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Research Article

Solving a Split Feasibility Problem by the Strong Convergence of Two Projection Algorithms in Hilbert Spaces

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The goal of this manuscript is to establish strong convergence theorems for inertial shrinking projection and CQ algorithms to solve a split convex feasibility problem in real Hilbert spaces. Finally, numerical examples were obtained to discuss the performance and effectiveness of our algorithms and compare the proposed algorithms with the previous shrinking projection, hybrid projection, and inertial forward-backward methods.

1. Introduction

Assume that Y is a real HS defined on the induced norm $\|.\|$ and the inner product $\langle .,. \rangle$. Let C be a NCC subset of Y.

The mapping $T : C \longrightarrow C$ is called NE, if for all $\kappa, \omega \in C$, the following inequality holds:

$$\|T\kappa - T\omega\| \leq \|\kappa - \omega\|. \quad (1)$$

For the mapping T,$F(T) = \{\kappa \in C : T\kappa = \kappa\}$ refers to the set of all FPs of it.

Here, we study the following inclusion problem:

$$\text{Find } \tilde{\kappa} \in Y \text{ so that } 0 \in A\tilde{\kappa} + B\tilde{\kappa}, \quad (2)$$

where $A : Y \longrightarrow Y$ and $B : Y \longrightarrow 2^Y$ are single-valued and set-valued operators, respectively.

There are many important applications enjoyed by approximating FP problems for NEMs, such as monotone variational inequalities, image restoration problems, convex optimization problems, and SCFPs, for example, see [1–3]. For more accuracy, these problems can be expressed as mathematical models such a machine learning and the linear inverse problem.

In the past, the solution of problem (2) was described by $(A + B)^{-1}(0)$ and it relied on the forward-backward splitting method [4–10]. This technique is described as follows: $\kappa_1 \in Y$ and

$$\kappa_{n+1} = (I + \tau B)^{-1}(\kappa_n - \tau A\kappa_n), \quad n \geq 1, \tau > 0. \quad (3)$$

In this scene, we do not mean the sum of A and B in the iterates, but each step of iterates includes only A as the forward term and B as the backward term. As special cases, this technique gets involved heavily in a study of the proximal point algorithm [11–13] and the gradient method [14–17].

In 1979, a good splitting iterative scheme in a real HS was introduced by Lions and Mercier [18]. It is described as follows:

$$\kappa_{n+1} = \left(2J_\tau^A - I\right)\left(2J_\tau^B - I\right)\kappa_n, \quad n \geq 1, \quad (4)$$

$$\kappa_{n+1} = J_\tau^A\left(2J_\tau^B - I\right)\kappa_n + \left(I - J_\tau^B\right)\kappa_n, \quad n \geq 1, \tag{5}$$

where $J_\tau^S = (I + \tau S)^{-1}$. In the previous literature, the algorithm (4) is called Peaceman-Rachford [7] and the scheme (5) is called Douglas-Rachford [19]. Generally, the convergence of both procedures is weak [7].

In 2001, a heavy ball method involved for studying maximal monotone operators is introduced by Alvarez and Attouch [20]; this idea was developed in [21, 22], where an inertial term was added. This procedure is called the inertial proximal point algorithm and it takes the shape

$$\begin{cases} \omega_n = \kappa_n + \theta_n(\kappa_n - \kappa_{n-1}), \\ \kappa_{n+1} = (I + \tau_n B)_n^{-1}\omega_n, \quad n \geq 1. \end{cases} \tag{6}$$

They got the weak convergence for the mapping B, if $\{\tau_n\}$ is nondecreasing and $\{\theta_n\} \subset [0, 1)$ with

$$\sum_{n=1}^{\infty} \theta_n \|\kappa_n - \kappa_{n-1}\|^2 < \infty. \tag{7}$$

In particular, condition (7) is true for $\theta_n < 1/3$. Here, θ_n is an extrapolation factor and the inertia is represented by the term $\theta_n(\kappa_n - \kappa_{n-1})$.

It should be noted that the inertial term improves and increases the convergence speed of the algorithm [23–25].

An inertial proximal point algorithm improved by Moudafi and Oliny [26], where the single-valued, cocoercive, and Lipschitz continuous operator was added, is as follows:

$$\begin{cases} \omega_n = \kappa_n + \alpha_n(\kappa_n - \kappa_{n-1}), \\ \kappa_{n+1} = (I + \tau_n B)_n^{-1}(\omega_n - \tau_n A\omega_n), \quad n \geq 1. \end{cases} \tag{8}$$

The problem of weak convergence still persists for the algorithm (8) via stipulation (7) and $\tau_n < 2/L$ where L is the Lipschitz constant of A.

Besides that, the strong convergence is of interest to many researchers, but the study of convergence via norm convergence in infinite-dimensional spaces is often much more desirable than weak convergence [27].

The first contribution of researchers to the strong convergence is the algorithm presented by Nakajo and Takahashi [28]. They added the CQ terms to the Mann algorithm as follows: for an arbitrary point $\kappa_0 \in C$, define the sequence $\{\kappa_n\}$ iteratively by

$$\begin{cases} \omega_n = \alpha_n\kappa_n + (1 - \alpha_n)T\kappa_n, \\ C_n = \{p \in C : \|\omega_n - p\| \leq \|\kappa_n - p\|\}, \\ Q_n = \{p \in C : \langle \kappa_0 - \kappa_n, \kappa_n - p\rangle \geq 0\}, \\ \kappa_{n+1} = P_{Q_n \cap C_n}\kappa_0, \quad n \geq 0. \end{cases} \tag{9}$$

They showed that the sequence $\{\kappa_n\}$ converges strongly to $P_{\mathrm{Fix}(T)}\kappa_0$, whenever the sequence $\{\alpha_n\}$ is bounded above by 1. We highly recommend seeing [24, 29], for more details on the CQ algorithms for NEMs.

