

Research Article

A Class of Variable-Order Fractional $p(\cdot)$ -Kirchhoff-Type Systems

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This paper is concerned with an elliptic system of Kirchhoff type, driven by the variable-order fractional $p(x)$ -operator. With the help of the direct variational method and Ekeland variational principle, we show the existence of a weak solution. This is our first attempt to study this kind of system, in the case of variable-order fractional variable exponents. Our main theorem extends in several directions previous results.

1. Introduction

In this article, we discuss the following variable-order fractional $p(\cdot)$ -Kirchhoff-type system:

$$\begin{cases} M_1 \left(\iint_{\mathbb{R}^{2N}} \frac{1}{p(x,y)} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+p(x,y)s(x,y)}} dx dy \right) (-\Delta)_{p(\cdot)}^{s(\cdot)} u(x) = f(u, v) + a(x) \text{ in } \Omega, \\ M_2 \left(\iint_{\mathbb{R}^{2N}} \frac{1}{p(x,y)} \frac{|v(x) - v(y)|^{p(x,y)}}{|x - y|^{N+p(x,y)s(x,y)}} dx dy \right) (-\Delta)_{p(\cdot)}^{s(\cdot)} v(x) = g(u, v) + b(x) \text{ in } \Omega, \\ u = v = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain with $N > p(x, y)s(x, y)$ for any $(x, y) \in \bar{\Omega} \times \bar{\Omega}$. Here, the main operator $(-\Delta)_{p(\cdot)}^{s(\cdot)}$ is the variable-order fractional $p(\cdot)$ -Laplacian given by

$$(-\Delta)_{p(\cdot)}^{s(\cdot)} \varphi(x) = P.V. \int_{\mathbb{R}^N} \frac{|\varphi(x) - \varphi(y)|^{p(x,y)-2} (\varphi(x) - \varphi(y))}{|x - y|^{N+p(x,y)s(x,y)}} dy, x \in \mathbb{R}^N, \quad (2)$$

along any $\varphi \in C_0^\infty(\mathbb{R}^N)$, where P.V. denotes the Cauchy principle value.

From now on, in order to simplify the notation, we denote

$$\begin{aligned} p^- &= \min_{(x,y) \in \mathbb{R}^{2N}} p(x, y), p^+ = \max_{(x,y) \in \mathbb{R}^{2N}} p(x, y), s^- \\ &= \min_{(x,y) \in \mathbb{R}^{2N}} s(x, y), s^+ = \max_{(x,y) \in \mathbb{R}^{2N}} s(x, y). \end{aligned} \quad (3)$$

We will assume that $M_1, M_2 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are continuous functions satisfying the condition

(M): there exist $m > 0$ and $\gamma > 1/p^-$ such that

$$M_1(t), M_2(t) > mt^{\gamma-1}, \text{ for all } t > 0. \quad (4)$$

Note that the Kirchhoff functions M_1, M_2 may be singular at $t = 0$ for $\gamma \in (0, 1)$.

Moreover, $H : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a C^1 -function verifying

$$(Z_1): \frac{\partial H}{\partial u}(u, v) = f(u, v) \text{ and } \frac{\partial H}{\partial v}(u, v) = g(u, v) \text{ for all } (u, v) \in \mathbb{R}^2; \tag{5}$$

(Z₂): there exists $K > 0$ such that

$$H(u, v) = H(u + K, v + K) \text{ for all } (u, v) \in \mathbb{R}^2. \tag{6}$$

Finally, we suppose that

(AB): $a(x), b(x) \in L^{q(x)}(\Omega)$, $(1/p(x)) + (1/q(x)) = 1$, $1 < q(x) < p_s^*(x)$, where $p_s^*(x) = N\bar{p}(x)/(N - \bar{s}(x)\bar{p}(x))$, $\bar{p}(x) = p(x, x)$, $\bar{s}(x) = s(x, x)$.

(PS): $p(\cdot) : \mathbb{R}^{2N} \rightarrow (1, \infty)$ is a *continuous* function fulfilling $0 < s^- < s^+ < 1 < p^- \leq p^+$, and $p(\cdot)$ and $s(\cdot)$ are symmetric, that is, $p(x, y) = p(y, x)$ and $s(x, y) = s(y, x)$ for any $(x, y) \in \mathbb{R}^{2N}$.

