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Research Article

Nontrivial Solutions for 4-Superlinear Schrödinger–Kirchhoff Equations with Indefinite Potentials

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This paper is devoted to the 4-superlinear Schrödinger–Kirchhoff equation $-(a+b\int_{\mathbb{R}^3}|\nabla u|^2 \mathrm{d}x)\Delta u + V(x)u = f(x,u)$, in $\mathbb{R}^3$, where $a>0$, $b\geq 0$. The potential V here is indefinite so that the Schrödinger operator $-\Delta + V$ possesses a finite-dimensional negative space. By using the Morse theory, we obtain nontrivial solutions for this problem.

1. Introduction and Main Results

In this work, we consider the Schrödinger–Kirchhoff type equation of the form

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2 \mathrm{d}x\right)\Delta u + V(x)u = f(x,u), x\in\mathbb{R}^3, \quad (1)$$

where $a>0$, $b\geq 0$ are constants. This equation arises when we look for stationary solutions of the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0 \quad (2)$$

proposed by Kirchhoff [1] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings, where ρ, h, P_0, and L are positive constants. In [2], J.L. Lions introduced an abstract functional analysis framework to the equation

$$u_{tt} - \left(a+b\int_\Omega|\nabla u|^2 \mathrm{d}x\right)\Delta u = f(x,u), \quad (3)$$

where $\Omega\subset\mathbb{R}^3$ is a bounded domain. After that, (3) received much attention. For more details on the physical and mathematical background of this problem, we refer to [1–3] for references.

Problem (1) has been studied extensively by many researchers. Some interesting studies by variational methods can be found in, for example, [3–13] and references therein. Under suitable conditions, it is well known that weak solutions to (1) correspond to critical points of the energy functional $\Phi : H^1(\mathbb{R}^3)\to\mathbb{R}$,

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla u|^2 + V(x)u^2\right)\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2 \mathrm{d}x\right)^2 - \int_{\mathbb{R}^3}F(x,u)\mathrm{d}x, \quad (4)$$

where $F(x,t)=\int_0^t f(x,s)\mathrm{d}s$. We emphasize that in the papers mentioned above, the authors only considered the case that the Schrödinger operator $-\Delta + V$ is positively definite. In this situation, a typical way to deal with (1) is to use the mountain pass theorem (cf. [14]). However, when the potential V is negative somewhere, the zero function $u=0$ is no longer a local minimizer of Φ. In this case, the functional Φ would not enjoy the general linking geometry due to the nonlocal term $\int_{\mathbb{R}^3}|\nabla u|^2 \mathrm{d}x$. Hence, the classical linking theorem (see, e.g., ([15], Theorem 2.12)) is also not applicable.

Such an indefinite situation was studied in [16]. To overcome these difficulties and the difficulty that the Sobolev embedding $H^1(\mathbb{R}^3)^\circ L^2(\mathbb{R}^3)$ is not compact, it is assumed in

[16] that

$$\mu\left(V^{-1}(-\infty,M]\right) < \infty \text{for all } M > 0, \, (V_0), \tag{5}$$

so that the related weighted Sobolev space is compactly embedded into $L^2(\mathbb{R}^3)$. Then, via Morse theory, they obtained nontrivial critical points of Φ.

In this paper, we will consider the case of more general V such that the abovementioned compact embedding may not be true. We assume the potential V satisfies.

(V) $V \in C(\mathbb{R}^3)$ is bounded such that the quadratic form

$$Q(u) := \int_{\mathbb{R}^3} \left(a|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x, \tag{6}$$

which is nondegenerate and the negative space of Q is finite-dimensional.

