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Research Article

Three-Dimensional Laplace Adomian Decomposition Method and Singular Pseudoparabolic Equations

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In this work, the solution of the linear, nonlinear, and coupled system fractional singular two-dimensional pseudoparabolic equation is examined by using a three-dimensional Laplace Adomian decomposition method (3-DLADM). Analysis of the method is discussed, and some demonstrative examples are mentioned to confirm the power and accuracy of the recommended method, and numerical analysis is applied to sketch the exact and approximate solution.

1. Introduction

The fractional derivative has been attracting much attention in physical and engineering problems, for instance, acoustics, viscoelasticity, and control. The sufficient condition for commutators of a fractional integral operator is discussed in [1]. The authors in [2] addressed the boundedness of commutators of the multidimensional Hardy-type operators with bounded mean oscillation coefficients. The fractional derivative of the Riemann zeta function was computed, using the Caputo derivative in the Ortigueira sense (for more details, see [3]). The author in [4] analyzed the fractional derivative of the Riemann function and discussed the functional equation with the distribution of prime numbers.

The parabolic equation occurred in several fields of applied mathematics, for example, the heat diffusion equation and fluid mechanics (for a model, see [5–8]). The solution fractional diffusion equation problems have been obtained via the Adomian decomposition method and series expansion method by the authors [9, 10]. Several articles have been found in previous studies, which are associated with qualities and applications of a fractional derivative [11–13]. The pseudoparabolic equation represents a diversity of physical operation. The author in [14] discussed the existence, uniqueness, and continuous dependence of powerful solutions of the one-dimensional pseudoparabolic equation. Overall, certainty of the nonlinear equations of real life is so far very hard to solve either theoretically or numerically. Currently, many researchers have suggested an exact solution to a one-dimensional coupled parabolic equation (for more details, see [15, 16]). The convergence of the Adomian method was discussed by many researchers (we refer the readers to see [17–20]). The author has modified the 2-D Laplace decomposition method to solve coupled pseudoparabolic equations in order to accelerate the convergence of the series solution [21].

Recently, the three-dimensional Laplace Adomian decomposition (3-DLADM) has been successfully applied to solve regular and singular coupled Burgers' equations (see [22]). The objective of this research is to find the solution of singular 2-D fractional pseudoparabolic equations by applying a more successful technique, which is called 3-DLADM.

2. Some Basic Idea of the Double and Triple Laplace Transform and Caputo Fractional Derivative

In this unit, we offer basic definitions, properties of fractional calculus, Mittag-Leffler function, and Laplace transform theory which should be used in this work.

Definition 1 (see [23]). The left-sided Caputo fractional derivative of $g, g \in C_{-1}^{m}, m \in N \cup \{0\}$, is given by

$$D_{*}^{\mu} g(t) = \frac{\partial^{\mu} g(t)}{\partial t^{\mu}} = \begin{cases} I^{m-\mu}\left[\dfrac{\partial^{m} g(t)}{\partial t^{m}}\right], m-1<\mu<m, m \in N, \\ \dfrac{\partial^{m} g(t)}{\partial t^{m}}, \mu = m. \end{cases} \quad (1)$$

Definition 2 (see [24]). The Caputo fractional partial derivative of function $f(x_1, t)$ with respect to t is given by

$$\frac{\partial^{\zeta} f(x_1,t)}{\partial t^{\zeta}} = \frac{1}{\Gamma(n-\zeta)} \int_0^t (t-\nu)^{n-\zeta-1} \frac{\partial^{n} f(x_1,\nu)}{\partial \nu^{n}} d\nu, \quad n-1<\zeta<n. \quad (2)$$

Definition 3 (see [25]). Let $g(x_1, x_2)$ be a continuous function of two variables x_1, x_2; then, the two-dimensional Laplace transform (2-DLT) of $g(x_1, x_2)$ is given

$$L_2[g(x_1,x_2)] = G(p_1,p_2) = \int_0^{\infty}\int_0^{\infty} e^{-p_1x_1-p_2x_2} g(x_1,x_2) dx_1\, dx_2, \quad (3)$$

where $x_1, x_2 > 0$, L_2 indicate 2-DLT, and p_1, p_2 are complex variables.

Definition 4 (see [26]). Let $g(x_1, x_2, t)$ be a piecewise continuous function on the interval $[0,\infty) \times 0,\infty) \times 0,\infty)$ of exponential order. Consider for some $a, b, c \in \mathbb{R}\sup x_1, x_2, t > 0$, $|\psi(x_1,x_2,t)|/e^{ax_1+bx_2+ct}$. Under these conditions, 3-DLT is defined by

$$\begin{aligned} L_3(g(x_1,x_2,t)) &= G(p_1,p_2,s) \\ &= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-p_1x_1-p_2x_2-st} g(x_1,x_2,t) dt dx_2 dx_1, \end{aligned} \quad (4)$$

where the symbol L_3 indicates 3-DLT and $p_1, p_2, s \in \mathbb{C}$.

Definition 5. The inverse 3-DLT of the function $F(p_1, p_2, s)$ is determined by

$$\begin{aligned} L_3^{-1}[G(p_1,p_2,s)] &= g(x_1,x_2,t) \\ &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{p_1x_1} dp_1 \frac{1}{2\pi i} \\ &\cdot \int_{c-i\infty}^{c+i\infty} e^{p_2x_2} dp_2 \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{st} G(p_1,p_2,s) ds, \end{aligned} \quad (5)$$

where L_3^{-1} indicate inverse 3-DLT with respect to p_1, p_2 and s.