On the one hand, when $s(\cdot) \equiv 1$, the operators in (1) reduce to the integer order, i.e., $p(\cdot)$ -Laplacian $\Delta_{p(\cdot)}$. This kind of variable exponent problem has a wide range of real applications, such as electrorheological fluids (see [1]), elastic mechanics ([2]), and image restoration ([3]). For this kind of operator combined with Kirchhoff function system problem, we recall [4–9], for example, Boulaaras and Allahem ([4]) studied the existence of positive solutions of a $p(x)$ -Kirchhoff system by using subsuper solutions concepts. A very interesting question arose is that whether there are other ways to solve this class of problems. And also, can we consider the nonlocal variable-order case? We know that the fractional variable-order derivatives proposed by Lorenzo and Hartley in [10] appeared in different nonlinear diffusion processes. Subsequently, many results of the variable-order problem have appeared in the literature [11–13].

Of course, when $p(x) \equiv p$ (or $p = 2$) and $s(\cdot) \equiv s$, the operators in (1) reduce to the classical non-local fractional p -Laplacian, i.e., $(-\Delta)_p^s$ ($(-\Delta)^s$). Similarly, for the Kirchhoff-type system cases, the papers [14–20] introduce a lot of related work in recent years, where many authors studied the existence and multiplicity of solutions by applying variational methods. For instance, based on the three critical points theorem, Azroul et al. ([15]) discussed an elliptic system with the homogeneous Dirichlet boundary conditions, and they obtained the existence of three weak solutions.

On the other hand, it is worth mentioning that Kirchhoff in 1883 (see [21]) presented a stationary version of differential equation, the so-called Kirchhoff equation:

$$\rho \frac{\partial^2 v}{\partial t^2} - \left(\frac{p_0}{l} + \frac{e}{2L} \int_0^L \left| \frac{\partial v}{\partial x} \right|^2 dx \right) \frac{\partial^2 v}{\partial x^2} = 0, \tag{7}$$

where ρ, l, e, L, p_0 are positive constants which represent the corresponding physical meanings. It is a generalization of D'Alembert equation. It is very interesting to combine this model with various operators due to its nonlocal nature.

Inspired by the above works, we consider a new fractional Kirchhoff-type system (1). As far as we know, this is the first

attempt on variable-order fractional situations to study a bi-nonlocal problem with variable exponent. In order to overcome the difficulty, we use the direct variational method and Ekeland variational principle to deal with it. Our result is new to the variable-order fractional system with variable exponent.

Now, we give the main result of this paper; our energy functional \mathcal{F} will be introduced in “Abstract Framework.”

Theorem 1. *Let Ω be a bounded smooth domain of \mathbb{R}^N , with $N > p(x, y)s(x, y)$ for any $(x, y) \in \bar{\Omega} \times \bar{\Omega}$, where $p(\cdot)$ and $s(\cdot)$ verify (PS). Assume that (M), (AB), and (Z₁) – (Z₂) are satisfied. Then, problem (1) admits a weak solution if \mathcal{F} is differentiable at (u_0, v_0) .*

The paper is organized as follows. In “Abstract Framework,” we state some interesting properties of variable exponent Lebesgue spaces and variable-order fractional Sobolev spaces with variable exponent. In “The Main Result,” we prove the functional \mathcal{F} is bounded from below and give the proof of Theorem 1.

2. Abstract Framework

In this section, first of all, we recall some basic properties about the variable exponent Lebesgue spaces in [22] and variable-order fractional Sobolev spaces. Secondly, we give some necessary lemmas that will be used in this paper. Finally, we introduce the definition of weak solutions for problem (1) and build the corresponding energy functional. Consider the set

$$C_+(\bar{\Omega}) = \{p \in C(\bar{\Omega}) : p(x) > 1 \text{ for all } x \in \bar{\Omega}\}. \tag{8}$$

For any $p \in C_+(\bar{\Omega})$, we define the variable exponent Lebesgue space as

$$L^{p(\cdot)}(\Omega) = \left\{ u : \text{the function } u : \Omega \rightarrow \mathbb{R} \text{ is measurable, } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}, \tag{9}$$

the vector space endowed with the *Luxemburg norm*

$$\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}. \tag{10}$$

Then, $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a separable reflexive Banach space (see [23], Theorem 2.5). Let $q \in C_+(\bar{\Omega})$ be the conjugate exponent of p , that is

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1, \text{ for all } x \in \bar{\Omega}. \tag{11}$$

Then, we have the following Hölder inequality, whose proof can be found in [23], (Theorem 2.1).