For the nonlinearity f, we make the following assumptions:

(f_1) $f \in C(\mathbb{R}^3 \times \mathbb{R})$ there exist $C > 0$ and $p \in (4,6)$ such that

$$|f(x,t)| \le C\left(1 + |t|^{p-1}\right) \text{for all } x \in \mathbb{R}^3 ; \tag{7}$$

(f_2) $f(x,t) = o(|t|)$ as $t \to 0$ uniformly in $x \in \mathbb{R}^3$ and

$$\lim_{t\to\infty} \frac{F(x,t)}{t^4} = +\infty \text{for all } x \in \mathbb{R}^3 ; \tag{8}$$

(f_3) $0 < 4F(x,t) \le tf(x,t)$ for $x \in \mathbb{R}^3$ and $t \ne 0$;

(f_4) for any $r > 0$, there holds

$$\lim_{|x|\to\infty} \sup_{0<|t|\le r} \left|\frac{f(x,t)}{t}\right| = 0. \tag{9}$$

Remark 1.

(i) (V) is a generic assumption on V. For example, if $V : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\lim_{|x|\to\infty} V(x) = V_\infty \in (0,\infty), \tag{10}$$

and 0 does not happen to be a spectrum point of the Schrödinger operator $-\Delta + V$, then such V satisfies our assumption (V).

(ii) Note that in (f_2), we have not required that the limit (8) holds uniformly

(iii) In order to produce critical points of Φ, eventually, we will encounter the compactness problem. For this issue, we assume assumption (f_4). It is easily see that let $a : \mathbb{R}^3 \to (0,\infty)$ be continuous, $\lim_{|x|\to\infty} a(x) = 0$ and $p \in (4,6)$, then

$$f(x,t) = a(x)|t|^{p-2}t \tag{11}$$

satisfies (f_1)–(f_4).

Now, we are ready to state our main results.

Theorem 2. *Assume that (V) and (f_1)–(f_4) are satisfied. Then, problem ((1)) has a nontrivial solution.*

Theorem 3. *Assume that (V) and (f_1)–(f_4) are satisfied. Moreover, if $f(x,\cdot)$ is odd, then problem ((1)) has a sequence of solutions $\{u_n\}$ such that the energy $\Phi(u_n) \to +\infty$.*

It is known that if the quadratic form Q is indefinite, usually, it is more difficult to verify the boundedness of the (PS) sequence. In [16], this is done by taking advantage of the compact embedding mentioned before. Under our present setting, the related Sobolev embedding $H^1(\mathbb{R}^3)^\circ L^2(\mathbb{R}^3)$ is not compact. We will illustrate a general technique to establish the boundedness of (PS) sequences. Moreover, it is also worth to point out that the weak limit of the bounded (PS) sequence is not obviously a critical point of Φ. Since we cannot easily see that Φ' is weakly sequentially continuous in $H^1(\mathbb{R}^3)$ by direct calculations due to the existence of nonlocal term $\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x$. Indeed, in general, we do not know $\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x \to \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x$ from $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$.

The paper is organized as follows. In the next section, we prove that the (PS) sequences of Φ are bounded and Φ satisfies the (PS) condition. In Section 3, we recall some concepts and results in infinite-dimensional Morse theory; then, we analyze the critical groups of Φ at infinity; finally, we give the proof of Theorem 11. Having established the (PS) condition, the proof of Theorem 3 is quite similar to that of ([16], Theorem 3); therefore, we omit it here.

2. Palais-Smale Condition

Throughout of this paper, we always denote $E = H^1(\mathbb{R}^3)$. In view of assumption (V), we may choose an equivalent norm $\|\cdot\|$ on E such that

$$Q(u) = \|u^+\|^2 - \|u^-\|^2, \, u = u^+ + u^-, \, u^\pm \in E^\pm, \tag{12}$$

where E^+ and E^- are positive and negative spaces of Q, respectively, and $E = E^+ \oplus E^-$. Here and in what follows, $u^\pm$ denotes the orthogonal projection of u on $E^\pm$. Then, the functional Φ can be rewritten as

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^2 - \int_{\mathbb{R}^3} F(x,u)\mathrm{d}x. \tag{13}$$

By (f_1), Φ is of class C^1 on E with the derivative given by

$$\left\langle \Phi'(u), v\right\rangle = \int_{\mathbb{R}^3} (a\nabla u \cdot \nabla v + V(x)uv)\mathrm{d}x + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x \int_{\mathbb{R}^3} \nabla u \cdot \nabla v \mathrm{d}x - \int_{\mathbb{R}^3} f(x,u)v\mathrm{d}x, \tag{14}$$

for all $u, v \in E$. Then, u solves (1) if and only if it is a critical point of Φ.