Based on the algorithm (9), Dong et al. [30] introduced a strong convergence result by implicating an inertial forward-backward algorithm for monotone inclusions as the following: assume that $A : Y \longrightarrow Y$ is an α-ISM operator and $B : Y \longrightarrow 2^Y$ is a MM operator so that $(A + B)^{-1}(0) \neq \varnothing$. Suppose $\{\alpha_n\} \in \mathbb{R}$ and $\{\kappa_n\} \in Y$ is a sequence maked iteratively by $\kappa_\circ, \kappa_1 \in Y$,

$$\begin{cases} \omega_n = \kappa_n + \alpha_n(\kappa_n - \kappa_{n-1}), \\ \upsilon_n = (I + \tau_n B)^{-1}(\omega_n - \tau_n A\omega_n), \\ C_n = \left\{p \in H : \|\upsilon_n - p\|^2 \leq \|\kappa_n - p\|^2 - 2\alpha_n\langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle + \alpha_n^2\|\kappa_{n-1} - \kappa_n\|^2\right\}, \\ Q_n = \{p \in H : \langle \kappa_0 - \kappa_n, \kappa_n - p\rangle \leq 0\}, \\ \kappa_{n+1} = P_{Q_n \cap C_n}\kappa_0, \quad n \geq 1. \end{cases} \tag{10}$$

Recently, nice convergence analysis for NEMs via suitable stipulations has been discussed by Dong et al. [31]. They extended the inertial Mann algorithm as follows:

$$\begin{cases} \omega_n = \kappa_n + \alpha_n(\kappa_n - \kappa_{n-1}), \\ \upsilon_n = \kappa_n + \beta_n(\kappa_n - \kappa_{n-1}), \\ \kappa_{n+1} = (1 - \lambda_n)\omega_n + \lambda_n T\upsilon_n, \quad n \geq 1, \end{cases} \tag{11}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\lambda_n\}$ are real sequences that justify the stipulations D_1 and D_2 [31].

Believing in the idea of strong convergence of algorithms in this manuscript, the two-step inertial shrinking projection algorithm is introduced which analyzes the strong convergence. As an application to our main results, the SCFP is solved. Finally, to see the behavior and performance of our algorithms in terms of convergence, numerical results are presented and discussed.

2. Preliminaries

This section is devoted to collect some important preliminaries, which we need in the sequel. Let C be a NCC subset of a real HS Y and $\{\kappa_n\}$ be a sequence in Y. Here, the strong convergence of $\{\kappa_n\}$ to a point κ is written as $\kappa_n \longrightarrow \kappa$. The metric projection of Y onto C is described by P_C, that is, $\|\kappa - P_C\kappa\| \le \|\kappa - \omega\|$ for all $\kappa \in Y$ and $\omega \in C$.

Lemma 1 (see [32]). *Let C be a NCC subset of a real HS Y, the metric projection P_C is firmly NE, i.e.,*

$$\|P_C\kappa - P_C\omega\|^2 \le \langle P_C\kappa - P_C\omega, \kappa - \omega\rangle, \tag{12}$$

for all $\kappa, \omega \in Y$. Furthermore, for all $\kappa \in Y$ and $\omega \in C$, $\langle \kappa - P_C\kappa, \omega - P_C\omega\rangle \le 0$ is satisfied.

Lemma 2 (see [32]). *Assume that Y is a real HS. Then, we get*

(i) $\|\kappa + \omega\|^2 \le \|\kappa\|^2 + 2\langle \omega, \kappa + \omega\rangle$

(ii) $\|\rho\kappa + (1-\rho)\omega\|^2 = \rho\|\kappa\|^2 + (1-\rho)\|\omega\|^2 - \rho(1-\rho)\|\kappa - \omega\|^2$

for each $\kappa, \omega \in Y$ and for a real number ρ.

Lemma 3 (see [33]). *Suppose that Y is a real HS and $\{\kappa_n\}$ is a sequence in Y. Then, the following hypotheses hold:*

(i) *If $\kappa_n \rightharpoonup \kappa$ and $\|\kappa_n\| \longrightarrow \|\kappa\|$ as $n \longrightarrow \infty$, then $\kappa_n \longrightarrow \kappa$ as $n \longrightarrow \infty$; that is, the HS Y has the Kadec-Klee property*

(ii) *If $\kappa_n \rightharpoonup \kappa$ as $n \longrightarrow \infty$, then $\|\kappa\| \le \liminf_{n\to\infty}\|\kappa_n\|$*

Lemma 4 (see [34]). *Let C be a NCC subset of a real HS Y. For each $\kappa, \omega, \upsilon \in Y$ and $b \in \mathbb{R}$, the following set is closed and convex:*

$$\left\{\delta \in C : \|\omega - \delta\|^2 \le \|\kappa - \delta\|^2 + \langle \upsilon, \delta\rangle + b\right\}. \tag{13}$$

Lemma 5 (see [28]). *Let C be a NCC subset of a real HS Y and $P_C : Y \longrightarrow C$ be the metric projection. Then, for all $\kappa \in Y$ and $\omega \in C$, the following inequality holds:*

$$\|\omega - P_C\kappa\|^2 + \|\kappa - P_C\kappa\|^2 \le \|\kappa - \omega\|^2. \tag{14}$$

Lemma 6 (see [35]). *Let T be a NE self-mapping of a NCC subset C of a real HS Y. The mapping $I - T$ is demiclosed, i.e., the sequence $\{\kappa_n\}$ in C weakly converges to some $\kappa \in C$ and the sequence $\{(I-T)(\kappa_n)\}$ strongly converges to some ω; it follows that $(I-T)(\kappa) = \omega$.*

Definition 7. Assume that $D(A) \subset Y$ is the domain of the mapping A, then for all $\kappa, \omega \in D(A)$, the mapping A is called

(i) monotone if $\langle \kappa - \omega, A\kappa - A\omega\rangle \ge 0$

(ii) σ-strongly monotone if there is $\beta > 0$ so that $\langle \kappa - \omega, A\kappa - A\omega\rangle \ge \sigma\|\kappa - \omega\|^2$

(iii) α-ISM if there is $\alpha > 0$ so that $\langle \kappa - \omega, A\kappa - A\omega\rangle \ge \alpha\|A\kappa - A\omega\|^2$

Lemma 8 (see [5]). *Let Y be a real HS, $A : Y \longrightarrow Y$ be an α-ISM operator, and $B : Y \longrightarrow 2^Y$ be a MM operator. For each $\ell > 0$, we consider*

$$T_\ell = J_\ell^B(I - \ell A) = (I + \ell B)^{-1}(I - \ell A), \tag{15}$$

then the following statements hold:

(i) *for $\ell > 0, F(T_\ell) = (A+B)^{-1}(0)$*

(ii) *for $0 < s \le \ell$ and $\kappa \in Y$, $\|\kappa - T_s\kappa\| \le 2\|\kappa - T_\ell\kappa\|$*

Lemma 9 (see [36]). *Let H be a real HS, $A : Y \longrightarrow Y$ be an α-ISM operator and $B : Y \longrightarrow 2^Y$ be a MM operator, then for all $\ell > 0$ and all $\kappa, \omega \in Y$, we have*

$$\|T_\ell\kappa - T_\ell\omega\|^2 \le \|\kappa - \omega\|^2 - \ell(2\alpha - \ell)\|A\kappa - A\omega\|^2. \tag{16}$$

3. Strong Convergence Results

From now on, we assume that C be a NCC subset of a real HS Y, $A : Y \longrightarrow Y$ is α-ISM operator, $B : Y \longrightarrow 2^Y$ is MM operator, $T : Y \longrightarrow Y$ is quasi-NEM so that $I - T$ is demiclosed at zero and $\Omega = (A+B)^{-1}(0)$.

