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Research Article

On Best Approximations in Hyperconvex Spaces

Reny George,[1,2] **Zoran D. Mitrović**,[3] **and Hassen Aydi**[4,5,6]

[1]*Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia*
[2]*Department of Mathematics and Computer Science, St. Thomas College, Bhilai, Chhattisgarh, India*
[3]*University of Banja Luka, Faculty of Electrical Engineering, 78000 Banja Luka, Patre 5, Bosnia and Herzegovina*
[4]*Institut Supérieur d'Informatique et des Techniques de Communication, Université de Sousse, H. Sousse, Tunisia*
[5]*Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa*
[6]*China Medical University Hospital, China Medical University, Taichung 40402, Taiwan*

Correspondence should be addressed to Reny George; renygeorge02@yahoo.com and Hassen Aydi; hassen.aydi@isima.rnu.tn

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In this manuscript, we present further extensions of the best approximation theorem in hyperconvex spaces obtained by Khamsi.

1. Introduction

The importance of fixed point theory emerges from the fact that it gives a unified approach and constitutes an essential tool in resolving problems which are not necessarily linear. A variant number of problems can be expressed as nonlinear equations of the form $f(u)=u$, where f is a self-mapping, see [1–6]. Nevertheless, an equation of the type $f(u)=u$ does not necessarily have a solution if f is a non-self-mapping. Let (X,d) be a metric space. Here, we search an optimal solution in the sense that $d(u,f(u))$ is minimum. That is, we resolve a problem of searching an element $u\in X$ so that u is in best proximity to f in some sense. A best proximity point result presents the condition under which the optimisation problem, i.e., $\inf_{u\in A} d(u,f(u))$, possesses a solution. The element u is called the best proximity point of $f:A\longrightarrow B$ if $d(u,f(u))=d(A,B)=\inf\{d(a,b),a\in A,b\in B\}$. Observe that the best proximity point is reduced to a fixed point if f is a self-mapping. For more related works, see [7–11].

The concept of a hyperconvex space was initiated in [12] by Aronszajn and Panitchpakdi. In hyperconvex spaces, many results on coincidence points, fixed points, best approximations, and coupled best approximations are obtained. See, for example, [13–23]. For more details on the best approximation and KKM principle, we refer readers to the classic book [24]. Due to Aronszajn and Panitchpakdi [12], the definition of a hyperconvex metric space is as follows.

A metric space (Λ,ω) is named to be a hyperconvex space if for any set of points $\{\ell_\alpha\}$ of Λ and for any family of nonnegative real numbers $\{r_\alpha\}$ with $\omega(\ell_\alpha,\ell_\beta)\le r_\alpha+r_\beta$, we have $\cap_\alpha B(\ell_\alpha,r_\alpha)\neq\varnothing$, where $B(\ell,r)=\{j\in\Lambda:\omega(\ell,j)\le r\}$ represents the closed ball with center $\ell\in\Lambda$ and radius r.

Suppose that a subset A of Λ is bounded. Consider,

$$\begin{aligned} co(A) &= \cap\{B\subseteq\Lambda : B \text{ is a closed ball so that } A\subseteq B\},\\ \mathscr{A}(\ell) &= \{A\subseteq\Lambda : A=co(A)\}, \end{aligned} \tag{1}$$

i.e., $A\in\mathscr{A}(\ell)$ iff A is an intersection of closed balls. Here, A is named to be an admissible subset of Λ. In the linear case, the notation $\operatorname{conv}(A)$ describes the convex hull of A. Note that $co(A)$ is always defined and is in $\mathscr{A}(\ell)$. If (Λ,ω) is a hyperconvex space, then it is complete [17].

Let Λ be a nonempty set. We denote by $\langle\Lambda\rangle$ and 2^Λ the set of all nonempty finite subsets of Λ and the set of all nonempty subsets of Λ, respectively. Let Λ and Ω be topological spaces with $A\subseteq\Lambda$ and $B\subseteq\Omega$. Given a set-valued map $F:\Lambda\longrightarrow 2^\Omega$, the image of A under F is the set $F(A)=\cup_{a\in A}F(a)$ and the inverse image of B under F is $F^-(B)=\{\ell\in\Lambda : F(\ell)\cap B\neq\varnothing\}$. The map F is lower (upper) semicontinuous if, for each open (closed) set $B\subseteq\Omega$,

$F^-(B)$ is open (closed) set in Λ. The map F is continuous if F is both upper semicontinuous and lower semicontinuous.