Lemma 4. *Let $\{u_n\}$ be a (PS) sequence of Φ , that is,*

$$\sup_n |\Phi(u_n)| < \infty, \Phi'(u_n) \to 0. \tag{15}$$

Then, $\{u_n\}$ is bounded in E.

Proof. Suppose by contradiction that $\|u_n\| \to \infty$. Let $v_n = \|u_n\|^{-1} u_n$. Then, passing to a subsequence, there exists $v \in E$ such that

$$v_n = v_n^+ + v_n^- \rightharpoonup v = v^+ + v^- \text{ in } E. \tag{16}$$

If $v = 0$, then $v_n^- \to v^-$ since $\dim E^- < \infty$. Noting that $\|v_n^+\|^2 + \|v_n^-\|^2 = 1$, we have

$$\|v_n^+\|^2 - \|v_n^-\|^2 \geq \frac{1}{2}. \tag{17}$$

By (f_3) we deduce that, for n large enough,

$$\begin{aligned} 1 + \sup_n |\Phi(u_n)| + \|u_n\| &\geq \Phi(u_n) - \frac{1}{4}\left\langle \Phi'(u_n), u_n \right\rangle \\ &= \frac{1}{4}\left(\|u_n^+\|^2 - \|u_n^-\|^2\right) + \int_{\mathbb{R}^3} \left(\frac{1}{4} f(x, u_n) u_n - F(x, u_n)\right) dx \\ &\geq \frac{1}{4}\|u_n\|^2 \left(\|v_n^+\|^2 - \|v_n^-\|^2\right) \geq \frac{1}{8}\|u_n\|^2, \end{aligned} \tag{18}$$

contradicting $\|u_n\| \to \infty$.

Now suppose that $v \neq 0$. Then, the set $\Theta = \{x \in \mathbb{R}^3 : v(x) \neq 0\}$ has positive Lebesgue measure. For $x \in \Theta$, we have $|u_n(x)| \to \infty$, and hence (8) implies

$$\frac{F(x, u_n(x))}{u_n^4(x)} v_n^4(x) \to +\infty. \tag{19}$$

Then, the Fatou lemma yields

$$\int_\Theta \frac{F(x, u_n)}{u_n^4} v_n^4 dx \to +\infty. \tag{20}$$

On the other hand, for large n,

$$\begin{aligned} \int_\Theta \frac{F(x, u_n)}{u_n^4} v_n^4 dx &= \frac{1}{\|u_n\|^4}\int_\Theta F(x, u_n) dx \leq \frac{1}{\|u_n\|^4}\int_{\mathbb{R}^3} F(x, u_n) dx \\ &= \frac{1}{\|u_n\|^4}\left[\frac{1}{2}\left(\|u_n^+\|^2 - \|u_n^-\|^2\right) + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2\right] \\ &\leq 1 + \frac{b}{4}\frac{|\nabla u_n|_2^4}{\|u_n\|^4} \leq 1 + M, \end{aligned} \tag{21}$$

where $M > 0$ is a constant and $|\cdot|_s$ denotes the standard norm in $L^s(\mathbb{R}^3)$. This is also a contradiction.

In conclusion, we deduce that the (PS) sequence $\{u_n\}$ is bounded.

To get a convergent subsequence of the (PS) sequence, we need the following lemma.