Now, we build our algorithms to finding an element in Ω as follows:

Now, we shall discuss the strong convergence of Algorithm 1: by introducing the following theorem.

Theorem 10. *Let the sequence $\{\alpha_n\}$, $\{\beta_n\}$ be bounded and $\{\gamma_n\}$ be a sequence in $(0,1]$ and $\{\tau_n\}$ be a sequence of positive real numbers so that the following two stipulations hold:*

(i) *$\inf_n\{\gamma_n\} \ge \gamma > 0$*

(ii) *$0 < \inf_n\{\tau_n\} \le \sup_n\{\tau_n\} < 2\alpha$*

If $\Omega \ne \emptyset$, then the sequence $\{\kappa_n\}$ created by Algorithm 1: converges strongly to $\Theta = P_\Omega(\kappa_1)$.

Proof. The proof will be divided into the following steps:

Step (i). For each $\kappa_1 \in H$, $\Omega \subset C_{n+1}$, and for $n \ge 0$. Prove that $P_{C_{n+1}}\kappa_1$ is well-defined.

From the stipulation (ii) and Lemma 9, we get $T_{\tau_n} = (I + \tau_n B)^{-1}(I - \tau_n A)$ is NEM. Thus, it follows from Lemma 8 that the set Ω is closed and convex. Moreover, Lemma 4 leads that for all $n \ge 1, C_{n+1}$ is closed and convex. Considering

Let $\{\alpha_n\}$,$\{\beta_n\}$, and $\{\gamma_n\}$ be sequences of real numbers. Select initial $\kappa_0, \kappa_1 \in C_1 = Y$.
Step (1). Compute

$$\omega_n = \kappa_n + \alpha_n(\kappa_n - \kappa_{n-1}),$$
$$\upsilon_n = \kappa_n + \beta_n(\kappa_n - \kappa_{n-1}).$$

Step (2). Compute

$$\mu_n = (1-\gamma_n)\omega_n + \gamma_n J^B_{\tau_n}(\upsilon_n - \tau_n A\upsilon_n).$$

Step (3). Compute

$$C_{n+1} = \{p \in C_n : \|\mu_n - p\|^2 \le \|\kappa_n - p\|^2 - 2(\alpha_n(1-\gamma_n)+\gamma_n\beta_n)\langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle + (\alpha_n^2(1-\gamma_n)+\gamma_n\beta_n^2)\|\kappa_{n-1}-\kappa_n\|^2\},$$
$$\kappa_{n+1} = P_{C_{n+1}}(\kappa_1), n \ge 1.$$

Algorithm 1: Shrinking projection algorithm.

$p \in \Omega$, we get

$$\|\omega_n - p\|^2 = \|(\kappa_n - p) - \alpha_n(\kappa_{n-1} - \kappa_n)\|^2 = \|\kappa_n - p\|^2 - 2\alpha_n\langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle + \alpha_n^2\|\kappa_{n-1} - \kappa_n\|^2. \quad (17)$$

By the same manner, one can write

$$\|\upsilon_n - p\|^2 = \|\kappa_n - p\|^2 - 2\beta_n\langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle + \beta_n^2\|\kappa_{n-1} - \kappa_n\|^2. \quad (18)$$

Furthermore, by Lemma 2 (ii) and Lemma 9, we obtain that

$$\begin{aligned}\|\mu_n - p\|^2 &= \left\|(1-\gamma_n)\omega_n + \gamma_n J^B_{\tau_n}(\upsilon_n - \tau_n A\upsilon_n) - p\right\|^2 \\ &= \left\|(1-\gamma_n)(\omega_n - p) + \gamma_n\left(T_{\tau_n}\upsilon_n - p\right)\right\|^2 \\ &= \gamma_n\left\|T_{\tau_n}\upsilon_n - p\right\|^2 + (1-\gamma_n)\|\omega_n - p\|^2 \\ &\quad - \gamma_n(1-\gamma_n)\left\|T_{\tau_n}\upsilon_n - \omega_n\right\|^2 \le (1-\gamma_n)\|\omega_n - p\|^2 \\ &\quad + \gamma_n\left\|T_{\tau_n}\upsilon_n - p\right\|^2 = (1-\gamma_n)\|\omega_n - p\|^2 \\ &\quad + \gamma_n\left\|T_{\tau_n}\upsilon_n - T_{\tau_n}p\right\|^2 \le (1-\gamma_n)\|\omega_n - p\|^2 \\ &\quad + \gamma_n\left(\|\upsilon_n - p\|^2 - \tau_n(2\alpha - \tau_n)\|A\upsilon_n - Ap\|^2\right) \\ &\le (1-\gamma_n)\|\omega_n - p\|^2 + \gamma_n\|\upsilon_n - p\|^2.\end{aligned} \quad (19)$$

Applying (17) and (18) in (19) and by stipulation (ii) of Theorem 10, we can write

$$\begin{aligned}\|\mu_n - p\|^2 &\le (1-\gamma_n)\left(\|\kappa_n - p\|^2 - 2\alpha_n\langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle + \alpha_n^2\|\kappa_{n-1} - \kappa_n\|^2\right) \\ &\quad + \gamma_n\left(\|\kappa_n - p\|^2 - 2\beta_n\langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle + \beta_n^2\|\kappa_{n-1} - \kappa_n\|^2\right) \\ &= \|\kappa_n - p\|^2 - 2(\alpha_n(1-\gamma_n) + \gamma_n\beta_n)\langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle \\ &\quad + \left[(1-\gamma_n)\alpha_n^2 + \gamma_n\beta_n^2\right]\|\kappa_{n-1} - \kappa_n\|^2.\end{aligned} \quad (20)$$

It can be easily obtained $\Omega \subset C_1 = Y$. For some $n \ge 1$, assume that $\Omega \subset C_n$, then $p \in C_n$, and by (20), we conclude that $p \in C_{n+1}$. Therefore, $\Omega \subset C_{n+1}$ for all $n \ge 1$, and this finishes the requirement of Claim 1.