Furthermore 3-DLT of the derivatives $\psi_{x_1}(x_1, x_2, t)$ and $\psi_t(x_1, x_2, t)$ are presented by

$$L_3\left[\psi_{x_1}(x_1,x_2,t)\right] = p_1\Psi(p_1,p_2,s) - \Psi(0,p_2,s), \quad (6)$$

$$L_3[\psi_t(x_1,x_2,t)] = s\psi(p_1,p_2,s) - \Psi(p_1,p_2,0). \quad (7)$$

The 2-DLT formulas for the partial fractional Caputo derivatives are denoted by

$$L_2\left(\frac{\partial^{\beta} f(x_1,t)}{\partial x^{\zeta}}\right) = p_1^{\beta} F(p_1,s) - \sum_{i=0}^{n-1} p_1^{\beta-1-i} L_t\left[\frac{\partial^{i}}{\partial x_1^{i}}\psi(0,t)\right], \quad (8)$$

$$L_2\left(\frac{\partial^{\zeta} f(x_1,t)}{\partial t^{\zeta}}\right) = s^{\zeta} F(p_1,s) - \sum_{i=0}^{n-1} s^{\zeta-1-i} L_{x_1}\left[\frac{\partial^{i}}{\partial t^{i}}\psi(x_1,0)\right], \quad (9)$$

where $\zeta, \beta > 0, m-1 < \zeta \le m, r-1 < \beta \le r, r, m \in N$ (for more details, see [27]).

In the following, we provided one and two parameters; the classical Mittag-Leffler function is useful in this work.

2.1. Mittag-Leffler Function. The Mittag-Leffler function of one parameter is established by

$$\Xi_{\eta}(\tau) = \sum_{i=0}^{\infty} \frac{\tau^{i}}{\Gamma(\eta i + 1)}, \quad \tau \in \mathbb{C}, \operatorname{Re}(\eta) > 0. \quad (10)$$

The Mittag-Leffler function with two parameters is determined by

$$\Xi_{\eta,\gamma}(\tau) = \sum_{i=0}^{\infty} \frac{\tau^{i}}{\Gamma(\eta i + \gamma)}, \quad \tau \in \mathbb{C}, \operatorname{Re}(\eta) > 0 \quad (11)$$

(see [28, 29]). If we set $\eta = 1$ in Equation (11), we get Equation (10). It appears from Equation (11) that

$$\Xi_{1,1}(\tau) = \sum_{i=0}^{\infty} \frac{\tau^{i}}{\Gamma(i+1)} = \sum_{i=0}^{\infty} \frac{\tau^{i}}{i!} = e^{\tau}, \quad (12)$$

$$\Xi_{1,2}(\tau)=\sum_{k=0}^{\infty}\frac{\tau^{i}}{\Gamma(i+2)}=\sum_{i=0}^{\infty}\frac{\tau^{i}}{(i+1)!}=\frac{1}{\tau}\sum_{k=0}^{\infty}\frac{\tau^{i+1}}{(i+1)}=\frac{e^{\tau}-1}{\tau}, \tag{13}$$

$$\Xi_{1,3}(\tau)=\sum_{i=0}^{\infty}\frac{\tau^{i}}{\Gamma(i+3)}=\sum_{i=0}^{\infty}\frac{\tau^{i}}{(i+2)!}=\frac{1}{\tau^{2}}\sum_{i=0}^{\infty}\frac{\tau^{i+2}}{(i+2)}=\frac{e^{\tau}-1-1}{\tau^{2}}; \tag{14}$$

hence, in general,

$$\Xi_{1,m}(\tau)=\frac{1}{\tau^{m-1}}\left[e^{\tau}-\sum_{i=0}^{m-2}\frac{\tau^{i}}{i!}\right]. \tag{15}$$

Differentiation of the Mittag-Leffler function is represented by

$$\frac{d^{n}}{dt^{n}}\left[t^{\eta-1}\Xi_{\eta,\zeta}\left(t^{\zeta}\right)\right]=t^{\eta-n-1}\Xi_{\eta,\zeta-n}\left(t^{\eta}\right) \tag{16}$$

(for more details, see [30]).

Next, we provide the Laplace transform (LT) of Mittag-Leffler functions helpful in this research:

$$\begin{aligned}
L_3\left[t^{\zeta}\Xi_{1,\zeta+1}(t)\right]&=\frac{1}{s^{\zeta}(s-1)},\\
L_3\left[t^{\zeta}\Xi_{1,\zeta+1}(t)\right]&=\frac{1}{s^{\zeta}(s-1)},\\
L_3\left[t^{2\zeta}\Xi_{1,2\zeta+1}(t)\right]&=\frac{1}{s^{2\zeta}(s-1)},\\
L_3\left[t^{\zeta-1}\Xi_{1,\zeta}(t)\right]&=\frac{1}{s^{\zeta-1}(s-1)},\\
L_3\left[t^{2\zeta-1}\Xi_{1,2\zeta}(\lambda t)\right]&=\frac{1}{s^{2\zeta-1}(s-\lambda)}.
\end{aligned} \tag{17}$$

In the same way, the single Laplace transform (LT) of two-parameter Mittag-Leffler functions

$$L_3\left[t^{\eta-1}\Xi_{\zeta,\eta}\left(\pm\lambda t^{\zeta}\right)\right]=\frac{s^{\zeta-\eta}}{\left(s^{\zeta}-\lambda\right)}. \tag{18}$$

3. Multidimensional Laplace Transforms ($n+1$-DLT)

Here, we deal with the multidimensional Laplace transform ($n+1$-DLT) which is very useful to this work.

Definition 6. Let $g(x_1,x_2,x_3,\cdots,x_n,t)$ be a piecewise continuous function on the interval $[0,\infty)\times 0,\infty)\times 0,\infty)\times\cdots\times[0,\infty)\times 0,\infty)$ of exponential order. Consider for some $a_1,a_2,a_3,\cdots,a_n,b\in\mathbb{R}\sup x_1,x_2,x_3,\cdots,x_n,t>0,|\psi(x_1,x_2,\ x_3,\cdots,x_n,t)|/e^{a_1x_1+a_2x_2+\cdots+a_nx_n+bt}$. Under these conditions, $n+1$-DLT is defined by

$$\begin{aligned}
L_nL_t(\xi)&=G(p_1,p_2,p_3,\cdots,p_n,s)\\
&=\int_0^{\infty}\cdots\int_0^{\infty}\int_0^{\infty}e^{-p_1x_1-p_2x_2\cdots-p_nx_n-st}\xi dx_1dx_2\cdots dx_ndt,
\end{aligned} \tag{19}$$

where $\xi=g(x_1,x_2,x_3,\cdots,x_n,t)$, the symbol L_nL_t indicate $n+1$-DLT, and $p_1,p_2,p_3,\cdots,p_n$ and s are complex variables.