Lemma 5. *Let $u_n \rightharpoonup u$ in E . Then,*

$$\underline{\lim}_{n\to\infty} \int_{\mathbb{R}^3} \nabla u \cdot \nabla (u_n - u) dx \geq 0. \tag{22}$$

Proof. Since $u_n \rightharpoonup u$ in E, we have $\nabla u_n \rightharpoonup \nabla u$ in $L^2(\mathbb{R}^3)$. Hence,

$$\int_{\mathbb{R}^3} \nabla u_n \cdot \nabla u dx \to \int_{\mathbb{R}^3} |\nabla u|^2 dx. \tag{23}$$

Define $\psi(u) = \int_{\mathbb{R}^3} |\nabla u|^2 dx$ in E. It is easy to see that ψ is continuous and convex. Hence, ψ is weakly lower semicontinuous in E, so that

$$\underline{\lim}_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \geq \int_{\mathbb{R}^3} |\nabla u|^2 dx. \tag{24}$$

Consequently,

$$\begin{aligned} \underline{\lim}_{n\to\infty} \int_{\mathbb{R}^3} \nabla u \cdot \nabla (u_n - u) dx &= \underline{\lim}_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx - \lim_{n\to\infty} \int_{\mathbb{R}^3} \nabla u_n \cdot \nabla u dx \\ &\geq \int_{\mathbb{R}^3} |\nabla u|^2 dx - \int_{\mathbb{R}^3} |\nabla u|^2 dx = 0. \end{aligned} \tag{25}$$

The lemma is proved.

Lemma 6. *Φ satisfies the (PS) condition.*

Proof. Let $\{u_n\}$ be a (PS) sequence. From Lemma 4, we know that $\{u_n\}$ is bounded in E. Up to a subsequence, we may assume that $u_n \rightharpoonup u$ in E. Hence,

$$\begin{aligned} \int_{\mathbb{R}^3} (a\nabla u_n \cdot \nabla u + V(x) u_n u) dx &\to \int_{\mathbb{R}^3} \left(a|\nabla u|^2 + V(x) u^2\right) dx \\ &= \|u^+\|^2 - \|u^-\|^2. \end{aligned} \tag{26}$$

Consequently,

$$\begin{aligned} o(1) = \left\langle \Phi'(u_n), u_n - u \right\rangle &= \int_{\mathbb{R}^3} (a\nabla u_n \cdot \nabla (u_n - u) + V(x) u_n (u_n - u)) dx + b\int_{\mathbb{R}^3} \\ &\quad \cdot |\nabla u_n|^2 dx \int_{\mathbb{R}^3} \nabla u_n \cdot \nabla (u_n - u) dx - \int_{\mathbb{R}^3} f(x, u_n)(u_n - u) dx \\ &= \int_{\mathbb{R}^3} \left(a|\nabla u_n|^2 + V(x) u_n^2\right) dx - \int_{\mathbb{R}^3} (a\nabla u_n \cdot \nabla u + V(x) u_n u) dx + b\int_{\mathbb{R}^3} \\ &\quad \cdot |\nabla u_n|^2 dx \int_{\mathbb{R}^3} \nabla u_n \cdot \nabla (u_n - u) dx - \int_{\mathbb{R}^3} f(x, u_n)(u_n - u) dx \\ &= o(1) + \left(\|u_n^+\|^2 - \|u^+\|^2\right) - \left(\|u_n^-\|^2 - \|u^-\|^2\right) + b\int_{\mathbb{R}^3} \\ &\quad \cdot |\nabla u_n|^2 dx \int_{\mathbb{R}^3} \nabla u_n \cdot \nabla (u_n - u) dx - \int_{\mathbb{R}^3} f(x, u_n)(u_n - u) dx. \end{aligned} \tag{27}$$

Because $\dim E^- < \infty$, we have $u_n^- \to u^-$ and thus $\|u_n^-\| \to \|u^-\|$. Collecting all infinitesimal terms, we obtain

$$\|u_n^+\|^2 - \|u^+\|^2 = o(1) + \int_{\mathbb{R}^3} f(x,u_n)(u_n-u)\mathrm{d}x - b\int_{\mathbb{R}^3} |\nabla u_n|^2\mathrm{d}x \int_{\mathbb{R}^3} \nabla u_n \cdot \nabla(u_n-u)\mathrm{d}x. \tag{28}$$

We claim that

$$\overline{\lim_{n\to\infty}} \int_{\mathbb{R}^3} f(x,u_n)(u_n-u)\mathrm{d}x \le 0. \tag{29}$$