Step (ii). Prove that the boundedness of $\{\kappa_n\}$. Because Ω is a NCC subset of Y, there is a unique $u \in \Omega$ so that $u = P_\Omega \kappa_1$.

From $\kappa_n = P_{C_n}\kappa_1, C_{n+1} \subset C_n$, and $\kappa_{n+1} \in C_n$ for all $n \ge 1$, we get

$$\|\kappa_n - \kappa_1\| \le \|\kappa_{n+1} - \kappa_1\|, \quad \text{for all } n \ge 1. \quad (21)$$

Also, since $\Omega \subset C_n$, we get

$$\|\kappa_n - \kappa_1\| \le \|u - \kappa_1\|, \quad \text{for all } n \ge 1. \quad (22)$$

By (21) and (22), we obtain that $\lim_{n\to\infty}\|\kappa_n - \kappa_1\|$ exists; this leads to $\{\kappa_n\}$ being bounded.

Step (iii). Prove that $\kappa_n \longrightarrow \Theta$ as $n \longrightarrow \infty$, for some $\Theta \in Y$. By the structure of C_n, for $m > n$, one sees that $\kappa_m = P_{C_m}\kappa_1 \in C_m \subset C_n$. From Lemma 5, we have

$$\|\kappa_m - \kappa_n\|^2 \le \|\kappa_m - \kappa_1\|^2 - \|\kappa_n - \kappa_1\|^2. \quad (23)$$

From Step (ii), we obtain that $\|\kappa_m - \kappa_n\|^2 \longrightarrow 0$ as $m, n \longrightarrow \infty$. This proves that the sequence $\{\kappa_n\}$ is a Cauchy. Therefore, $\kappa_n \longrightarrow \Theta$ as $n \longrightarrow \infty$. In particular, we can obtain

$$\lim_{n\to\infty}\|\kappa_{n+1} - \kappa_n\| = 0. \quad (24)$$

Step (iv). Prove that $\Theta \in \Omega$. Because $\{\alpha_n\}$ and $\{\beta_n\}$ are bounded, then by (24), we have

$$\|\omega_n - \kappa_n\| = |\alpha_n|\|\kappa_n - \kappa_{n-1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \quad (25)$$

$$\|\upsilon_n - \kappa_n\| = |\beta_n|\|\kappa_n - \kappa_{n-1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (26)$$

From (24), (25), and (26), we have

$$\|\kappa_{n+1} - \omega_n\| \le \|\kappa_{n+1} - \kappa_n\| + \|\omega_n - \kappa_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \quad (27)$$

$$\|\kappa_{n+1} - \upsilon_n\| \le \|\kappa_{n+1} - \kappa_n\| + \|\upsilon_n - \kappa_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (28)$$

Let $\{\alpha_n\}$,$\{\beta_n\}$, and $\{\gamma_n\}$ be sequences of real numbers. Select initial $\kappa_0, \kappa_1 \in Y$.
Step (1). Compute
$$\omega_n = \kappa_n + \alpha_n(\kappa_n - \kappa_{n-1}),$$
$$\upsilon_n = \kappa_n + \beta_n(\kappa_n - \kappa_{n-1}).$$
Step (2). Compute
$$\mu_n = (1-\gamma_n)\omega_n + \gamma_n J^B_{\tau_n}(\upsilon_n - \tau_n A\upsilon_n).$$
Step (3). Compute
$$C_n = \{p \in Y : \|\mu_n - p\|^2 \le \|\kappa_n - p\|^2 - 2(\alpha_n(1-\gamma_n) + \gamma_n\beta_n)\langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle + (\alpha_n^2(1-\gamma_n) + \gamma_n\beta_n^2)\|\kappa_{n-1} - \kappa_n\|^2\},$$
$$Q_n = \{p \in Y : \langle p - \kappa_n, \kappa_1 - \kappa_n\rangle \le 0\},$$
$$\kappa_{n+1} = P_{C_n\cap Q_n}(\kappa_1), n \ge 1.$$

ALGORITHM 2: CQ algorithm.

Since $\kappa_{n+1} \in C_{n+1}$, we obtain that

$$\begin{aligned}\|\mu_n - \kappa_{n+1}\| &\le \|\kappa_n - \kappa_{n+1}\|^2 - 2(\alpha_n(1-\gamma_n) + \gamma_n\beta_n)\langle \kappa_n - \kappa_{n+1}, \kappa_{n-1} - \kappa_n\rangle \\ &+ \left[(1-\gamma_n)\alpha_n^2 + \gamma_n\beta_n^2\right]\|\kappa_{n-1} - \kappa_n\|^2 \le \|\kappa_n - \kappa_{n+1}\|^2 \\ &+ 2|\alpha_n(1-\gamma_n) + \gamma_n\beta_n|\|\kappa_n - \kappa_{n+1}\|\|\kappa_{n-1} - \kappa_n\| \\ &+ \left[(1-\gamma_n)\alpha_n^2 + \gamma_n\beta_n^2\right]\|\kappa_{n-1} - \kappa_n\|^2.\end{aligned} \tag{29}$$

Thus, from the boundedness of $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ and (24), (29), we obtain that

$$\|\mu_n - \kappa_{n+1}\| \longrightarrow 0. \tag{30}$$

By (24), (30), (27), and (28) and the following inequalities

$$\begin{aligned}\|\mu_n - \kappa_n\| &\le \|\mu_n - \kappa_{n+1}\| + \|\kappa_n - \kappa_{n+1}\|, \\ \|\mu_n - \omega_n\| &\le \|\mu_n - \kappa_{n+1}\| + \|\omega_n - \kappa_{n+1}\|, \\ \|\mu_n - \upsilon_n\| &\le \|\mu_n - \kappa_{n+1}\| + \|\upsilon_n - \kappa_{n+1}\|,\end{aligned} \tag{31}$$

we get that

$$\|\mu_n - \kappa_n\| \longrightarrow 0, \|\mu_n - \omega_n\| \longrightarrow 0, \|\mu_n - \upsilon_n\| \longrightarrow 0. \tag{32}$$