Definition 7. The inverse $n+1$-DLT of the function $G(p_1,p_2,p_3,\cdots,p_n,s)$ is determined by

$$\begin{aligned}
L_n^{-1}L_s^{-1}[F]&=g(x_1,x_2,x_3,\cdots,x_n,t\mid)\\
&=\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}e^{p_1x_1}dp_1\cdots\frac{1}{2\pi i}\int_{c_n-i\infty}^{c_n+i\infty}e^{p_nx_n}dp_n\frac{1}{2\pi i}\\
&\quad\cdot\int_{d-i\infty}^{d+i\infty}e^{st}Fds,
\end{aligned} \tag{20}$$

where $F=G(p_1,p_2,p_3,\cdots,p_n,s)$ and $L_n^{-1}L_s^{-1}$ indicate inverse $n+1$-DLT with respect to $p_1,p_2,p_3,\cdots,p_n$ and s.

3.1. Existence Condition for the Multi-Laplace Transform. If $f(x_1,x_2,x_3,\cdots,x_n,t)$ is said to be of exponential order $a_1,a_2,a_3,\cdots,a_n$ (>0) and b (>0) on $0\le x_1,x_2,x_3,\cdots,x_n<\infty$, $0\le t<\infty$, if there exists a positive constant K such that for all $x_1>X_1,x_2>X_2,X_3>x_3,\cdots,x_n>X_n$ and $t>T$:

$$|f(x_1,x_2,x_3,\cdots,x_n,t)|\le Ke^{a_1x_1+a_2x_2+a_3x_3+\cdots a_nx_n+bt}, \tag{21}$$

where $f(x_1,x_2,x_3,\cdots,x_n,t)$ can be written as

$$f(x_1,x_2,x_3,\cdots,x_n,t)=O\left(e^{a_1x_1+a_2x_2+a_3x_3+\cdots a_nx_n+bt}\right), \tag{22}$$

at $x_1,x_2,x_3,\cdots,x_n\longrightarrow\infty$ and $t\longrightarrow\infty$. Or, equivalently,

$$\begin{aligned}
&\lim_{\substack{x_1,x_2,x_3,\cdots,x_n\to\infty\\ t\to\infty}}e^{-\zeta_1x_1-\zeta_2x_2-\zeta_3x_3-\cdots-\zeta_nx_n-\beta t}|f(x_{11},x_{12},x_{13},\cdots,x_{1n},t)|\\
&=k\lim_{\substack{x_1,x_2,x_3,\cdots,x_n\to\infty\\ t\to\infty}}e^{-(\zeta_1-a_1)x_1-(\zeta_2-a_2)x_2-(\zeta_3-a_3)x_3-\cdots-(\zeta_n-a_n)x_n-(\beta-b)t}\\
&=0,\zeta_1>a_1,\zeta_2>a_2,\zeta_3>a_3\cdots\zeta_n>a_n,\beta>b,
\end{aligned} \tag{23}$$

where the function $f(x_1,x_2,x_3,\cdots,x_n,t)$ is simply called an exponential order as $x_1,x_2,x_3,\cdots,x_n\longrightarrow\infty$, $t\longrightarrow\infty$, and clearly, it does not grow faster than $K e^{a_1x_1+a_2x_2+a_3x_3+\cdots a_nx_n+bt}$ as $x_1,x_2,x_3,\cdots,x_n\longrightarrow\infty$, $t\longrightarrow\infty$.

Theorem 8. *If a function $f(x_1,x_2,x_3,\cdots,x_n,t)$ is a continuous function in every finite intervals $(0,X_1),(0,X_2),(0,X_3),\cdots,(0,X_n)$ and $(0,T)$ and of exponential order*

$e^{a_1x_1+a_2x_2+a_3x_3+\cdots a_nx_n+bt}$, *then the double Laplace transform of* $f(x_1, x_2, x_3, \cdots, x_n, t)$ *exists for all* $p_1, p_2, p_3, \cdots, p_n$ *and s provided* $\operatorname{Re} p_1 > a_1$, $\operatorname{Re} p_2 > a_2$, $\operatorname{Re} p_3 > a_3, \cdots$, $\operatorname{Re} p_n > a_n$ *and* $\operatorname{Re} s > b$.

Proof. We have

$$\begin{aligned}|F(p_1, p_2, p_3, \cdots, p_n, s)| &= \left|\int_0^\infty \cdots \int_0^\infty \int_0^\infty e^{-p_1x_1-p_2x_2-\cdots-p_nx_n-st} f(x_1, x_2, x_3, \cdots, x_n, t)dx_1dx_2 \cdots dx_ndt\right| \\ &\leq \int_0^\infty \cdots \int_0^\infty e^{-(p_1-a_1)x_1-(p_2-a_2)x_2-(p_3-a_3)x_3-\cdots-(p_n-a_n)x_n-(s-b)t}dx_1dx_2 \cdots dx_ndt \\ &= \frac{K}{(p_1-a_1)(p_2-a_2)(p_3-a_3)\cdots(p_n-a_n)(s-b)}.\end{aligned} \tag{24}$$

For $\operatorname{Re} p_1 > a_1$, $\operatorname{Re} p_2 > a_2$, $\operatorname{Re} p_3 > a_3, \cdots$, $\operatorname{Re} p_n > a_n$ and $\operatorname{Re} s > b$.

In the following theorem, we generalize Equations (8) and (9) to a multidimensional Laplace transform.