Let A be an admissible subset of Λ. The set-valued map $F : A \longrightarrow 2^\Lambda$ is named to be quasiconvex if for any admissible set A of Λ, $F^-(A)$ is also admissible (see [15]). Observe that if F is a quasiconvex map, then the set $F^-(B(\ell, r))$ is admissible for each closed ball $B(\ell, r)$. Note that, if $A \in \mathscr{A}(\ell)$, then $A + r \in \mathscr{A}(\ell)$ (see [25]), where $A + r = \cup_{a\in A} B(a, r)$.

Khamsi [17] presented a hyperconvex version of the KKM principle in hyperconvex spaces. As an application, he gave a hyperconvex version of the best approximation result of Fan for continuous single-valued maps. In this manuscript, we ensure the existence of a solution of a best approximation problem for set-valued maps F and G: for a set K, find $\ell_0 \in K$ so that

$$\omega(G(\ell_0), F(\ell_0)) \le \omega(G(\ell), F(\ell_0)) \text{ for all } \ell \in K. \quad (2)$$

Let Λ be a metric space and $K \subset \Lambda$. A multivalued map $H : K \longrightarrow 2^\Lambda$ is said to be a KKM map if

$$co(A) \subset H(A) \text{ for any } A \in \langle K \rangle. \quad (3)$$

Theorem 1 (see [17], KKM principle). *Let Λ be a hyperconvex space, K be an arbitrary subset of Λ, and $H : K \longrightarrow 2^\Lambda$ be a KKM map so that $H(\ell_0)$ is compact for some $\ell_0 \in K$ and $H(\ell)$ is closed for any $\ell \in K$. Then, $\bigcap_{\ell\in K} H(\ell) \neq \varnothing$.*

Theorem 2 (see [17], best approximation). *Let Λ be a hyperconvex space and $K \in \mathscr{A}(\ell)$ be compact. Given a continuous map $f : K \longrightarrow \Lambda$, there is $\ell_0 \in K$ so that*

$$\omega(\ell_0, f(\ell_0)) = \inf_{\ell\in K} \omega(\ell, f(\ell_0)). \quad (4)$$

This result has been generalized to other forms of maps. For more details, see [13–16, 18, 22].

Now, we give the definition of a measure of noncompactness of Pasicki [26].

Definition 3 (see [26]). Let Λ be a metric space. An arbitrary function $\theta : 2^\Lambda \longrightarrow [0,\infty]$ is named to be a measure of noncompactness on Λ if

(1) $\theta(A) = 0$ iff A is a totally bounded set

(2) for $A, B \in 2^\Lambda A \subseteq B$, implies $\theta(A) \le \theta(B)$

(3) for all $A \subseteq \Lambda$ and $\ell \in \Lambda$, $\theta(A \cup \{\ell\}) = \theta(A)$

Definition 4 (see [19]). Let Λ be a metric space, θ be a measure of noncompactness on Λ, and $K \subset \Lambda$. The map $H : K \longrightarrow 2^\Lambda$ is condensing if for any $\varepsilon > 0$, there is $A \in \langle K \rangle$ so that $\theta(\bigcap_{a\in A} H(a)) < \varepsilon$. A condensing map $H : K \longrightarrow 2^\Lambda$ is a condensing KKM map if it is a KKM map.

In this paper, we present further extensions of the best approximation result (Theorem 2) obtained by Khamsi. Finally, we present a problem related to the Schauder conjecture.

2. Results

The following result generalizes Theorem 1. The proof is essentially the same as Theorem 3.1 in [19].

Theorem 5. *Let θ be a measure of noncompactness on Λ a hyperconvex space, K be an arbitrary subset of Λ, and $H : K \longrightarrow 2^\Lambda$ be a condensing KKM map such that each $H(\ell)$ is closed, then $\bigcap_{\ell\in K} H(\ell)$ is nonempty and compact set.*

We introduce the concept of a φ-quasiconvex map in hyperconvex spaces.

Definition 6. Let Λ be a hyperconvex space and $K \in \mathscr{A}(\ell)$. A set-valued map $G : K \longrightarrow 2^\Lambda$ is said to be a φ-quasiconvex if for any $\ell \in K$ and $r > 0$,

$$coG^-(B(\ell, r)) \subseteq G^-(B(\ell, \varphi(r))), \quad (5)$$

where $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is a continuous monotone increasing function so that $\varphi(u) \ge u$ for any $u \ge 0$ and $\varphi(0) = 0$.

Let $i_\omega : [0,\infty) \longrightarrow [0,\infty)$ be the identity map and $I_\omega : K \longrightarrow 2^K$ be the identity set-valued map so that $I_\omega(\ell) = \{\ell\}$ for any $\ell \in K$. Note that a quasiconvex map is i_ω-quasiconvex and I_ω is i_ω-quasiconvex in hyperconvex spaces.