We now have

$$\begin{aligned}\left\|T_{\tau_n}\upsilon_n - \upsilon_n\right\| &= \left\|\frac{1}{\gamma_n}[\mu_n - (1-\gamma_n)\omega_n] - \upsilon_n\right\| \\ &= \frac{1}{\gamma_n}\|\gamma_n\upsilon_n + (1-\gamma_n)\omega_n - \mu_n\| \\ &= \frac{1}{\gamma_n}\|\gamma_n(\upsilon_n - \mu_n) + (1-\gamma_n)(\omega_n - \mu_n)\| \\ &\le \frac{1}{\gamma_n}[\gamma_n\|\upsilon_n - \mu_n\| + (1-\gamma_n)\|\omega_n - \mu_n\|].\end{aligned} \tag{33}$$

Again using the stipulation (i) and (32), we get

$$\lim_{n\to\infty}\left\|T_{\tau_n}\upsilon_n - \upsilon_n\right\| = 0. \tag{34}$$

As $\lim\inf_{n\to\infty}\tau_n > 0$, there is $\varepsilon > 0$ so that $\tau_n \ge \varepsilon$ and $\varepsilon \in (0, 2\alpha)$ for all $n \ge 1$. Hence, from Lemma 8 (ii) and (34), one can write

$$\|T_\varepsilon\upsilon_n - \upsilon_n\| \le 2\left\|T_{\tau_n}\upsilon_n - \upsilon_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{35}$$

Based on (32), as $\kappa_n \longrightarrow \Theta$, we get $\upsilon_n \longrightarrow \Theta$. Because T_ε is NE, T_ε is continuous mapping. Hence, using (35), we have $\Theta \in \Omega$.

Step (v). Prove that $\Theta = P_\Omega(\kappa_1)$. Since $\kappa_n = P_{C_n}\kappa_1$, and $\Omega \subset C_n$, we obtain that

$$\langle \kappa_1 - \kappa_n, \kappa_n - p\rangle \ge 0, \quad \forall p \in \Omega. \tag{36}$$

By taking the limit in (36), we have

$$\langle \kappa_1 - \Theta, \Theta - p\rangle \ge 0, \quad \forall p \in \Omega. \tag{37}$$

This shows that $\Theta = P_\Omega\kappa_1$ and this finishes the requirement.

Next, we shall discuss the strong convergence of Algorithm 2: by presenting the following theorem.

Theorem 11. *Let the sequence $\{\alpha_n\}$, $\{\beta_n\}$ be bounded and $\{\gamma_n\}$ be a sequence in $(0, 1]$ and $\{\tau_n\}$ be a sequence of positive real numbers so that the following two stipulations hold:*

(i) $\inf_n\{\gamma_n\} \ge \gamma > 0$

(ii) $0 < \inf_n\{\tau_n\} \le \sup_n\{\tau_n\} < 2\alpha$

If $\Omega \ne \varnothing$, then the sequence $\{\kappa_n\}$ marked by Algorithm 2: converges strongly to $\Theta = P_\Omega(\kappa_1)$.

Proof. In the same way as proving Theorem 10, we will discuss the following steps:

Step (i). Prove that for all $\kappa_1 \in Y$, $\Omega \subset Q_n \cap C_n$ and for each $n \ge 0.\{\kappa_n\}_{n=0}^\infty$ is well-defined.

It is clear that C_n is closed and a convex subset of Y (by Lemma 4). So, we can rewrite the set Q_n in the shape

$$Q_n = \{p \in Y : \langle \kappa_1 - \kappa_n, p\rangle \le \langle \kappa_1 - \kappa_n, \kappa_n\rangle\}. \tag{38}$$

It follows that Q_n is closed and convex subset of Y too. Thus, $Q_n \cap C_n$ is also closed and convex, for each $n \geq 0$.

Let $p \in \Omega$. One can obtain by similar way of Theorem 10 that

$$\begin{aligned}\|\mu_n - p\|^2 \leq \|\kappa_n - p\|^2 - 2(\alpha_n(1-\gamma_n) + \gamma_n\beta_n)\langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle \\ + \left[(1-\gamma_n)\alpha_n^2 + \gamma_n\beta_n^2\right]\|\kappa_{n-1} - \kappa_n\|^2.\end{aligned} \tag{39}$$

Hence, $p \in C_n$ for all $n \geq 1$; this implies that $\Omega \subset C_n$.

When $n = 1$, we get $Q_1 = Y$, and hence, $\Omega \subset C_1 \cap Q_1$. Suppose that $\Omega \subset C_l \cap Q_l$ for some $l \geq 1$. It follows from $\kappa_{l+1} = P_{C_l \cap Q_l}(\kappa_1)$ that

$$\langle \kappa_{l+1} - a, \kappa_1 - \kappa_{l+1}\rangle \geq 0, \tag{40}$$

for each $a \in C_l \cap Q_l$. Since $\Omega \subseteq C_l \cap Q_l$, and $p \in \Omega$, we have

$$\langle \kappa_{l+1} - p, \kappa_1 - \kappa_{l+1}\rangle \geq 0. \tag{41}$$

This yields $p \in Q_{l+1}$, and hence, $\Omega \subseteq Q_{l+1}$. This implies that $\Omega \subseteq C_{l+1} \cap Q_{l+1}$, and hence, $\{\kappa_n\}$ is well-defined as well as $\Omega \subset C_n \cap Q_n$.

Step (ii). Prove that the boundedness of $\{\kappa_n\}$. Based on Algorithm 2:, one gets

$$\langle \xi - \kappa_n, \kappa_1 - \kappa_n\rangle \leq 0, \quad \forall \xi \in Q_n, n \geq 1. \tag{42}$$

This implies that $\kappa_n = P_{Q_n}(\kappa_1)$. Since $\Omega \subset Q_n$, we have

$$\|\kappa_n - \kappa_1\| \leq \|\kappa_1 - \xi\|, \quad \forall \xi \in \Omega. \tag{43}$$

Also, since $\kappa_{n+1} \in Q_n$, we can write

$$\|\kappa_n - \kappa_1\| \leq \|\kappa_{n+1} - \kappa_1\|. \tag{44}$$

By (43) and (44), we obtain $\lim_{n\to\infty}\|\kappa_n - \kappa_1\|$ exists, and hence, $\{\kappa_n\}$ is bounded.