Theorem 9. *Let* $\zeta, \beta > 0$, $m-1 < \zeta \leq m$, $r-1 < \beta \leq r$, $r, m \in N$, *be so that* $\psi \in C^l(\mathbb{R}^+ \times \mathbb{R}^+ \times \cdots \times \mathbb{R}^+)$, $l = \max\{m, r\}$, $\psi^{(l)} \in L_1((0, a_1) \times (0, a_2) \times (0, a_3) \times \cdots \times (0, b))$, *for any* $a_1, a_2, a_3, \cdots, a_n$ *(>0) and b (>0):*

$$|\psi(x_1, x_2, x_3, \cdots, x_n, t)| \leq Ke^{a_1x_1+a_2x_2+a_3x_3+\cdots a_nx_n+bt}, x_1 > a_1 > 0, x_2 > a_2 > 0, \cdots, t > b > 0, \tag{25}$$

hold for any constant $k, \tau_1, \tau_2, \tau_3 \cdots \tau_n \geq 0$, *then*

$$\begin{aligned}L_nL_t\left[D_t^\zeta \psi(x_1, x_2, x_3, \cdots, x_n, t)\right] &= s^\zeta(\Psi(p_1, p_2, p_3, \cdots, p_n, s)) - \sum_{i=0}^{m-1} s^{\zeta-1-i}L_n\left(\frac{\partial^i \psi(x_1, x_2, x_3, \cdots, x_n, 0)}{\partial t^i}\right), \\ L_nL_t\left[D_{x_1}^\beta \psi(x_1, x_2, x_3, \cdots, x_n, t)\right] &= p_1^\beta \Psi(p_1, p_2, p_3, \cdots, p_n, s) - \sum_{i=0}^{r-1} p_1^{\beta-1-i}L_{n-1}L_t\left(\frac{\partial^i \psi(0, x_2, x_3, \cdots, x_r, t)}{\partial x_1^i}\right), \\ L_nL_t\left[D_{x_2}^\beta \psi(x_1, x_2, x_3, \cdots, x_n, t)\right] &= p_2^\beta \Psi(p_1, p_2, p_3, \cdots, p_n, s) - \sum_{i=0}^{r-1} p_2^{\beta-1-i}L_{n-1}L_t\left(\frac{\partial^i \psi(x_1, 0, x_3, \cdots, x_r, t)}{\partial x_2^i}\right),\end{aligned} \tag{26}$$

$$L_nL_t\left[D_x^\beta \psi(x_1, x_2, x_3, \cdots, x_n, t)\right] = p_3^\beta \Psi(p_1, p_2, p_3, \cdots, p_n, s) - \sum_{i=0}^{r-1} p_3^{\beta-1-i}L_{n-1}L_t\left(\frac{\partial^i \psi(x_1, x_2, 0, \cdots, x_r, t)}{\partial x_2^i}\right), \tag{27}$$

where $\Psi(p_1, p_2, p_3, \cdots, p_n, s)$ *is* $n+1$*-DLT of* $\psi(x_1, x_2, x_3, \cdots, x_n, t)$.

Proof. With the use of Theorem 9 and definition of multidimensional Laplace transform, we can deduce the proof easily.

Theorem 10. *Let f be piecewise continuous on* $[0,\infty) \times 0,\infty) \times 0,\infty) \times \cdots \times [0,\infty) \times 0,\infty)$,$n+1$*-DLT of the partial derivatives of order* ζth $\prod_{i=1}^n x_i(\partial^\zeta \psi/\partial t^\zeta)$ *and* $\prod_{i=1}^n x_i f(x_1, x_2, x_3, \cdots, x_n, t)$ *are given by*

$$L_nL_t\left[\prod_{i=1}^n x_i \frac{\partial^\zeta}{\partial t^\zeta}\psi(x_1, x_2, x_3, \cdots, x_n, t)\right] = (-1)^n \frac{\partial^n}{\partial p_1 \partial p_2 \partial p_3 \cdots \partial p_n}[\Lambda], \tag{28}$$

where

$$\Lambda = s^\zeta \Psi(p_1, p_2, p_3, \cdots, p_n, s) \sum_{i=0}^{n-1} s^{\zeta-1-i}L_n\left[\frac{\partial^i}{\partial t^i}\psi(x_1, x_2, x_3, \cdots, x_n, 0)\right], \tag{29}$$

$$L_nL_t\left[\prod_{i=1}^n x_i f(xx_1, x_2, x_3, \cdots, x_n, t)\right] = (-1)^n \frac{\partial^n}{\partial p_1 \partial p_2 \partial p_3 \cdots \partial p_n}[F(p_1, p_2, p_3, \cdots, p_n, s)]. \tag{30}$$

Proof. By employing definition of n-DLT for $\partial^\zeta \psi/\partial t^\zeta$, we get

$$L_nL_t\left(\frac{\partial^\zeta \psi}{\partial t^\zeta}\right) = \int_0^\infty \cdots \int_0^\infty \int_0^\infty e^{-p_1x_1-p_2x_2\cdots-p_nx_n-st}\frac{\partial^\zeta \psi}{\partial t^\zeta} \cdot (x_1, x_2, x_3, \cdots, x_n, t)dx_{11}dx_{12} \cdots dx_{1n}dt\,; \tag{31}$$

by taking partial derivatives $\partial^n/\partial p_1 \partial p_2 \partial p_3 \cdots \partial p_n$ for both sides of Equation (31), we have

$$\frac{\partial^n}{\partial p_1 \partial p_2 \partial p_3 \cdots \partial p_n}\left(L_nL_t\left(\frac{\partial^\zeta \psi}{\partial t^\zeta}\right)\right) = \int_0^\infty e^{-st}\frac{\partial^\zeta \psi}{\partial t^\zeta} \cdot \left(\int_0^\infty \cdots \int_0^\infty \frac{\partial^n}{\partial p_1 \partial p_2 \partial p_3 \cdots \partial p_n}\left(e^{-p_1x_1-p_2x_2\cdots-p_nx_n}dx_1dx_2 \cdots dx_n\right)\right)dt. \tag{32}$$