Lemma 2. Assume that $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, then

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) \|u\|_{p(\cdot)} \|v\|_{q(\cdot)} \leq 2 \|u\|_{p(\cdot)} \|v\|_{q(\cdot)}. \quad (12)$$

The variable-order fractional Sobolev spaces with variable exponent is defined by

$$W^{s(\cdot), p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+p(x,y)s(x,y)}} dx dy < \infty \right\}, \quad (13)$$

with the norm $\|u\|_{s,p(\cdot)} = \|u\|_{p(\cdot)} + [u]_{s,p(\cdot)}$, where

$$[u]_{s,p(\cdot)} = \inf \left\{ \lambda > 0 : \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{N+p(x,y)s(x,y)}} dx dy < 1 \right\}. \quad (14)$$

For a more detailed introduction of this space, we can refer to [24]. For the reader's convenience, we now list some of the results in reference [24] which will be used in our paper. We define the new variable-order fractional Sobolev spaces with variable exponent

$$X = \left\{ u : \mathbb{R}^N \longrightarrow \mathbb{R} : u|_{\Omega} \in L^{p(\cdot)}(\Omega), \iint_Q \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{N+p(x,y)s(x,y)}} dx dy < \infty, \text{ for some } \lambda > 0 \right\}, \quad (15)$$

where $Q := \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)$. The space X is endowed with the norm

$$\|u\|_X = \|u\|_{p(\cdot)} + [u]_X, \quad (16)$$

where

$$[u]_X = \inf \left\{ \lambda > 0 : \iint_Q \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{N+p(x,y)s(x,y)}} dx dy < 1 \right\}. \quad (17)$$

We know that the norms $\|\cdot\|_{s,p(\cdot)}$ and $\|\cdot\|_X$ are not the same due to the fact that $\Omega \times \Omega \subset Q$ and $\Omega \times \Omega \neq Q$. This makes the variable-order fractional Sobolev space with variable exponent $W^{s(\cdot), p(\cdot)}(\Omega) \times W^{s(\cdot), p(\cdot)}(\Omega)$ not sufficient for investigating the class of problems like (1).

For this, we set space as

$$X_0 = \{ u \in X : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}. \quad (18)$$

The space X_0 is a separable reflexive Banach space, see [25], with respect to the norm

$$\begin{aligned} \|u\|_{X_0} &= \inf \left\{ \lambda > 0 : \iint_Q \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{N+p(x,y)s(x,y)}} dx dy \right. \\ &= \left. \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{N+p(x,y)s(x,y)}} dx dy < 1 \right\}, \end{aligned} \quad (19)$$

where last equality is a consequence of the fact that $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$.

In the following Lemma, we give a compact embedding result. For the proof, we refer the reader to [24].

Lemma 3. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $s(\cdot) \in (0, 1)$. Let $p(x, y)$ be continuous variable exponents with $s(x, y)p(x, y) < N$ for $(x, y) \in \bar{\Omega} \times \bar{\Omega}$. Assume that $q : \bar{\Omega} \longrightarrow (1, \infty)$ is a continuous function such that

$$p^*(x) > q(x) \geq q^- > 1, \text{ for all } x \in \bar{\Omega}. \quad (20)$$

Then, there exists a constant $C = C(N, s, p, q, \Omega)$ such that for every $u \in X_0$, it holds that

$$\|u\|_{q(x)} \leq C \|u\|_{X_0}. \quad (21)$$

The space X_0 is continuously embedded in $L^{q(x)}(\Omega)$. Moreover, this embedding is compact.

We define the fractional modular function $\rho_{p(\cdot)}^{s(\cdot)} : X_0 \longrightarrow \mathbb{R}$, by

$$\rho_{p(\cdot)}^{s(\cdot)}(u) = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+p(x,y)s(x,y)}} dx dy. \quad (22)$$

We also have the next result of ([24], Proposition 2.2).