Step (iii). Prove that $\|\kappa_{n+1} - \kappa_n\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Because $\kappa_{n+1} \in Q_n$ and $\kappa_n = P_{Q_n}(\kappa_1)$, it follows from Lemma 5 that

$$\|\kappa_{n+1} - \kappa_n\|^2 \leq \|\kappa_{n+1} - \kappa_1\|^2 - \|\kappa_n - \kappa_1\|^2 \longrightarrow 0. \tag{45}$$

This implies that $\|\kappa_{n+1} - \kappa_n\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Step (iv). Prove that $\Theta = P_\Omega(\kappa_1)$. In the same manner as the proof of Step (iv) of Theorem 10, we can write

$$\|T_\varepsilon \nu_n - \nu_n\| \longrightarrow 0, \|\nu_n - \kappa_n\| \longrightarrow 0, \tag{46}$$

where $\varepsilon \in (0, 2\alpha)$. It follows from the nonexpansivity of T_ε that

$$\begin{aligned}\|T_\varepsilon\kappa_n - \kappa_n\| &= \|T_\varepsilon\kappa_n - T_\varepsilon\nu_n\| + \|T_\varepsilon\nu_n - \nu_n\| + \|\nu_n - \kappa_n\| \\ &\leq 2\|\nu_n - \kappa_n\| + \|T_\varepsilon\nu_n - \nu_n\|.\end{aligned} \tag{47}$$

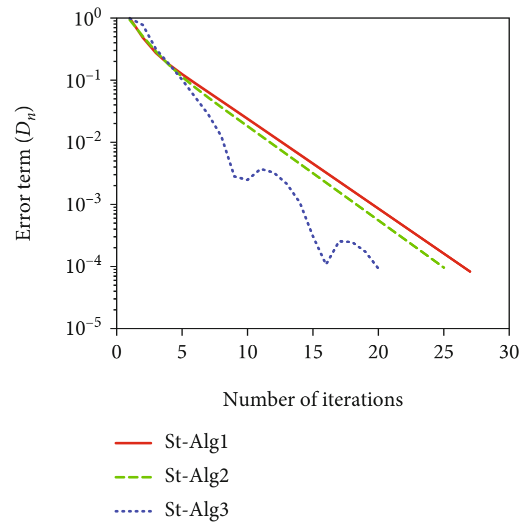

Figure 1: Numerical computational behavior of Algorithm 1:, Algorithm 2:, and Algorithm (4.2) while $\kappa_0 = \kappa_1 = 2t + 3$.

From (46) and (47), we can get

$$\|T_\varepsilon\kappa_n - \kappa_n\| \longrightarrow 0. \tag{48}$$

By the boundedness of $\{\kappa_n\}$, there exists a subsequence $\{\kappa_{n_k}\}$ of $\{\kappa_n\}$ so that $\kappa_{n_k} \rightharpoonup \kappa^*$. This combines with (48), and from Lemma 6, we have $\kappa^* \in F(T_\varepsilon)$; this means that $\kappa^* \in \Omega$.

Since $\Theta = P_\Omega(\kappa_1)$ and $\kappa^* \in \Omega$, (43) and Lemma 3 (ii) imply that

$$\begin{aligned}\|\kappa_1 - \Theta\| \leq \|\kappa_1 - \kappa^*\| &\leq \liminf_{k\to\infty}\left\|\kappa_{n_k} - \kappa_1\right\| \\ &\leq \limsup_{k\to\infty}\left\|\kappa_{n_k} - \kappa_1\right\| \leq \|\kappa_1 - \Theta\|.\end{aligned} \tag{49}$$

Since the nearest point Θ is unique, then we have $\Theta = \kappa^*$. Also, we get $\|\kappa_{n_k} - \kappa_1\| \longrightarrow \|\kappa_1 - \Theta\|$. Applying Lemma 3 (i), we have $\kappa_{n_k} \longrightarrow \Theta$ as $k \longrightarrow \infty$. Again, the uniqueness of Θ leads to $\kappa_n \longrightarrow \Theta$ as $n \longrightarrow \infty$.

This finishes the requirement.

4. Application to Solve Split Convex Feasibility Problem

This part is devoted to applying our methods to find a solution to the SCFP. Assume that $T : Y_1 \longrightarrow Y_2$ is a bounded linear operator and T^* its adjoint defined on real HSs Y_1 and Y_2. Let $C \subset Y_1$ and $Q \subset Y_2$ be a NCC sets. Censor and Elfving [37] formulated the SCFP as follows:

$$\text{Find } \bar{\kappa} \in C \text{ so that } T(\bar{\kappa}) \in Q. \tag{50}$$

Censor and Elfving in [37] have introduced the SCFP in HSs while using a multidistance approach to find an adaptive approach for resolving it. Many of the problems that emerge from state retrieval and restoration of medical image can be formulated as split variational feasibility problems [38, 39].

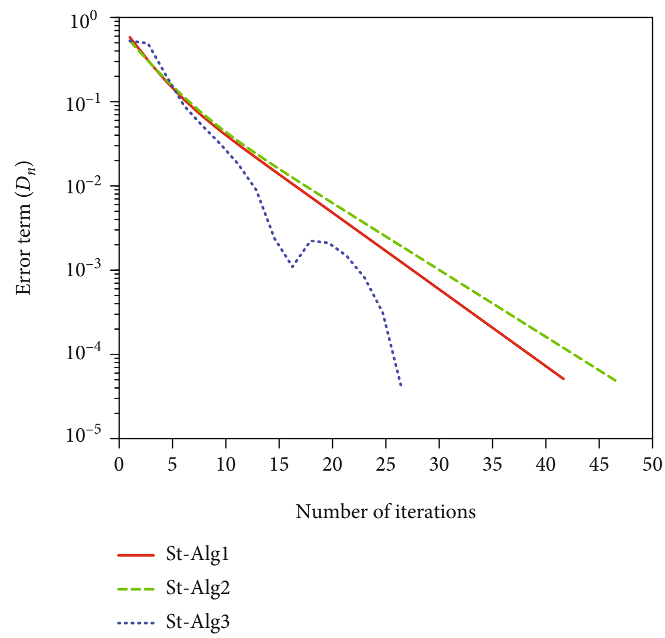

FIGURE 2: Numerical computational behavior of Algorithm 1:, Algorithm 2:, and Algorithm (4.2) while $\kappa_0 = \kappa_1 = (5 + t^2)/5$.

This problem is also used in a variety of disciplines like image restoration, dynamic emission tomographic image reconstruction, and radiation therapy treatment planning [40–42]. Let consider

$$A(\kappa) = \nabla\left(\frac{1}{2}\left\|T\kappa - P_Q(T\kappa)\right\|^2\right) = T^*\left(I - P_Q\right)T\kappa, \quad (51)$$

where P_Q is the metric projection on to Q, ∇ is the gradient, and $B = \partial i_C$. Due to the above construction, the problem (50) has an inclusion format as described in (2). It can be seen that$A$ is Lipschitz which continues with constant $L = \|T\|^2$ and B is MM, see, for example, [43].

