s + 24\delta_{12} \omega - 60\delta_{11} \omega + 120\delta_{10} \omega,$$

$$c_{15} = 6\delta_{12}^2 \delta_{10} f + 6\delta_{12} \delta_{11}^2 f - 36\delta_{12} \delta_{11} \delta_{10} f + 36\delta_{12} \delta_{10}^2 f + 36\delta_{12} e k^2 - 6\delta_{11}^3 f + 36\delta_{11}^2 \delta_{10} f - 60\delta_{11} \delta_{10}^2 f - 36\delta_{11} e k^2 + 30\delta_{10}^3 f + 3\delta_{12}^2 s - 18\delta_{12} \delta_{11} s + 36\delta_{12} \delta_{10} s + 18\delta_{11}^2 s - 60\delta_{11} \delta_{10} s + 45\delta_{10}^2 s - 36\delta_{12} \omega + 60\delta_{11} \omega - 90\delta_{10} \omega,$$

$$c_{16} = 144\delta_{11} k^4 \mu + 6\delta_{12}^2 \delta_{11} f - 12\delta_{12}^2 \delta_{10} f - 12\delta_{12} \delta_{11}^2 f + 36\delta_{12} \delta_{11} \delta_{10} f - 24\delta_{12} \delta_{10}^2 f - 72\delta_{12} e k^2 + 6\delta_{11}^3 f - 24\delta_{11}^2 \delta_{10} f + 30\delta_{11} \delta_{10}^2 f + 36\delta_{11} e k^2 - 12\delta_{10}^3 f - 6\delta_{12}^2 s + 18\delta_{12} \delta_{11} s - 24\delta_{12} \delta_{10} s - 12\delta_{11}^2 s + 30\delta_{11} \delta_{10} s - 18\delta_{10}^2 s + 24\delta_{12} \omega - 30\delta_{11} \omega + 36\delta_{10} \omega,$$

$$c_{17} = 720\delta_{12} k^4 \mu - 144\delta_{11} k^4 \mu + 2\delta_{12}^3 f - 6\delta_{12}^2 \delta_{11} f + 6\delta_{12}^2 \delta_{10} f + 6\delta_{12} \delta_{11}^2 f - 12\delta_{12} \delta_{11} \delta_{10} f + 6\delta_{12} \delta_{10}^2 f + 36\delta_{12} e k^2 - 2\delta_{11}^3 f + 6\delta_{11}^2 \delta_{10} f - 6\delta_{11} \delta_{10}^2 f - 12\delta_{11} e k^2 + 2\delta_{10}^3 f + 3\delta_{12}^2 s - 6\delta_{12} \delta_{11} s + 6\delta_{12} \delta_{10} s + 3\delta_{11}^2 s - 6\delta_{11} \delta_{10} s + 3\delta_{10}^2 s - 6\delta_{12} \omega + 6\delta_{11} \omega - 6\delta_{10} \omega. \quad (17)$$

By assuming that the coefficients of same powers concerning z in Eq. (16) are zero, then we have numbers of equations:

$$c_{1i} = 0, (i = 1, 2, \dots, 7). \quad (18)$$

By solving number of these equations, we obtain

$$\delta_{12} = -\frac{6\sqrt{10}\sqrt{\mu}k^2}{\sqrt{f}}, \delta_{11} = 0, \delta_{10} = 0, \quad (19)$$

then

$$W_{r10}(z) = -\frac{6\sqrt{10}\sqrt{\mu}k^2/\sqrt{f}}{(z-1)^2}, \quad (20)$$

where $\omega = 0$ and $s = -(1/5)(\sqrt{f}e\sqrt{10}/\sqrt{\mu})$;

$$\delta_{12} = \frac{24k^2 e}{s}, \delta_{11} = 0, \delta_{10} = \frac{6e^2}{5s\mu}, \quad (21)$$

then

$$W_{r20}(z) = \frac{24k^2 e/s}{(z-1)^2} + \frac{6e^2}{5s\mu}, \quad (22)$$

where $f = -(5/8)(\mu s^2/e^2)$ and $\omega = (3/10)(e^2/\mu)$.