The integral inside bracket is determined by

$$\int_0^\infty \cdots \int_0^\infty \frac{\partial^n}{\partial p_1 \partial p_2 \partial p_3 \cdots \partial p_n} \left(e^{-p_1x_1-p_2x_2\cdots-p_nx_n} dx_1 dx_2 \cdots dx_n\right) = (-1)^n \int_0^\infty \cdots \int_0^\infty (\Delta) dx_1 dx_2 \cdots dx_n, \tag{33}$$

where $\Delta = \prod_{i=1}^n x_{1i} e^{-p_1x_{11}-p_2x_{12}\cdots-p_nx_{1n}}$; hence, we find that

$$\begin{aligned} &\frac{\partial^n}{\partial p_1 \partial p_2 \partial p_3 \cdots \partial p_n}\left(L_n L_t\left(\frac{\partial^\zeta \psi}{\partial t^\zeta}\right)\right) \\ &= (-1)^n \int_0^\infty \cdots \int_0^\infty \int_0^\infty {}_{i=1}^{n} x_{1i} e^{-p_1x_1-p_2x_2\cdots-p_nx_n-st} \frac{\partial^\zeta \psi}{\partial t^\zeta} dx_1 dx_2 \cdots dx_n dt \\ &= (-1)^n L_n L_t \left[\prod_{i=1}^n x_{1i} \frac{\partial^\zeta}{\partial t^\zeta} \psi(x_1, x_2, x_3, \cdots, x_n, t)\right]; \end{aligned} \tag{34}$$

by substituting Equations (33) and (34) into Equation (34) and using Theorem 9, we achieved

$$\begin{aligned} &L_n L_t \left[\prod_{i=1}^n x_{1i} \frac{\partial^\zeta}{\partial t^\zeta} \psi(x_1, x_2, x_3, \cdots, x_n, t)\right] \\ &= (-1)^n \frac{\partial^n}{\partial p_1 \partial p_2 \partial p_3 \cdots \partial p_n} [\Lambda]. \end{aligned} \tag{35}$$

Similarly, we can obtain Equation (30).

In particular, at $n = 2$, we have

$$L_2 L_t \left[x_1 x_2 \frac{\partial^\zeta \psi}{\partial t^\zeta}\right] = \frac{\partial^2}{\partial p_1 \partial p_2}\left[s^\zeta \Psi(p_1, p_2, s) - s^{\zeta-1} \Psi(p_1, p_2, 0)\right], \tag{36}$$

$$L_2 L_t [x_1 x_2 f(x_1, x_2, t)] = \frac{\partial^2}{\partial p_1 \partial p_2} (L_2 L_t [f(x_1, x_2, t)]). \tag{37}$$

4. Singular $2 + 1$-D Fractional Pseudoparabolic Equation and 3-DLADM

The following is the procedure demonstrating two problems that are related to the linear and nonlinear singular $2 + 1$-D pseudoparabolic equation:

Problem 11. The 3-DLADM is an effective technique for solving linear singular two-dimensional pseudoparabolic equation.

We consider $0 < \zeta \leq 1$; let us consider a general fractional singular $2 + 1$-D pseudoparabolic equation:

$$D_t^\zeta \psi - \frac{1}{x_1}\left(x_1 \frac{\partial}{\partial x_1}\psi\right)_{x_1} - \frac{1}{x_2}\left(x_2 \frac{\partial}{\partial x_2}\psi\right)_{x_2} - \frac{1}{x_1}\left(x_1 \psi_{x_1}\right)_{x_1 t} = f(x_1, x_2, t), \tag{38}$$

$$\text{subject to } \psi(x_1, x_2, 0) = f_1(x_1, x_2), \tag{39}$$

where the linear parts $1/x_1(x_1(\partial/\partial x_1))_{x_1}$ and $1/x_2(x_2(\partial/\partial x_2))_{x_2}$ are called Bessel's operator and $f(x_1, x_2, t)$, $f_1(x_1, x_2)$ are known functions. For the objective to solve Equation (38), we are implementing the following steps.

Step 1. By multiplying the two sides of Equation (38) by $x_1 x_2$

$$x_1 x_2 D_t^\zeta \psi - x_2\left(x_1 \frac{\partial}{\partial x_1}\psi\right)_{x_1} - x_1\left(x_2 \frac{\partial}{\partial x_2}\psi\right)_{x_2} - x_2\left(x_1 \frac{\partial}{\partial x_1}\psi\right)_{x_1 t} = x_1 x_2 f(x_1, x_2, t). \tag{40}$$

Step 2. By implementing Equations (36), (37), and (7) for the equation in the first step and 2-DLT for condition, we get

$$\begin{aligned} \frac{\partial^2}{\partial p_1 \partial p_2}\Psi(p_1, p_2, s) = {} & \frac{1}{s}\frac{\partial^2}{\partial p_1 \partial p_2} L_{x_1} L_{x_2}[f_1(x_1, x_2)] \\ & + \frac{1}{s^\zeta}\frac{\partial^2}{\partial p_1 \partial p_2}(L_3[f(x_1, x_2, t)]) \\ & + \frac{1}{s^\zeta} L_3\left[x_2\left(x_1 \frac{\partial}{\partial x_1}\psi\right)_{x_1} + x_1\left(x_2 \frac{\partial}{\partial x_2}\psi\right)_{x_2}\right. \\ & \left. + x_2\left(x_1 \frac{\partial}{\partial x_1}\psi\right)_{x_1 t}\right]. \end{aligned} \tag{41}$$

Step 3. Operating the integral form of Equation (41), from 0 to p_1 and 0 to p_2 with respect to p_1, p_2, respectively, we get

$$\begin{aligned} \Psi(p_1, p_2, s) = {} & \frac{1}{s}\int_0^{p_1}\int_0^{p_2}\left(\frac{\partial^2}{\partial p_1 \partial p_2} L_{x_1} L_{x_2}[f_1(x_1, x_2)]\right) dp_1 dp_2 \\ & + \frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2} L_3\left[x_2\left(x_1 \frac{\partial}{\partial x_1}\psi\right)_{x_1} + x_1\left(x_2 \frac{\partial}{\partial x_2}\psi\right)_{x_2}\right. \\ & \left. + x_2\left(x_1 \frac{\partial}{\partial x_1}\psi\right)_{x_1 t}\right] dp_1 dp_2 \\ & + \frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(\frac{\partial^2}{\partial p_1 \partial p_2}(L_3[f(x_1, x_2, t)])\right) dp_1 dp_2. \end{aligned} \tag{42}$$