If (Λ, ω) is a linear metric space, then I_ω may not be i_ω-quasiconvex.

Example 1. Denote by S the linear space of real sequences. The Fréchet metric ω_F for S is given as follows (see [27]):

Let $\ell = (\ell_1, \ell_2, \cdots, \ell_n, \cdots)$, $j = (j_1, j_2, \cdots, j_n, \cdots)$, and $0 = (0, 0, \cdots, 0, \cdots)$,

$$d_F(\ell, j) = \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{|\ell_n - j_n|}{1 + |\ell_n - j_n|}, \quad (6)$$

then, we obtain

$$\text{conv}B\left(0, \frac{1}{3}\right) \subseteq B\left(0, \frac{1}{3}\right). \quad (7)$$

Namely, for $\ell = (1, 0, 1, 0, \cdots)$ and $j = (1/2, 1/2, \cdots)$, we have that $\ell \in B(0, 1/3)$, $j \in B(0, 1/3)$, and $(\ell + j)/2 \notin B(0, 1/3)$.

Note that for any $\ell \in S$ and $r > 0$, one writes

$$\text{conv}B(\ell, r) \subseteq B(\ell, 2r). \quad (8)$$

So, in the linear metric space (S, ω_F), the map I_ω is not i_ω-quasi-convex and I_ω is φ-quasiconvex, where $\varphi(u) = 2u$, $u \in [0,\infty)$.

Theorem 7. *Let (Λ, ω) be a hyperconvex space, $K \in \mathscr{A}(\ell)$ be closed, θ be a measure of noncompactness on Λ, $F, G : K$*

$\longrightarrow \mathscr{A}(\ell)$ be continuous maps with compact values and G be φ -quasi-convex. If for any $u > 0$ there is $j \in K$ so that

$$\theta(\{\ell \in K : \omega(G(\ell), F(\ell)) \le \varphi(\omega(G(j), F(\ell)))\}) \le u, \tag{9}$$

then, there is $\ell_0 \in K$ so that

$$\omega(G(\ell_0), F(\ell_0)) \le \inf_{\ell \in K} \varphi(\omega(G(\ell), F(\ell_0))). \tag{10}$$

Proof. Define $H : K \longrightarrow 2^{\Lambda}$ by

$$H(j) = \{\ell \in K : \omega(G(\ell), F(\ell)) \le \varphi(\omega(G(j), F(\ell)))\}. \tag{11}$$

From condition $\varphi(t) \ge t$, we obtain

$$\omega(G(j), F(j)) \le \varphi(\omega(G(j), F(j))) \text{for all } j \in K, \tag{12}$$

so $H(j)$ is a nonempty set for all $j \in K$, because $j \in H(j)$.

From condition (9), one asserts that H is a condensing map.

Since F and G are continuous maps and φ is a continuous function, we get that $H(j)$ is a closed set for all $j \in K$.

The map $j \longmapsto H(j)$ is KKM. Indeed, suppose that for some $A \in \langle K \rangle$,

$$co(A) \nsubseteq H(A). \tag{13}$$

Then, there is $j \in co(A)$ so that $j \notin H(a)$ for any $a \in A$. Thus,

$$\omega(G(j), F(j)) > \varphi(\omega(G(a), F(j))) \text{for all } a \in A. \tag{14}$$

The function φ is increasing, so

$$\varphi^{-1}(\omega(G(j), F(j))) > \omega(G(a), F(j)) \text{ for all } a \in A. \tag{15}$$

Let $\varepsilon > 0$ be so that

$$\varphi^{-1}(\omega(G(j), F(j))) - \varepsilon \ge \omega(G(a), F(j)) \text{ for all } a \in A. \tag{16}$$

Then,

$$A \subseteq G^{-}\left(F(j) + \varphi^{-1}(\omega(G(j), F(j))) - \varepsilon\right), \tag{17}$$

and hence,

$$co(A) \subseteq co\left(G^{-}\left(F(j) + \varphi^{-1}(\omega(G(j), F(j))) - \varepsilon\right)\right). \tag{18}$$

Since G is φ-quasiconvex, one gets from (18),

$$co(A) \subseteq G^{-}\left(F(j) + \varphi\left(\varphi^{-1}(\omega(G(j), F(j))) - \varepsilon\right)\right). \tag{19}$$

Since $j \in co(A)$, we deduce that

$$\omega(G(j), F(j)) \le \varphi\left(\varphi^{-1}(\omega(G(j), F(j))) - \varepsilon\right). \tag{20}$$

Consequently,

$$\varphi^{-1}(\omega(G(j), F(j)))) \le \varphi^{-1}(\omega(G(j), F(j))) - \varepsilon, \tag{21}$$

which is not possible. Therefore, H must be a KKM map. Now, form Theorem 5, there is $\ell_0 \in K$ so that

$$\ell_0 \in \bigcap_{\ell \in K} H(\ell). \tag{22}$$

Therefore,

$$\omega(G(\ell_0), F(\ell_0)) \le \inf_{\ell \in K} \varphi(\omega(G(\ell), F(\ell_0))). \tag{23}$$

Taking the set K to be compact, we state from Theorem 7 the next results.