$W(z) = R(\eta)$ is a rational function of $\eta = e^{\alpha z} (\alpha \in \mathbb{C})$, applying it into Eq. (14) then

$$-\omega R + s \frac{R^2}{2} + f \frac{R^3}{3} + k^2 e \alpha^2 (R'' \eta^2 + R' \eta) + k^4 \alpha^4 \mu (R^{(4)} \eta^4 + 6R''' \eta^3 + 7R'' \eta^2 + R' \eta) = 0, \quad (23)$$

substituting

$$W_s(z) = \frac{\delta_{12}}{(\eta-1)^2} + \frac{\delta_{11}}{\eta-1} + \delta_{10}, \quad (24)$$

into the Eq. (23), we attain that

$$\frac{1}{6} \sum_{i=1}^7 \frac{c_{2i} \alpha^2 \eta^{7-i}}{(\eta-1)^6} = 0, \quad (25)$$

where

$$c_{21} = 2\delta_{10}^3 f + 3\delta_{10}^2 s - 6\delta_{10} \omega,$$

$$c_{22} = 6\alpha^4\delta_{11}k^4\mu + 6\alpha^2\delta_{11}ek^2 + 6\delta_{11}\delta_{10}^2f - 12\delta_{10}^3f + 6\delta_{11}\delta_{10}s - 18\delta_{10}^2s - 6\delta_{11}\omega + 36\delta_{10}\omega,$$

$$c_{23} = 96\delta_{12}\alpha^4k^4\mu + 60\alpha^4\delta_{11}k^4\mu + 24\delta_{12}\alpha^2ek^2 - 12\alpha^2\delta_{11}ek^2 + 6\delta_{12}\delta_{10}^2f + 6\delta_{11}\delta_{10}^2f - 30\delta_{11}\delta_{10}^2f + 30\delta_{10}^3f + 6\delta_{12}\delta_{10}s + 3\delta_{11}^2s - 30\delta_{11}\delta_{10}s + 45\delta_{10}^2s - 6\delta_{12}\omega + 30\delta_{11}\omega - 90\delta_{10}\omega,$$

$$c_{24} = 396\delta_{12}\alpha^4k^4\mu - 36\delta_{12}\alpha^2ek^2 + 12\delta_{12}\delta_{11}\delta_{10}f - 24\delta_{12}\delta_{10}^2f + 2\delta_{11}^3f - 24\delta_{11}^2\delta_{10}f + 60\delta_{11}\delta_{10}^2f - 40\delta_{10}^3f + 6\delta_{12}\delta_{11}s - 24\delta_{12}\delta_{10}s - 12\delta_{11}^2s + 60\delta_{11}\delta_{10}s - 60\delta_{10}^2s + 24\delta_{12}\omega - 60\delta_{11}\omega + 120\delta_{10}\omega,$$

$$c_{25} = 2\delta_{12}^3f - 6\delta_{12}^2\delta_{11}f + 6\delta_{12}^2\delta_{10}f + 6\delta_{12}\delta_{11}^2f - 12\delta_{12}\delta_{11}\delta_{10}f + 6\delta_{12}\delta_{10}^2f - 2\delta_{11}^3f + 6\delta_{11}^2\delta_{10}f - 6\delta_{11}\delta_{10}^2f + 2\delta_{10}^3f + 3\delta_{12}^2s - 6\delta_{12}\delta_{11}s + 6\delta_{12}\delta_{10}s + 3\delta_{11}^2s - 6\delta_{11}\delta_{10}s + 3\delta_{10}^2s - 6\delta_{12}\omega + 6\delta_{11}\omega - 6\delta_{10}\omega,$$

$$c_{26} = 216\delta_{12}\alpha^4k^4\mu - 60\alpha^4\delta_{11}k^4\mu + 12\alpha^2\delta_{11}ek^2 + 6\delta_{12}^2\delta_{10}f + 6\delta_{12}\delta_{11}^2f - 36\delta_{12}\delta_{11}\delta_{10}f + 36\delta_{12}\delta_{10}^2f - 6\delta_{11}^3f + 36\delta_{11}^2\delta_{10}f - 60\delta_{11}\delta_{10}^2f + 30\delta_{10}^3f + 3\delta_{12}^2s - 18\delta_{12}\delta_{11}s + 36\delta_{12}\delta_{10}s + 18\delta_{11}^2s - 60\delta_{11}\delta_{10}s + 45\delta_{10}^2s - 36\delta_{12}\omega + 60\delta_{11}\omega - 90\delta_{10}\omega,$$