Step 4. The series solution of the singular $2 + 1$-D pseudoparabolic equation is therefore entirely determined by

$$\psi(x_1, x_2, t) = \sum_{m=0}^\infty \psi_m(x_1, x_2, t). \tag{43}$$

Step 5. Working with the 3-DLT for both sides of Equation (42) and applying Equation (43), we obtain

$$\sum_{m=0}^{\infty} \psi_m(x_1, x_2, t) = f_1(x_1, x_2) + L_3^{-1} \cdot \left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(\frac{\partial^2}{\partial p_1 \partial p_2} L_{x_1} L_{x_2}[f_1(x_1, x_2, t)]\right) dp_1 dp_2\right] + L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(L_3\left[x_2\left(x_1 \frac{\partial}{\partial x_1}\left(\sum_{m=0}^{\infty}\psi_m\right)\right)_{x_1}\right]\right) dp_1\right] + L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(L_3\left[x_1\left(x_2 \frac{\partial}{\partial x_2}\left(\sum_{n=0}^{\infty}\psi_m\right)\right)_{x_2}\right]\right) dp_1 dp_2\right] + L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(L_3\left[x_2\left(x_1 \frac{\partial}{\partial x_1}\left(\sum_{n=0}^{\infty}\psi_m\right)\right)_{x_1 t}\right]\right) dp_1 dp_2\right]; \tag{44}$$

in view of the first approximation,

$$\psi_0 = f_1(x_1, x_2) + L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(\frac{\partial^2}{\partial p_1 \partial p_2} L_{x_1} L_{x_2}[f_1(x_1, x_2, t)]\right) dp_1 dp_2\right], \tag{45}$$

and the remaining components ψ_{m+1}, $m \geq 0$, are denoted by

$$\psi_{m+1} = +L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(L_3\left[x_2\left(x_1\left(\psi_{mx_1}\right)\right)_{x_1}\right]\right) dp_1\right] + L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(L_3\left[x_1\left(x_2\left(\psi_{mx_2}\right)\right)_{x_2}\right]\right) dp_1 dp_2\right] + L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(L_3\left[x_2\left(x_1\left(\psi_{mx_1}\right)\right)_{x_1 t}\right]\right) dp_1 dp_2\right]. \tag{46}$$

Here, we consider that the inverse 3-DLT with respect to p_1, p_2 and s of Equations (45) and (46) exists. To display the applicability of the method explained previously, we current the following example.

Example 12. Singular $2+1$-D pseudoparabolic equation is given by

$$D_t^\zeta \psi - \frac{1}{x_1}\left(x_1 \psi_{x_1}\right)_{x_1} - \frac{1}{x_2}\left(x_2 \psi_{x_2}\right)_{x_2} - \frac{1}{x_1}\left(x_1 \psi_{x_1}\right)_{x_1 t} = \left(x_1^2 - x_2^2\right)e^t - 4e^t 0 \leq x_1, x_2, t < \infty, 0 < \zeta \leq 1, \tag{47}$$

$$\text{subject to } \psi(x_1, x_2, 0) = x_1^2 - x_2^2. \tag{48}$$

By applying the above steps and Theorem 8 for Equation (47), all terms of the sequence are computed as follows:

$$\psi_0 = x_1^2 - x_2^2 + x_1^2 t^\zeta \Xi_{1,\zeta+1}(t) - x_2^2 t^\zeta \Xi_{1,\zeta+1}(t) - 4t^\zeta \Xi_{1,\zeta+1}(t), \tag{49}$$

$$\psi_{m+1} = L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(L_3\left[x_2\left(x_1\left(\psi_{mx_1}\right)\right)_{x_1}\right]\right) dp_1\right] + L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(L_3\left[x_1\left(x_2\left(\psi_{mx_2}\right)\right)_{x_2}\right]\right) dp_1 dp_2\right] + L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(L_3\left[x_2\left(x_1\left(\psi_{mx_1}\right)\right)_{x_1 t}\right]\right) dp_1 dp_2\right]; \tag{50}$$

according to 3-DLADM, we get the following components:

$$\begin{aligned}\psi_1 &= L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(L_3\left[x_2\left(x_1\left(\psi_{0x_1}\right)\right)_{x_1} + x_1\left(x_2\left(\psi_{0x_2}\right)_{x_2}\right) + x_2\left(x_1\left(\psi_{0x_1}\right)_{x_1 t}\right)\right]\right) dp_1 dp_2\right] \\ &= L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(L_3\left[4x_1x_2t^{\zeta-1}\Xi_{1,\zeta}(t)\right]\right) dp_1 dp_2\right] \\ &= L_3^{-1}\left[\frac{4}{p_1p_2s^{2\zeta-1}(s-1)}\right]\psi_1 = 4t^{2^{\zeta}-1}\Xi_{1,2\zeta}(t).\end{aligned} \tag{51}$$

In the same way, we receive that

$$\psi_2 = L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2} L_3\left[x_2\left(x_1\left(\psi_{1x_1}\right)\right)_{x_1} + x_1\left(x_2\left(\psi_{1x_2}\right)_{x_2}\right) + x_2\left(x_1\left(\psi_{1x_1}\right)_{x_1 t}\right)\right] dp_1 dp_2\right] = L_3^{-1}[0] = 0, \tag{52}$$