For any NCC subset C of a real HS Y, the indicator function i_C of C is defined by

$$i_C(\kappa) = \begin{cases} 0, & \text{if } \kappa \in C, \\ \infty, & \text{otherwise}. \end{cases} \quad (52)$$

Now, on the basis of the main results, we can deduce the following results for a SCFP.

Theorem 12. *Let $\{\kappa_n\}$ be a sequence iterated as follows: choose initial points $\kappa_0, \kappa_1 \in C_1 = Y$ and let $\{\alpha_n\}$, $\{\beta_n\}$ be bounded sequences and $\{\gamma_n\}$ be a sequence in $(0, 1]$. Consider that $\{\tau_n\}$ is a sequence of positive real numbers so that the following assumptions are fulfilled:*

(i) $\inf_n\{\gamma_n\} \geq \gamma > 0$

(ii) $0 < \inf_n\{\tau_n\} \leq \sup_n\{\tau_n\} < 2\alpha$

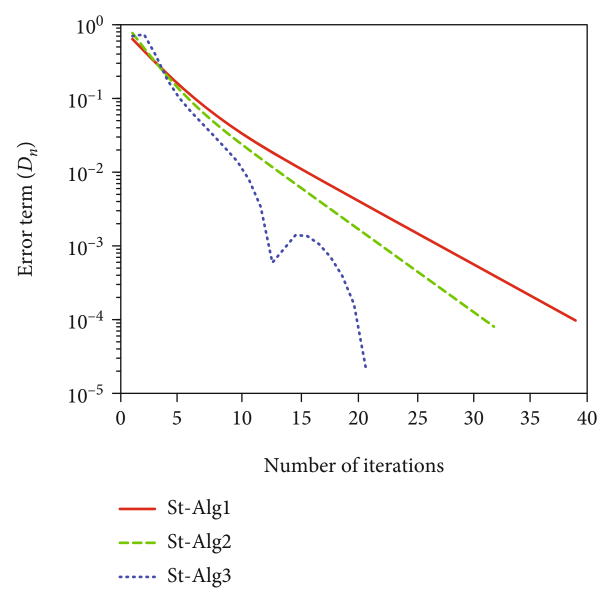

FIGURE 3: Numerical computational behavior of Algorithm 1:, Algorithm 2:, Algorithm (4.2) while $\kappa_0 = \kappa_1 = 3e^{2t}t^4$.

Step (1). Compute

$$\begin{aligned} \omega_n &= \kappa_n + \alpha_n(\kappa_n - \kappa_{n-1}), \\ \upsilon_n &= \kappa_n + \beta_n(\kappa_n - \kappa_{n-1}). \end{aligned} \quad (53)$$

Step (2). Compute

$$\mu_n = (1 - \gamma_n)\omega_n + \gamma_n P_C\left[\upsilon_n - \tau_n T^*\left(I - P_Q\right)T\upsilon_n\right]. \quad (54)$$

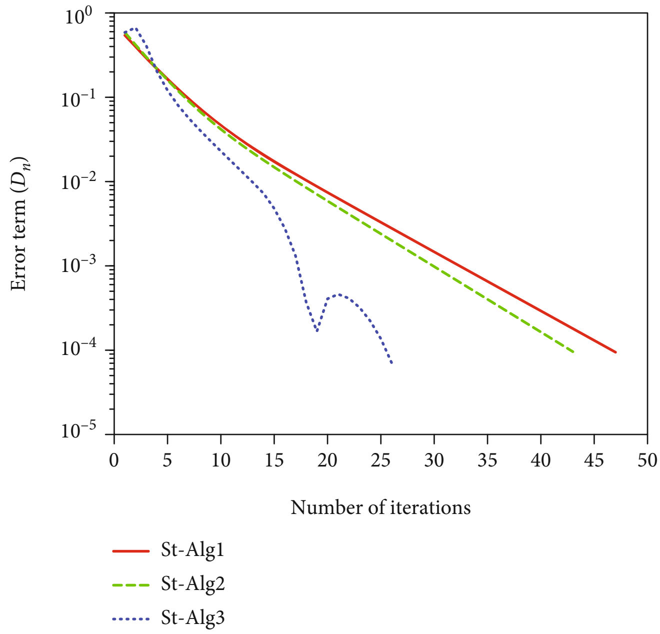

FIGURE 4: Numerical computational behavior of Algorithm 1:, Algorithm 2:, and Algorithm (4.2) while $\kappa_0 = \kappa_1 = 3e^t \sin(t)$.

Step (3). Compute

$$C_{n+1} = \{p \in C_n : \|\mu_n - p\|^2 \le \|\kappa_n - p\|^2 - 2(\alpha_n(1-\gamma_n) + \gamma_n\beta_n) \cdot \langle \kappa_n - p, \kappa_{n-1} - \kappa_n \rangle + (\alpha_n^2(1-\gamma_n) + \gamma_n\beta_n^2)\|\kappa_{n-1} - \kappa_n\|^2\},$$
$$\kappa_{n+1} = P_{C_{n+1}}(\kappa_1), \quad n \ge 1. \tag{55}$$

If the solution set Γ_{SCFP} is nonempty, then the sequence $\{\kappa_n\}$ converges weakly to an element of solution set Γ_{SCFP}.