Lemma 4. Assume that $u \in X_0$ and $\{u_j\} \subset X_0$, then

$$\|u\|_{X_0} < 1 \text{ (resp. } = 1, > 1) \Leftrightarrow \rho_{p(\cdot)}^{s(\cdot)}(u) < 1 \text{ (resp. } = 1, > 1),$$

$$\|u\|_{X_0} < 1 \Rightarrow \|u\|_{X_0}^{p^+} \leq \rho_{p(\cdot)}^{s(\cdot)}(u) \leq \|u\|_{X_0}^{p^-},$$

$$\|u\|_{X_0} > 1 \Rightarrow \|u\|_{X_0}^{p^-} \leq \rho_{p(\cdot)}^{s(\cdot)}(u) \leq \|u\|_{X_0}^{p^+},$$

$$\lim_{j \rightarrow \infty} \|u_j\|_{X_0} = 0(\infty) \Leftrightarrow \lim_{j \rightarrow \infty} \rho_{p(\cdot)}^{s(\cdot)}(u_j) = 0(\infty),$$

$$\lim_{j \rightarrow \infty} \|u_j - u\|_{X_0} = 0 \Leftrightarrow \lim_{j \rightarrow \infty} \rho_{p(\cdot)}^{s(\cdot)}(u_j - u) = 0. \quad (23)$$

Finally, we define our workspace $S = X_0 \times X_0$, which is endowed with the norm

$$\|(u, v)\|_S = \|u\|_{X_0} + \|v\|_{X_0}. \quad (24)$$

We say that a pair of functions $(u, v) \in S$ is the weak solution of problem (1), if for all $(\phi, \varphi) \in S$ one has

$$\begin{aligned} M_1\left(\delta_{p(\cdot)}(u)\right) & \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)-2}(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{N+p(x,y)s(x,y)}} dx dy \\ & = \int_{\Omega} ((f(u, v) + a(x))\phi) dx, \\ M_2\left(\delta_{p(\cdot)}(v)\right) & \iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+p(x,y)s(x,y)}} dx dy \\ & = \int_{\Omega} ((g(u, v) + b(x))\varphi) dx, \end{aligned} \quad (25)$$

where

$$\delta_{p(\cdot)}(u) = \iint_{\mathbb{R}^{2N}} \frac{1}{p(x, y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}} dx dy. \quad (26)$$

Let us consider the following functional associated to problem (1), defined by $\mathcal{F} : S \rightarrow \mathbb{R}$

$$\begin{aligned} \mathcal{F}(u, v) & = \widetilde{M}_1\left(\delta_{p(\cdot)}(u)\right) - \widetilde{M}_2\left(\delta_{p(\cdot)}(v)\right) - \int_{\Omega} H(u, v) dx \\ & \quad - \int_{\Omega} a(x)u dx - \int_{\Omega} b(x)v dx, \end{aligned} \quad (27)$$

for all $(u, v) \in S$, where $\widetilde{M}_i(t) = \int_0^t M_i(\tau) d\tau$. Obviously, the continuity of M yields that \mathcal{F} is well defined and of class C^1 on $S \setminus \{0, 0\}$. Furthermore, for every $(u, v) \in S \setminus \{0, 0\}$, the derivative of \mathcal{F} is given by

$$\begin{aligned} \langle \mathcal{F}'(u, v), (\phi, \varphi) \rangle & = M_1\left(\delta_{p(\cdot)}(u)\right) \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+p(x,y)s(x,y)}} dx dy \\ & \quad + M_2\left(\delta_{p(\cdot)}(v)\right) \iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+p(x,y)s(x,y)}} dx dy \\ & \quad - \int_{\Omega} ((f(u, v) + a(x))\phi) dx - \int_{\Omega} ((g(u, v) + b(x))\varphi) dx, \end{aligned} \quad (28)$$

for any $(\phi, \varphi) \in S$. Therefore, the weak solution $(u, v) \in S \setminus \{0, 0\}$ of problem (1) is a nontrivial critical point of \mathcal{F} .

Now, we recall the following well-known Ekeland variational principle found in [7], which will be used to prove our conclusion, that is Theorem 1.