$$\begin{aligned}\psi_3 &= 0, \\ \psi_4 &= 0, \cdots;\end{aligned} \tag{53}$$

by adding all the terms, we have

$$\psi(x_1, x_2, t) = \psi_0 + \psi_1 + \psi_2 + \cdots; \tag{54}$$

therefore, the approximation solution of Equation (47) is denoted by

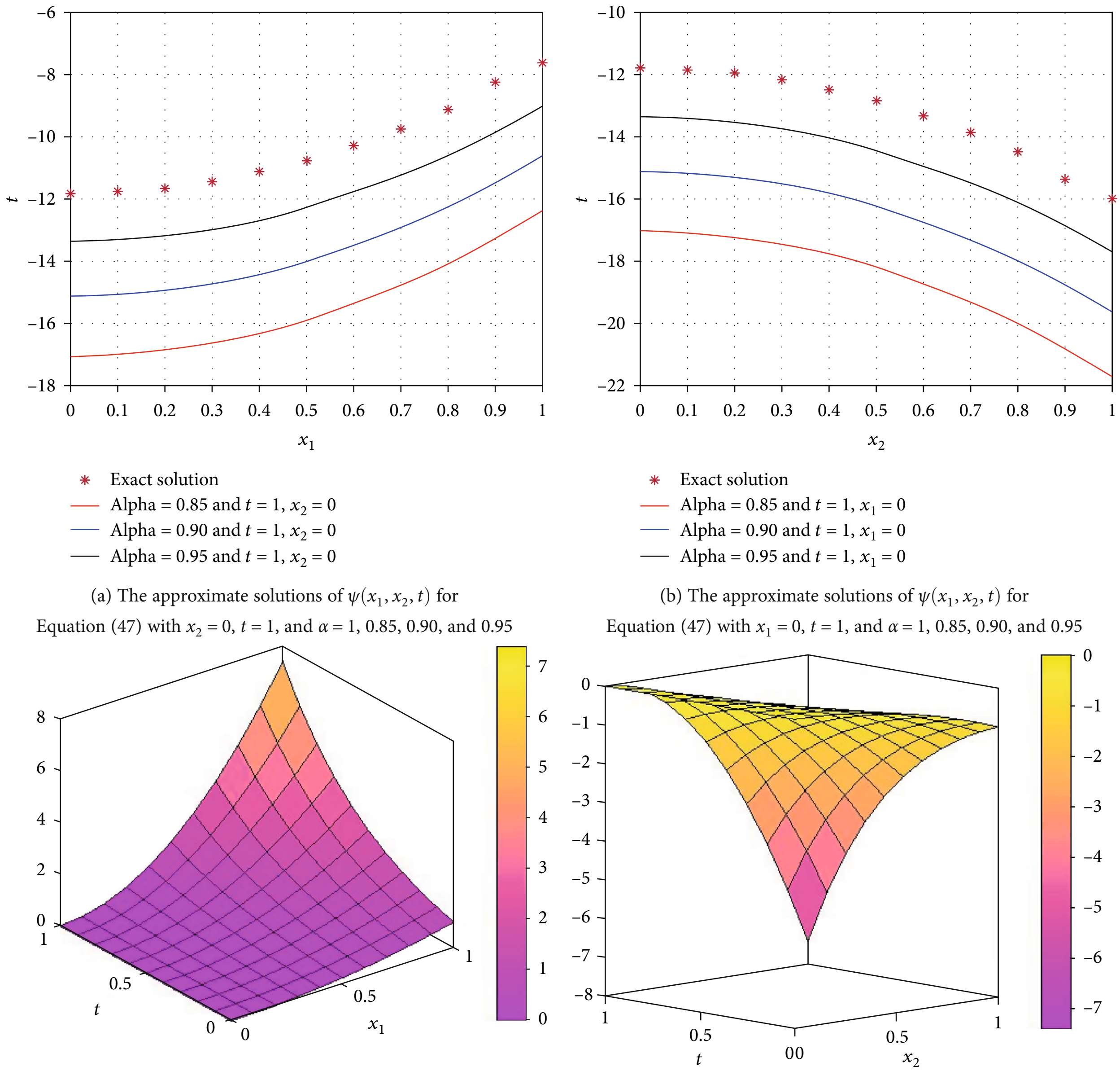
$$\begin{aligned}\psi(x_1, x_2, t) = x_1^2 - x_2^2 + x_1^2 t^\zeta \Xi_{1,\zeta+1}(t) - x_2^2 t^\zeta \Xi_{1,\zeta+1}(t) \\ - 4t^\zeta \Xi_{1,\zeta+1}(t) + 4t^{2^{\zeta}-1}\Xi_{1,2\zeta}(t).\end{aligned} \tag{55}$$

By using $\zeta = 1$ and Equations (13) and (14), the approximation solution becomes

$$\psi(x_1, x_2, t) = \left(x_1^2 - x_2^2\right)e^t. \tag{56}$$

Figures 1(a) and 1(b) show the approximate and exact solutions of Equation (47); at $t = 1$ and $\alpha = 1$, we obtain the exact solution of Equation (47); by taking different values of α such as $\alpha = 0.85$, $\alpha = 0.90$, and $\alpha = 0.95$, we get the approximate solution. Figures 1(c) and 1(d) show the plot of function $\psi(x_1, x_2, t)$ in three dimensions.

In the next problem, we apply the previous method.



(a) The approximate solutions of $\psi(x_1, x_2, t)$ for Equation (47) with $x_2 = 0$, $t = 1$, and $\alpha = 1, 0.85, 0.90$, and 0.95

(b) The approximate solutions of $\psi(x_1, x_2, t)$ for Equation (47) with $x_1 = 0$, $t = 1$, and $\alpha = 1, 0.85, 0.90$, and 0.95

(c) The surface shows $\psi(x_1, x_2, t)$ for Equation (47) with $x_2 = 0$

(d) The surface shows $\psi(x_1, x_2, t)$ for Equation (47) with $x_1 = 0$

FIGURE 1

Problem 13. Consider the next nonlinear singular 2 + 1-D pseudoparabolic equation:

$$D_t^{\zeta}\psi - \frac{1}{x_1}\left(x_1 \frac{\partial}{\partial x_1}\psi\right)_{x_1} - \frac{1}{x_2}\left(x_2 \frac{\partial}{\partial x_2}\psi\right)_{x_2} - \frac{1}{x_1}\frac{\partial^2}{\partial x_1 \partial t}\left(x_1 \frac{\partial}{\partial x_1}\psi_{x_1}\right) = -2\mu(x_1)\psi_{x_1}\psi_{x_1 x_1} + \nu(x_1)\left(\psi_{x_1}\right)^2 + f(x_1, x_2, t), \quad 0 < \zeta \leq 1, 0 \leq x_1, x_2, t < \infty, \tag{57}$$

$$\text{subject to } \psi(x_1, x_2, 0) = f_1(x_1, x_2). \tag{58}$$

By applying the previous technique, the first approximation is given by

$$\psi_0 = f_1(x_1, x_2) + L_3^{-1}\left[\frac{1}{s^{\zeta}}\int_0^{p_1}\int_0^{p_2}\left(\frac{\partial^2}{\partial p_1 \partial p_2}(L_3[f(x_1, x_2, t)])\right)dp_1 dp_2\right], \tag{59}$$