Given $\varepsilon > 0$. For $r \ge 1$, by (f_1) and Hölder's inequality, we have

$$\begin{aligned} \int_{|u_n|\ge r} f(x,u_n)(u_n^+-u^+)\mathrm{d}x &\le 2C\int_{|u_n|\ge r} |u_n|^{p-1}|u_n^+-u^+|\mathrm{d}x \\ &\le 2Cr^{p-6}\int_{|u_n|\ge r} |u_n|^5|u_n^+-u^+|\mathrm{d}x \\ &\le 2Cr^{p-6}|u_n|_6^5|u_n^+-u^+|_6. \end{aligned} \tag{30}$$

Since $p < 6$, we may fix r large enough such that,

$$\int_{|u_n|\ge r} f(x,u_n)(u_n^+-u^+)\mathrm{d}x \le \frac{\varepsilon}{3}, \tag{31}$$

for all n. Moreover, it follows from (f_4) that there exists $R > 0$ such that

$$\int_{\substack{|x|\ge R \\ |u_n|\le r}} f(x,u_n)(u_n^+-u^+)\mathrm{d}x \le |u_n|_2|u_n^+-u^+|_2 \sup_{|t|\le r,|x|\ge R}\left|\frac{f(x,t)}{t}\right| \le \frac{\varepsilon}{3}, \tag{32}$$

for all n. Finally, from (f_1) and (f_2) we deduce that, for any $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$\begin{aligned} |F(x,t)| &\le \varepsilon t^2 + c_\varepsilon |t|^p, \\ |f(x,t)| &\le \varepsilon|t| + c_\varepsilon|t|^{p-1}. \end{aligned} \tag{33}$$

Note that $u_n^+ \to u^+$ in $L^s(B_R(\mathbf{0}))$ for every $s \in 2, 2^*)$. Consequently, for n large enough,

$$\begin{aligned} \int_{\substack{|x|\le R \\ |u_n|\le r}} f(x,u_n)(u_n^+-u^+)\mathrm{d}x &\le \int_{\substack{|x|\le R \\ |u_n|\le r}} |u_n||u_n^+-u^+|\mathrm{d}x + c_1\int_{\substack{|x|\le R \\ |u_n|\le r}} |u_n|^{p-1}|u_n^+-u^+|\mathrm{d}x \\ &\le |u_n|_2|u_n^+-u^+|_{L^2(B_R(\mathbf{0}))} + c_1|u_n|_p^{p-1}|u_n^+-u^+|_{L^p(B_R(\mathbf{0}))} \le \frac{\varepsilon}{3}. \end{aligned} \tag{34}$$

Combining (31), (32), and (34), we obtain that

$$\int_{\mathbb{R}^3} f(x,u_n)(u_n^+-u^+)\mathrm{d}x \le \varepsilon, \tag{35}$$

for n large enough. Since $\dim E^- < \infty$, we obtain that (29) holds by the arbitrariness of ε.

Now using Lemma 5, we deduce from (28) that

$$\begin{aligned} \overline{\lim_{n\to\infty}}\left(\|u_n^+\|^2 - \|u^+\|^2\right) &= \overline{\lim_{n\to\infty}}\left(\int_{\mathbb{R}^3} f(x,u_n)(u_n-u)\mathrm{d}x - b\int_{\mathbb{R}^3} \cdot |\nabla u_n|^2\mathrm{d}x \int_{\mathbb{R}^3}\nabla u_n \cdot \nabla(u_n-u)\mathrm{d}x\right) \\ &= \overline{\lim_{n\to\infty}}\int_{\mathbb{R}^3} f(x,u_n)(u_n-u)\mathrm{d}x \\ &\quad - \underline{\lim_{n\to\infty}}\, b\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x\int_{\mathbb{R}^3}\nabla u_n \cdot \nabla(u_n-u)\mathrm{d}x \\ &\le \overline{\lim_{n\to\infty}}\int_{\mathbb{R}^3} f(x,u_n)(u_n-u)\mathrm{d}x \le 0. \end{aligned} \tag{36}$$

Combining this with the weakly lower semicontinuous of the norm, we have

$$\|u^+\|^2 \le \underline{\lim_{n\to\infty}}\|u_n^+\|^2 \le \overline{\lim_{n\to\infty}}\|u_n^+\|^2 \le \|u^+\|^2. \tag{37}$$

That is $\|u_n^+\| \to \|u^+\|$. Reminding $\|u_n^-\| \to \|u^-\|$, we get $\|u_n\| \to \|u\|$. Thus, $u_n \to u$ in E.