$$c_{27} = 12\delta_{12}\alpha^4k^4\mu - 6\alpha^4\delta_{11}k^4\mu + 12\delta_{12}\alpha^2ek^2 - 6\alpha^2\delta_{11}ek^2 + 6\delta_{12}^2\delta_{11}f - 12\delta_{12}^2\delta_{10}f - 12\delta_{12}\delta_{11}^2f + 36\delta_{12}\delta_{11}\delta_{10}f - 24\delta_{12}\delta_{10}^2f + 6\delta_{11}^3f - 24\delta_{11}^2\delta_{10}f + 30\delta_{11}\delta_{10}^2f - 12\delta_{10}^3f - 6\delta_{12}^2s + 18\delta_{12}\delta_{11}s - 24\delta_{12}\delta_{10}s - 12\delta_{11}^2s + 30\delta_{11}\delta_{10}s - 18\delta_{10}^2s + 24\delta_{12}\omega - 30\delta_{11}\omega + 36\delta_{10}\omega. \quad (26)$$

By assuming that the coefficients of the same powers concerning η in Eq. (25) are zero, then obtain the numbers of equations:

$$c_{2i} = 0, (i = 1, 2, \dots, 7). \quad (27)$$

Solve the numbers of these equations, then attain

$$\delta_{12} = -\frac{6\sqrt{10}\sqrt{\mu}k^2\alpha^2}{\sqrt{f}}, \delta_{11} = -\frac{6\sqrt{10}\sqrt{\mu}k^2\alpha^2}{\sqrt{f}}, \delta_{10} = 0, \quad (28)$$

where $\eta = e^{\alpha z} (\alpha \in \mathbb{C})$.

$$W_s(e^{\alpha z}) = -\frac{6\sqrt{10}\sqrt{\mu}k^2\alpha^2/\sqrt{f}}{(e^{\alpha z} - 1)^2} - \frac{6\sqrt{10}\sqrt{\mu}k^2\alpha^2/\sqrt{f}}{e^{\alpha z} - 1},$$

$$W_s(e^{\alpha z}) = -\frac{(6\sqrt{10}\sqrt{\mu}k^2\alpha^2/\sqrt{f})e^{\alpha z}}{(e^{\alpha z} - 1)^2}, \quad (29)$$

so, we obtain the simply periodic solutions of Eq. (14) with pole at $z = 0$

$$W_{s10}(z) = -\frac{3\sqrt{10}\sqrt{\mu}k^2\alpha^2}{2\sqrt{f}} \left(\coth^2 \frac{\alpha}{2} z - 1 \right), \quad (30)$$

where $\omega = \alpha^4k^4\mu + \alpha^2ek^2$ and $s = (1/5)((5\alpha^2k^2\mu + e)\sqrt{10}\sqrt{f}/\sqrt{\mu})$. Furthermore,

$$\delta_{12} = -\frac{12\alpha^2ek^2}{s}, \delta_{11} = -\frac{12\alpha^2ek^2}{s}, \delta_{10} = -\frac{2\alpha^2ek^2}{s}, \quad (31)$$

where $\eta = e^{\alpha z} (\alpha \in \mathbb{C})$.

$$W_s(e^{\alpha z}) = -\frac{12\alpha^2ek^2/s}{(e^{\alpha z} - 1)^2} - \frac{12\alpha^2ek^2/s}{e^{\alpha z} - 1} - \frac{2\alpha^2ek^2}{s},$$

$$W_s(e^{\alpha z}) = -\frac{(12\alpha^2ek^2/s)e^{\alpha z}}{(e^{\alpha z} - 1)^2} - \frac{2\alpha^2ek^2}{s}, \quad (32)$$

so, we attain again the simply periodic solutions of Eq. (14) with pole at $z = 0$

$$W_{s20}(z) = -\frac{3\alpha^2ek^2}{s} \left(\coth^2 \frac{\alpha}{2} z \right) - \frac{5\alpha^2ek^2}{s}, \quad (33)$$

where $\omega = -\alpha^2ek^2$ and $f = 0$.

By the weak $\langle 1, 2 \rangle$ condition, so, we introduce here the elliptic solutions by (9) with $z = 0$ pole.

$$W_{d0}(z) = \delta_{12}\wp(z) + \delta_{10}, \quad (34)$$

substitute $W_{d0}(z)$ into Eq. (14); then, we have

$$\sum_{i=0}^3 c_{3i}\wp^i(z) = 0, \quad (35)$$

FIGURE 1: Perspective view of 3D graph of $W_{r,1}(z)$ for the fixed values $\omega = 2$, $z_0 = 0.5$, $\mu = 1$, $k = 1$, and $f = 1$ represents the exact solutions.