Theorem 8. *Let (Λ, ω) be a hyperconvex space, $K \in \mathscr{A}(\ell)$ be compact, $F, G : K \longrightarrow \mathscr{A}(\ell)$ be continuous maps with compact values, and G be φ-quasiconvex. Then, there is $\ell_0 \in K$ so that*

$$\omega(G(\ell_0), F(\ell_0)) \le \inf_{\ell \in K} \varphi(\omega(G(\ell), F(\ell_0))). \tag{24}$$

Theorem 9. *Let (Λ, ω) be a hyperconvex space, $K \in \mathscr{A}(\ell)$ be compact, $F, G : K \longrightarrow \mathscr{A}(K)$ be continuous maps with compact values, and G be φ-quasiconvex onto a map. Then, there is $\ell_0 \in K$ so that $G(\ell_0) \cap F(\ell_0) \neq \varnothing$.*

Theorem 10. *Let (Λ, ω) be a hyperconvex space, $K \in \mathscr{A}(\ell)$ be compact, and $F, G : K \longrightarrow \mathscr{A}(\ell)$ be continuous maps with compact values. If there is $\lambda \ge 1$ such that for any $\ell \in X$ and $r > 0$,*

$$coG^{-}(B(\ell, r)) \subseteq G^{-}(B(\ell, \lambda r)), \tag{25}$$

then there is $\ell_0 \in K$ so that

$$\omega(G(\ell_0), F(\ell_0)) \le \inf_{\ell \in K} \lambda \omega(G(\ell), F(\ell_0)). \tag{26}$$

Theorem 11. *Let (Λ, ω) be a hyperconvex space, $K \in \mathscr{A}(\ell)$ be compact, and $F : K \longrightarrow \mathscr{A}(\ell)$ be a continuous map with compact values so that $F(\ell) \cap K \neq \varnothing$ for any $\ell \in K$. Then, there is $\ell_0 \in K$ so that $\ell_0 \in F(\ell_0)$.*

Remark 12. If F is a single-valued map, we deduce from Theorem 11 the main theorem of Park [22] (Theorem 5. (iv)).

Theorem 13 (see [18], Theorem 3.2). *Let (Λ, ω) be a hyperconvex space, $K \in \mathscr{A}(\ell)$ be compact, $F, G : K \longrightarrow \mathscr{A}(\ell)$ be continuous maps with compact values, and G be quasiconvex. Then, there is $\ell_0 \in K$ so that*

$$\omega(G(\ell_0), F(\ell_0)) \le \inf_{\ell \in K} \omega(G(\ell), F(\ell_0)). \tag{27}$$

Theorem 14 (see [16], Theorem 2.9). *Let (Λ, ω) be a hyperconvex space, $K \in \mathscr{A}(\ell)$ be compact, and $F : K \longrightarrow \mathscr{A}(\ell)$ be* *a continuous map with compact values. Then, there is $\ell_0 \in K$ so that*

$$\omega(\ell_0, F(\ell_0)) = \inf_{\ell \in K} \omega(\ell, F(\ell_0)). \tag{28}$$

Finally, we give the following problems.

Problem 15. Does for every linear metric space (Λ, ω) and for a compact subset K of Λ, there is a map φ so that the identity map $I_\omega : K \longrightarrow 2^K$ (i.e., $I_\omega(\ell) = \{\ell\}, \ell \in K$) is φ-quasiconvex?

In other words, does for a linear metric space, there is a continuous monotone increasing function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ so that $\varphi(u) \geq u$ for any $u \geq 0$, $\varphi(0) = 0$ and

$$\text{conv} B(\ell, r) \subseteq B(\ell, \varphi(r)), \tag{29}$$

for any $\ell \in K$ and $r > 0$?

In 2001, Cauty [28] obtained the affirmative solution to the Schauder conjecture as follows:

Problem 16. Let K be a compact convex subset of a (metrizable) topological vector space. Does any continuous map $f : K \longrightarrow K$ have a fixed point?

Remark 17. Note that if Problem 15 is affirmative, then Problem 16 is affirmative.

Data Availability

Data sharing is not applicable to this article as no data set was generated or analyzed during the current study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors have read and agreed to the published version of the manuscript.

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