3. Critical Groups and the proof of Theorem 11

Before giving the proof of Theorem 11, we recall some concepts and results of infinite dimensional Morse theory [17].

Let X be a Banach space, $\varphi : X \to \mathbb{R}$ be a C^1 functional, u be an isolated critical point of φ and $\varphi(u) = c$. Then,

$$C_q(\varphi,u) := H_q(\varphi_c, \varphi_c \setminus \{0\}), q = 0,1,2,\cdots, \tag{38}$$

is called the qth critical group of φ at u, where $\varphi_c := \varphi^{-1}(-\infty,c]$ and H_* stands for the singular homology with coefficients in $\mathbb{Z}$.

If φ satisfies the (PS) condition and the critical values of φ are bounded from below by α, then following Bartsch and Li [18], we call

$$C_q(\varphi,\infty) := H_q(X,\varphi_\alpha), q = 0,1,2,\cdots \tag{39}$$

the qth critical group of φ at infinity. It is well known that the homology on the right hand does not depend on the choice of α.

Proposition 7 (see [18]). *If $\varphi \in C^1(X,\mathbb{R})$ satisfies the (PS) condition and $C_k(\varphi,0) \ne C_k(\varphi,\infty)$ for some $k \in \mathbb{N}$, then φ has a nonzero critical point.*

Proposition 8 (see [19]). *Suppose $\varphi \in C^1(X,\mathbb{R})$ has a local linking at 0 , i.e., $X = Y \oplus Z$ and*

$$\begin{cases} \varphi(u) \le 0 & \text{for } u \in Y \cap B_\rho, \\ \varphi(u) \ge 0 & \text{for } u \in (Z\setminus\{0\}) \cap B_\rho, \end{cases} \tag{40}$$

for some $\rho > 0$, where $B_\rho = \{u \in X : \|u\| \le \rho\}$. If $k = \dim Y < \infty$, then $C_k(\varphi, 0) \neq 0$.

For the proof of Theorem 11, we may assume the Φ has only finitely many critical points. Since Φ satisfies the (PS) condition, the critical group $C_*(\Phi,\infty)$ of Φ at infinity makes sense. To study $C_*(\Phi,\infty)$, we need the following lemma.

Before state it, we point out that the proof of the following lemma is quite different and more general ([16], Lemma 2.4), because in our case, the working Sobolev space is $H^1(\mathbb{R}^3)$, which can not compactly embedded into $L^2(\mathbb{R}^3)$.

Lemma 9. *There exists $A > 0$ such that, if $\Phi(u) \le -A$, then*

$$\left.\frac{d}{dt}\right|_{t=1} \Phi(tu) < 0. \tag{41}$$

Proof. Otherwise, there exists a sequence $\{u_n\} \subset E$ such that $\Phi(u_n) \le -n$ but

$$\left\langle \Phi'(u_n), u_n \right\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} \Phi(tu) \ge 0. \tag{42}$$

Consequently,

$$\begin{aligned} \|u_n^+\|^2 - \|u_n^-\|^2 &\le \left(\|u_n^+\|^2 - \|u_n^-\|^2\right) + \int_{\mathbb{R}^3} [f(x,u_n)u_n - 4F(x,u_n)]\mathrm{d}x \\ &= 4\Phi(u_n) - \left\langle \Phi'(u_n), u_n \right\rangle \le -4n. \end{aligned} \tag{43}$$

Let $v_n = \|u_n\|^{-1} u_n$ and $v_n^\pm$ be the orthogonal projection of v_n on $E^\pm$. Then, $v_n^- \to v^-$ for some $v^- \in E^-$, because $\dim E^- < \infty$.