FIGURE 2: Perspective view of 3D graph of $W_{r,1}(z)$ for the fixed values $\omega = 2$, $z_0 = -0.5$, $\mu = 1$, $k = 1$, and $f = 1$ represents the exact solution.

where

$$\begin{aligned} c_{30} &= -\delta_{10}\omega + \frac{1}{2}\delta_{10}^2s + \frac{1}{3}\delta_{10}^3f - \frac{1}{2}ek^2ag_3 - 12\delta_{12}g_3k^4\mu, \\ c_{31} &= -18\delta_{12}g_2k^4\mu + \delta_{12}\delta_{10}^2f + \delta_{12}\delta_{10}s - \delta_{10}\omega, \\ c_{32} &= \frac{1}{2}\delta_{12}^2s + \delta_{12}^2\delta_{10}f + 6ek^2\delta_{12}, \\ c_{33} &= \frac{1}{3}\delta_{12}^3f + 120\delta_{12}\mu k^4. \end{aligned} \quad (36)$$

By assuming that the coefficients of the same powers concerning $\wp(z)$ in Eq. (35) are zero, then obtain the num-

FIGURE 3: Perspective view of 3D graph of $W_{s,1}(z)$ for the fixed values $\omega = 9$, $z_0 = 1/6$, $\mu = 1$, $k = 1$, $f = 1$, and $\alpha = 1$ represents the exact solutions.

FIGURE 4: Perspective view of 3D graph of $W_{s,1}(z)$ for the fixed values $\omega = 9$, $z_0 = -1/6$, $\mu = 1$, $k = 1$, $f = 1$, and $\alpha = 1$ represents the exact solutions.

bers of equations:

$$c_{3i} = 0, \quad (i = 0, 1, \dots, 3). \quad (37)$$

Solve these equation; then, we have

$$\delta_{12} = -\frac{6\sqrt{10}\sqrt{\mu}k^2}{\sqrt{f}}, \quad \delta_{10} = 0, \quad (38)$$

FIGURE 5: Perspective view of 3D graph of $W_{r,2}(z)$ for the fixed values $\omega = 5$, $z_0 = 1/3$, $\mu = 1$, $k = 1$, $s = 1$, and $e = 1$ represents exact solutions.

FIGURE 6: Perspective view of 3D graph of $W_{r,2}(z)$ for the fixed values $\omega = 5$, $z_0 = -1/3$, $\mu = 1$, $k = 1$, $s = 1$, and $e = 1$ represents exact solutions.

and then we have

$$W_{d0}(z) = -\frac{6\sqrt{10}\sqrt{\mu}k^2}{\sqrt{f}}\wp(z), \quad (39)$$

hence, the elliptic general solutions of Eq. (14) with arbitrary pole are expressed such as

$$W_{d,1}(z) = -\frac{6\sqrt{10}\sqrt{\mu}k^2}{\sqrt{f}}\wp(z - z_0), \quad (40)$$

where $z_0 \in \mathbb{C}$.

FIGURE 7: Perspective view of 3D graph of $W_{s,2}(z)$ for the fixed values $\omega = 6$, $z_0 = 1/8$, $\mu = 1$, $k = 1$, $s = 1$, $e = 1$, and $\alpha = 1$ represents exact solutions.

FIGURE 8: Perspective view of 3D graph of $W_{s,2}(z)$ for the fixed values $\omega = 6$, $z_0 = -1/8$, $\mu = 1$, $k = 1$, $s = 1$, $e = 1$, and $\alpha = 1$ represents exact solutions.

Applying the additional formula to the $W_{d,1}(z)$, and we attain

$$W_{d,1}(z) = -\frac{6\sqrt{10}\sqrt{\mu}k^2}{\sqrt{f}}\left(-\wp(z) + \frac{1}{4}\left(\wp'(z) + G_1\right)^2\right) + \frac{6\sqrt{10}\sqrt{\mu}k^2}{\sqrt{f}}H_1, \quad (41)$$

where $e = -24k^2\mu$, $s = -(24/5)k^2\sqrt{\mu}\sqrt{10}\sqrt{f}$, $\omega = -18g_2k^4\mu$, $G_1 = 4H_1^3 - g_2H_1 - g_3$, and g_2 and g_3 are the arbitrary constants.