Theorem 5. Let X be a Banach space and $\mathcal{F} : X \rightarrow \mathbb{R}$ be a C^1 function which is bounded from below. Then, for any $\varepsilon > 0$, there exists $\bar{\omega}_\varepsilon \in X$ such that

$$\mathcal{F}(\bar{\omega}_\varepsilon) \leq \inf_X \mathcal{F} + \varepsilon \text{ and } \|\mathcal{F}'(\bar{\omega}_\varepsilon)\|_{X^*} \leq \varepsilon. \quad (29)$$

Throughout the paper, for simplicity, we use $\{c_i, i \in \mathbb{N}\}$ to denote different nonnegative or positive constant.

3. The Main Result

Lemma 6. Under the same assumptions of Theorem 1, then \mathcal{F} is coercive and bounded from below.

Proof. Firstly, we know that functional \mathcal{F} is well defined. Indeed, it is sufficient to prove that the functional $T : S \rightarrow \mathbb{R}$, $T(u, v) = \int_{\Omega} H(u, v) dx$, is well defined. Since H is continuous on $[0, K] \times [0, K]$ and $H(u, v) = H(u + K, v + K)$ for all $(u, v) \in \mathbb{R}^2$, we get $|H(u, v)| \leq c_1$ for all $(u, v) \in \mathbb{R}^2$. Thus,

$$|T(u, v)| \leq \int_{\Omega} |H(u, v)| dx \leq c_1 |\Omega|, \text{ for all } (u, v) \in \mathbb{R}^2, \quad (30)$$

i.e., T is well defined, where $|\Omega|$ is the Lebesgue measure of Ω . Next, we will prove that \mathcal{F} is coercive and bounded from below. Let $(u, v) \in S$, and we have

$$\begin{aligned} \mathcal{F}(u, v) & = \widetilde{M}_1\left(\delta_{p(\cdot)}(u)\right) - \widetilde{M}_2\left(\delta_{p(\cdot)}(v)\right) - \int_{\Omega} H(u, v) dx \\ & \quad - \int_{\Omega} a(x)u dx - \int_{\Omega} b(x)v dx \\ & \geq \widetilde{M}_1\left(\delta_{p(\cdot)}(u)\right) - \widetilde{M}_2\left(\delta_{p(\cdot)}(v)\right) - c_1 |\Omega| \\ & \quad - \int_{\Omega} a(x)u dx - \int_{\Omega} b(x)v dx. \end{aligned} \quad (31)$$

By the condition (AB) and Lemma 2, we get

$$\begin{aligned} \mathcal{F}(u, v) & \geq \widetilde{M}_1\left(\delta_{p(\cdot)}(u)\right) - \widetilde{M}_2\left(\delta_{p(\cdot)}(v)\right) - c_1 |\Omega| \\ & \quad - 2\|a(x)\|_{q(x)} \|u\|_{p(x)} - 2\|b(x)\|_{q(x)} \|v\|_{p(x)}. \end{aligned} \quad (32)$$

It follows from (M) and Lemmas 3 and 4 that

$$\begin{aligned} \mathcal{F}(u, v) & \geq m \int_0^{\frac{1}{p^+ \rho_{p(\cdot)}^s(u)}} \tau^{\gamma-1} d\tau + m \int_0^{\frac{1}{p^+ \rho_{p(\cdot)}^s(v)}} \tau^{\gamma-1} d\tau - c_3 \|u\|_{X_0} \\ & \quad - c_4 \|v\|_{X_0} - c_2 = \frac{m}{\gamma(p^+)^{\gamma}} \left(\left(\rho_{p(\cdot)}^s(u) \right)^{\gamma} + \left(\rho_{p(\cdot)}^s(v) \right)^{\gamma} \right) \\ & \quad - c_3 \|u\|_{X_0} - c_4 \|v\|_{X_0} - c_2 \\ & \geq \frac{m}{\gamma(p^+)^{\gamma}} \left(\min \left\{ \|u\|_{X_0}^{\gamma p^-}, \|u\|_{X_0}^{\gamma p^+} \right\} + \min \left\{ \|v\|_{X_0}^{\gamma p^-}, \|v\|_{X_0}^{\gamma p^+} \right\} \right) \\ & \quad - \max \{c_3, c_4\} (\|u\|_{X_0} + \|v\|_{X_0}) - c_2. \end{aligned} \quad (33)$$

Since $\gamma p^+ > \gamma p^- > 1$, when $\|(u, v)\|_S \rightarrow +\infty$, at least one of $\|u\|_{X_0}$ and $\|v\|_{X_0}$ converges to infinity. So, \mathcal{F} is coercive and bounded from below. The proof of Lemma 6 is complete.