If $v^- \neq 0$, then for some $v \in E \setminus \{0\}$ we have $v_n \rightharpoonup v$ in E. Similar to (20), we obtain

$$\int_{\mathbb{R}^3} \frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x \ge 4\int_{\mathbb{R}^3} \frac{F(x,u_n)}{\|u_n\|^4}\mathrm{d}x \ge 4\int_{v\neq 0} \frac{F(x,u_n)}{u_n^4} v_n^4 \mathrm{d}x \to +\infty. \tag{44}$$

Hence, by (42), we get

$$\begin{aligned} 0 &\le \frac{\left\langle \Phi'(u_n), u_n \right\rangle}{\|u_n\|^4} = \frac{\|u_n^+\|^2 - \|u_n^-\|^2}{\|u_n\|^4} + \frac{b|\nabla u_n|_2^4}{\|u_n\|^4} - \int_{\mathbb{R}^3} \frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x \\ &\le 1 + M - \int_{\mathbb{R}^3} \frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x \to -\infty, \end{aligned} \tag{45}$$

a contradiction. Therefore, $v^- = 0$. From

$$\|v_n^+\|^2 + \|v_n^-\|^2 = 1, \tag{46}$$

we see that $\|v_n^+\| \to 1$. Consequently, for n large enough,

$$\|u_n^+\| = \|u_n\|\|v_n^+\| \ge \|u\|\|v_n^-\| = \|u_n^-\|, \tag{47}$$

violating (43). Hence, the desired result is proved.

Lemma 10. *$C_q(\Phi,\infty) = 0$ for all $q = 0, 1, 2, \cdots$.*

Proof. Let $B = \{v \in E : \|v\| \le 1\}$, $S = \partial B$ be the unit sphere in E, and $A > 0$ be the number given in Lemma 9. Without loss of generality, we may assume that

$$-A < \inf_{\|u\|\le 2} \Phi(u). \tag{48}$$

Using (8), it is clear that for any $v \in S$,

$$\Phi(sv) \to -\infty, \text{as } s \to +\infty. \tag{49}$$

So there is $s_v > 0$ such that $\Phi(s_v v) = -A$. Set $u = s_v v$, then a direct computation and Lemma 9 gives

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=s_v} \Phi(sv) = \frac{1}{s_v}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} \Phi(tu) < 0. \tag{50}$$

By the implicit function theorem, $T : v \mapsto s_v$ is a continuous function on S. Using the function T, as in [20–22], we can construct a strong deformation retract $\eta : E \setminus B \to \Phi_{-A}$,

$$\eta(u) = \begin{cases} u, & \text{if } \Phi(u) \le -A, \\ T\left(\dfrac{u}{\|u\|}\right)\dfrac{u}{\|u\|}, & \text{if } \Phi(u) > -A \end{cases}, \tag{51}$$

and deduce

$$C_q(\Phi,\infty) = H_q(E, \Phi_{-A}) \cong H_q(E, E\setminus B) = 0, q = 0, 1, 2, \cdots. \tag{52}$$

Now, we are ready to prove our main result.

Proof of Theorem 11. It follows from (V) and (f_2) that as $\|u\| \to 0$,

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) + o\left(\|u\|^2\right). \tag{53}$$

Hence, there exists $\rho > 0$ such that Φ is positive in $(E^+ \setminus \{0\}) \cap B_\rho$ and negative in $(E^- \setminus \{0\}) \cap B_\rho$, that is, Φ has a local linking with respect to the decomposition $E = E^+ \oplus E^-$. Therefore, Proposition 8 yields

$$C_k(\Phi, 0) \neq 0, \tag{54}$$

where $k = \dim E^-$. By Lemma 10, $C_k(\Phi,\infty) = 0$. Applying Proposition 7, we obtain that Φ has a nonzero critical point. The proof of Theorem 11 is completed.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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