By the above approach, so, we obtain the meromorphic solutions of Eq. (14) with arbitrary pole as mention as follows:

$$W_{r,1}(z) = -\frac{6\sqrt{10}\sqrt{\mu}k^2/\sqrt{f}}{(z-z_0-1)^2}, \quad (42)$$

where $\omega = 0$, $s = -(1/5)(\sqrt{f}e\sqrt{10}/\sqrt{\mu})$, and $z_0 \in \mathbb{C}$.

$$W_{r,2}(z) = \frac{24k^2e/s}{(z-z_0-1)^2} + \frac{6e^2}{5s\mu}, \quad (43)$$

where $f = -(5/8)(\mu s^2/e^2)$, $\omega = (3/10)(e^2/\mu)$, and $z_0 \in \mathbb{C}$.

$$W_{s,1}(z) = -\frac{3\sqrt{10}\sqrt{\mu}k^2\alpha^2}{2\sqrt{f}} \left(\coth^2 \frac{\alpha}{2}(z-z_0) - 1 \right), \quad (44)$$

where $\omega = \alpha^4 k^4 \mu + \alpha^2 e k^2$, $s = (1/5)((5\alpha^2 k^2 \mu + e)\sqrt{10}\sqrt{f}/\sqrt{\mu})$, and $z_0 \in \mathbb{C}$.

$$W_{s,2}(z) = -\frac{3\alpha^2 e k^2}{s} \left(\coth^2 \frac{\alpha}{2}(z-z_0) \right) - \frac{5\alpha^2 e k^2}{s}, \quad (45)$$

where $\omega = -\alpha^2 e k^2$, $f = 0$, and $z_0 \in \mathbb{C}$.

4. Description about Figures

Here, we display the exact solutions for $W_{r,1}(z)$, $W_{s,1}(z)$, $W_{r,2}(z)$, and $W_{s,2}(z)$ by graphical phenomena as in Figures 1–8. These graphs are represented by the source of maple to persuade important results. Figures 1–8 display different multisolitary wave solutions that are obtained by different values of z_0 and ω , whereas other parameters are constant.

Figures 1 and 2 indicate the exact solutions for $W_{r,1}(z)$, adjust the values $\omega = 2$, $z_0 = 0.5$, $\mu = 1$, $k = 1$, and $f = 1$ and $\omega = 2$, $z_0 = -0.5$, $\mu = 1$, $k = 1$, and $f = 1$.

Figures 3 and 4 indicate the exact solutions for $W_{s,1}(z)$, adjust the values $\omega = 9$, $z_0 = 1/6$, $\mu = 1$, $k = 1$, $f = 1$, and $\alpha = 1$ and $\omega = 9$, $z_0 = -1/6$, $\mu = 1$, $k = 1$, $f = 1$, and $\alpha = 1$.

Figures 5 and 6 indicate the exact solutions for $W_{r,2}(z)$, adjust the values $\omega = 5$, $z_0 = 1/3$, $\mu = 1$, $k = 1$, $s = 1$, and $e = 1$ and $\omega = 5$, $z_0 = -1/3$, $\mu = 1$, $k = 1$, $s = 1$, and $e = 1$.

Figures 7 and 8 indicate the exact solutions for $W_{s,2}(z)$, adjust the values $\omega = 6$, $z_0 = 1/8$, $\mu = 1$, $k = 1$, $s = 1$, $e = 1$, and $\alpha = 1$ and $\omega = 6$, $z_0 = -1/8$, $\mu = 1$, $k = 1$, $s = 1$, $e = 1$, and $\alpha = 1$.

5. Comparison and Conclusion

Khan et al. [25] represented the modified simple equation technique for the analytical treatment of generalized fifth-order KdV equation. This proposed technique provides fresh exact solutions in the area of engineering and mathematical physics. The results demonstrated the remarkable exact solutions for this technique. For this purpose, we create the com-

parison between the modified simple equation technique and the extended complex approach.

We employed the extended complex technique to explore the exact analytical solutions of the generalized fifth-order KdV equation. The graphical phenomena are showed by setting the values of arbitrary parameters, and the graphical representations are revealed the mechanism of wave behavior, for example, Figures 1–8 depict that different multisolitary wave solutions are attained by different values of z_0 and ω , whereas other parameters are constant. The extended complex approach is calculated by the source of maple software. This approach is a powerful analytical technique since it provides different new exact solutions which are indicated by the forms of rational, periodic, and elliptic function solutions. These results have been obtained by the extended complex technique to show a deeper understanding of physical structures and provide remarkable exact solutions of higher degree NPDEs.

Data Availability

The data used to support the finding of this study are mentioned in the article.

Conflicts of Interest

The authors mentioned here that they have no conflict of interests.

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