Proof of Theorem 1. Obviously, since $\mathcal{F} \in C^1(S, \mathbb{R})$ is weakly lower semicontinuous and bounded from below, by means of the Ekeland variational principle, we have $(u_j, v_j) \subset S$ such that

$$\mathcal{F}(u_j, v_j) \rightarrow \inf_S \mathcal{F} \text{ and } \mathcal{F}'(u_j, v_j) \rightarrow 0. \quad (34)$$

Furthermore, by the above expression, we get $|\mathcal{F}(u_j, v_j)| \leq c_5$. Thus, it follows from (33) that

$$c_6 \leq |\mathcal{F}(u_j, v_j)| \leq c_5, \quad (35)$$

which implies that the sequences $\{u_j\}$ and $\{v_j\}$ are bounded in X_0 . So, without the loss of generality, there exist subsequences $\{u_j\}$ and $\{v_j\}$ such that $u_j \rightharpoonup u_0$ and $v_j \rightharpoonup v_0$ in X_0 , and thus

$$\int_{\Omega} a(x)u_j dx \rightarrow \int_{\Omega} a(x)u_0 dx \text{ and } \int_{\Omega} b(x)v_j dx \rightarrow \int_{\Omega} a(x)v_0 dx. \quad (36)$$

According to compact embedding theorem, which is Lemma 3, we obtain

$$u_j(x) \rightarrow u_0(x) \text{ and } v_j(x) \rightarrow v_0(x), \text{ a.e. } x \in \Omega. \quad (37)$$

Again, by continuity of H , we get

$$H(u_j(x), v_j(x)) \rightarrow H(u_0(x), v_0(x)) \text{ a.e. } x \in \Omega. \quad (38)$$

And because H is bounded, we get the following convergence from the Lebesgue dominated convergence theorem:

$$\int_{\Omega} H(u_j, v_j) dx \rightarrow \int_{\Omega} H(u_0, v_0) dx. \quad (39)$$

By (34), we note that

$$\begin{aligned} \inf_S \mathcal{F} &= \lim \mathcal{F}(u_j, v_j) = \lim \left(\widetilde{M}_1 \left(\delta_{p(\cdot)}(u_j) \right) - \widetilde{M}_2 \left(\delta_{p(\cdot)}(v_j) \right) \right) \\ &\quad - \int_{\Omega} H(u_j, v_j) dx - \int_{\Omega} a(x)u_j dx - \int_{\Omega} b(x)v_j dx. \end{aligned} \quad (40)$$

In view of Fatou's lemma, we have

$$\delta_{p(\cdot)}(u_0) \leq \liminf \delta_{p(\cdot)}(u_j) \text{ and } \delta_{p(\cdot)}(v_0) \leq \liminf \delta_{p(\cdot)}(v_j). \quad (41)$$

By the continuous monotone increasing property of \widetilde{M}_1 and \widetilde{M}_2 , we get

$$\begin{aligned} \widetilde{M}_1 \left(\delta_{p(\cdot)}(u_0) \right) &\leq \lim \widetilde{M}_1 \left(\delta_{p(\cdot)}(u_j) \right) \text{ and } \widetilde{M}_2 \left(\delta_{p(\cdot)}(v_0) \right) \\ &\leq \lim \widetilde{M}_2 \left(\delta_{p(\cdot)}(v_j) \right). \end{aligned} \quad (42)$$

In conclusion,

$$\begin{aligned} \inf_S \mathcal{F} &\geq \widetilde{M}_1 \left(\delta_{p(\cdot)}(u_0) \right) - \widetilde{M}_2 \left(\delta_{p(\cdot)}(v_0) \right) - \int_{\Omega} H(u_0, v_0) dx \\ &\quad - \int_{\Omega} a(x)u_0 dx - \int_{\Omega} b(x)v_0 dx = \mathcal{F}(u_0, v_0), \end{aligned} \quad (43)$$

which implies $\mathcal{F}(u_0, v_0) = \inf_S \mathcal{F}$. Thus, $(u_0, v_0) \in S$ is a weak solution of problem (1) if \mathcal{F} is differentiable at (u_0, v_0) . The proof is complete.

Data Availability

We do not involve any data in our work.

Conflicts of Interest

The authors declare that they have no competing interests.

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