Theorem 13. *Let $\{\kappa_n\}$ be a sequence iterated as follows: choose initial points $\kappa_0, \kappa_1 \in C_1 = Y$ and let $\{\alpha_n\}$, $\{\beta_n\}$ be bounded sequences and $\{\gamma_n\}$ be a sequence in $(0, 1]$. Consider that $\{\tau_n\}$ is a sequence of positive real numbers so that the following assumptions are fulfilled:*

(i) $\inf_n\{\gamma_n\} \ge \gamma > 0$

(ii) $0 < \inf_n\{\tau_n\} \le \sup_n\{\tau_n\} < 2\alpha$

Step (1). Compute

$$\omega_n = \kappa_n + \alpha_n(\kappa_n - \kappa_{n-1}),$$
$$\upsilon_n = \kappa_n + \beta_n(\kappa_n - \kappa_{n-1}). \tag{56}$$

Step (2). Compute

$$\mu_n = (1-\gamma_n)\omega_n + \gamma_n P_C[\upsilon_n - \tau_n T^*(I - P_Q)T\upsilon_n]. \tag{57}$$

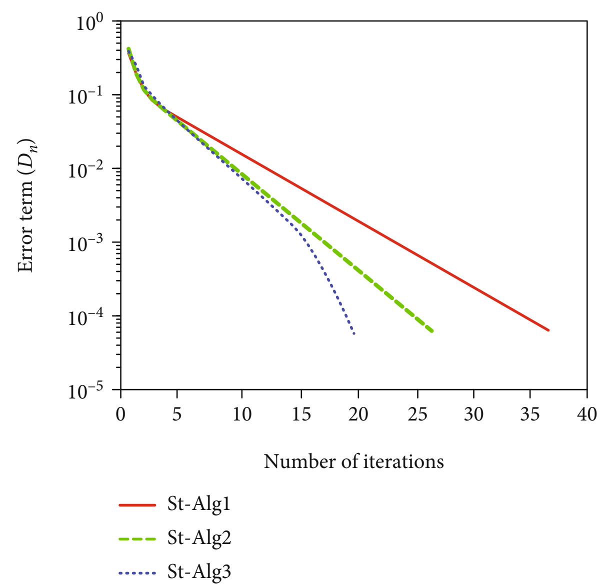

FIGURE 5: Numerical computational behavior of Algorithm 1:, Algorithm 2:, and Algorithm (4.2) while $\kappa_0 = \kappa_1 = (2t^3 - 3e^t)\cos(t)$.

Step (3). Compute

$$C_n = \{p \in Y : \|\mu_n - p\|^2 \le \|\kappa_n - p\|^2 - 2(\alpha_n(1-\gamma_n) + \gamma_n\beta_n) \cdot \langle \kappa_n - p, \kappa_{n-1} - \kappa_n \rangle + (\alpha_n^2(1-\gamma_n) + \gamma_n\beta_n^2)\|\kappa_{n-1} - \kappa_n\|^2\},$$
$$Q_n = \{p \in Y : \langle p - \kappa_n, \kappa_1 - \kappa_n \rangle \le 0\}, \kappa_{n+1} = P_{C_n \cap Q_n}(\kappa_1), \quad n \ge 1. \tag{58}$$

If the solution set Γ_{SCFP} is nonempty, then the sequence $\{\kappa_n\}$ converges weakly to an element of solution set Γ_{SCFP}.

Table 1: Numerical computational results of Algorithm 1:, Algorithm 2:, and Algorithm (4.2).

	Number of iterations			Execution time in seconds		
Initial point ($\kappa_0 = \kappa_1$)	St-Alg1	St-Alg2	St-Alg3	St-Alg1	St-Alg2	St-Alg3
$2t + 3$	27	25	20	0.0497	0.0184	0.0184
$5 + t^2/5$	25	28	16	0.0497	0.0184	0.0184
$3e^{2t}t^4$	39	32	21	0.0497	0.0184	0.0184
$3e^t \sin(t)$	47	43	26	0.0497	0.0184	0.0184
$(2t^3 - 3e^t)\cos(t)$	55	40	30	0.0497	0.0184	0.0184

5. Supportive Numerical Examples

This section is the mainstay of the paper as it studies the behavior and performance of our algorithms numerically and graphically. The program used here is MATLAB R2014a running on an HP Compaq 510, Core™ 2 Duo CPU T5870 with 2.0 GHz and 2 GB RAM.

Example 14. Let $Y_1 = Y_2 = L_2([0, 2\pi])$ be two HSs with an inner product

$$\langle x, y \rangle := \int_0^{2\pi} x(t)y(t)dt, \quad \forall x, y \in L_2([0, 2\pi]), \tag{59}$$

and the induced norm defined by

$$\|x\| := \sqrt{\int_0^{2\pi} |x(t)|^2 dt}, \quad \forall x \in L_2([0, 2\pi]). \tag{60}$$

Next, consider the feasible set $C \subset Y_1$ as

$$C = \left\{ x \in Y_1 : \int_0^{2\pi} x(t)dt \leq 1 \right\}, \tag{61}$$

and $Q \subset Y_2$ is

$$Q = \left\{ x \in Y_2 : \int_0^{2\pi} |x(t) - \sin(t)|^2 dt \leq 16 \right\}. \tag{62}$$

Consider the mapping $T : Y_1 \longrightarrow Y_2$ so that $(Tx)(s) = x(s), x \in Y_1$. Then, $(T^* x)(s) = x(s)$ and $\|T\| = 1$. So, we wish to solve the following problem:

$$\text{Find } \bar{\kappa} \in C \text{ so that } T(\bar{\kappa}) \in Q. \tag{63}$$

We can also observe that since $(Tx)(s) = x(s), x \in Y_1$, the above problem is actually a SCFP in the form of

$$\text{Find } \bar{\kappa} \in C \cap Q. \tag{64}$$

Figures 1–5 and Table 1 show the numerical computational results of Algorithm (4.2) by Dong et al. in [30] (St-Alg1), Algorithm 1: (St-Alg2), and Algorithm 2: (St-Alg3) by assuming $D_n = \|\kappa_n - \kappa_{n-1}\| \leq 10^{-4}$.

Remark 15. It is important to note that the different choices of initial points have substantial effect on the CPU (time) and a number of iterations on the proposed algorithms and also for those existing algorithms that are used for comparison. These facts can be seen from Figures 1–5 and Table 1. We conclude from these numerical results that our algorithms are faster in convergence than their counterpart presented by Dong et al. in [30].

6. Conclusion

The quality of the algorithm is measured by two main factors: velocity in convergence and time. When the convergence is faster in a short time, the results are faster and more accurate. Given the importance of algorithms in many applications in real society, many researchers have studied this logic and try to obtain a strong convergence, which has a prominent role in studying the efficiency and effectiveness of these algorithms. On the basis of this principle, in this paper was studied the effect of shrinking projection and CQ term on two inertial terms to get the strong convergence of new algorithms called two inertial shrinking projection and CQ algorithms. These results have been implicated to obtain a solution to SCFP in HSs. Finally, some numerical results were formulated to illustrate the efficiency and effectiveness of algorithms.

Abbreviations

HSs: Hilbert spaces
NCC: Nonempty closed convex
NEMs: Nonexpansive mappings
FPs: Fixed points
SCFPs: Split convex feasibility problems
ISM: Inverse strongly monotone
MM: Maximal monotone.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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