and the rest of the terms are given by

$$\psi_{m+1} = L_3^{-1}\left[\frac{1}{s^{\zeta}}\int_0^{p_1}\int_0^{p_2}\left(L_3\left[x_2\left(x_1\psi_{mx_1}\right)_{x_1} + x_1\left(x_2\psi_{mx_2}\right)_{x_2} + x_2\left(x_1\psi_{mx_1}\right)_{x_1 t}\right]\right)dp_1 dp_2\right] - L_3^{-1}\left[\frac{1}{s^{\zeta}}\int_0^{p_1}\int_0^{p_2}(L_3[(x_1 x_2 \mu(x_1)A_m) - (x_1 x_2 \nu(x_1)B_m)])dp_1 dp_2\right], \tag{60}$$

where nonlinear terms A_m and B_m are decomposed as

$$A_m = \sum_{m=0}^{\infty} \psi_{mx_1}\psi_{mx_1x_1}, \quad B_m = \sum_{m=0}^{\infty} \left(\psi_{mx_1}\right)^2. \tag{61}$$

The nonlinear terms $\psi_{x_1}\psi_{x_1x_1}$ and $(\psi_{x_1})^2$ are denoted by

$$A_0 = \psi_{0x_1}\psi_{0x_1x_1}, \tag{62}$$

$$A_1 = \psi_{0x_1}\psi_{1x_1x_1} + \psi_{1x_1}\psi_{0x_1x_1}, \tag{63}$$

$$A_2 = \psi_{0x_1}\psi_{2x_1x_1} + \psi_{1x_1}\psi_{1x_1x_1} + \psi_{2x_1}\psi_{0x_1x_1}, \tag{64}$$

$$A_3 = \psi_{0x_1}\psi_{3x_1x_1} + \psi_{1x_1}\psi_{2x_1x_1} + \psi_{2x_1}\psi_{1x_1x_1} + \psi_{3x_1}\psi_{0x_1x_1}, \tag{65}$$

$$B_0 = \left(\psi_{0x_1}\right)^2, \tag{66}$$

$$B_1 = 2\psi_{0x_1}\psi_{1x_1}, \tag{67}$$

$$B_2 = 2\psi_{0x_1}\psi_{2x_1} + \left(\psi_{1x_1}\right)^2, \tag{68}$$

$$B_3 = 2\psi_{0x_1}\psi_{3x_1} + 2\psi_{1x_1}\psi_{2x_1}. \tag{69}$$

To demonstrate this technique for a nonlinear problem, we examine the following example.

Example 14. Consider the following nonlinear pseudoparabolic equation:

$$\begin{aligned} D_t^\zeta\psi - \frac{1}{x_1}\left(x_1\frac{\partial}{\partial x_1}\psi\right)_{x_1} - \frac{1}{x_2}\left(x_2\frac{\partial}{\partial x_2}\psi\right)_{x_2} - \frac{1}{x_1}\left(x_1\frac{\partial}{\partial x_1}\psi_{x_1}\right)_{x_1t} \\ = -x_1\psi_{x_1}\psi_{x_1x_1} + \left(\psi_{x_1}\right)^2 + 2\left(x_1^2 - x_2^2\right)e^{2t} - 8e^{2t}, \quad 0 \le x_1, x_2, t < \infty, 0 < \zeta \le 1, \end{aligned} \tag{70}$$

$$\text{subject to}\ \psi(x_1, x_2, 0) = x_1^2 - x_2^2. \tag{71}$$

By using the mentioned 3-DLADM and Theorem 8, we have

$$\psi_0 = x_1^2 - x_2^2 + x_1^2 t^\zeta \Xi_{1,\zeta+1}(2t) - x_2^2 t^\zeta \Xi_{1,\zeta+1}(2t) - 8t^\zeta \Xi_{1,\zeta+1}(2t), \tag{72}$$

$$\begin{aligned} \psi_{m+1} = L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(L_3\left[x_2\left(x_1\psi_{mx_1}\right)_{x_1} + x_1\left(x_2\psi_{mx_2}\right)_{x_2} \right.\right.\right. \\ \left.\left.\left. + x_2\left(x_1\psi_{mx_1}\right)_{x_1t}\right]\right)dp_1dp_2\right] \\ - L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(L_3\left[x_1^2x_2A_m - x_1x_2B_m\right]\right)dp_1dp_2\right], \end{aligned} \tag{73}$$

where A_m and B_m are defined in Equations (65) and (69). The subsequent terms are introduced by

$$\begin{aligned} \psi_1 &= L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(L_3\left[x_2\left(x_1\psi_{0x_1}\right)_{x_1} + x_1\left(x_2\psi_{0x_2}\right)_{x_2} \right.\right.\right. \\ &\quad \left.\left.\left. + x_2\left(x_1\psi_{0x_1}\right)_{x_1t}\right]\right)dp_1dp_2\right] \\ &\quad - L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(L_3\left[\left(x_1^2x_2A_0\right) - x_1x_2B_0\right]\right)dp_1dp_2\right] \\ &= L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(L_3\left[8x_1x_2t^{\zeta-1}\Xi_{1,\zeta}(2t)\right]\right)dp_1dp_2\right] \\ &= L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(\frac{8}{p_1^2p_2^2s^{\zeta-1}(s-2)}\right)dp_1dp_2\right] \\ &= L_3^{-1}\left[\frac{8}{p_1p_2s^{2\zeta-1}(s-2)}\right]\psi_1 = 8t^{2\zeta-1}\Xi_{1,2\zeta}(2t). \end{aligned} \tag{74}$$

Following in a similar manner, we have

$$\psi_2 = 0,\ \psi_3 = 0, \psi_4 = 0, \cdots. \tag{75}$$

Hence, according to Equation (43), we have

$$\begin{aligned} \psi(x_1, x_2, t) = x_1^2 - x_2^2 + x_1^2 t^\zeta \Xi_{1,\zeta+1}(2t) - x_2^2 t^\zeta \Xi_{1,\zeta+1}(2t) \\ - 8t^\zeta \Xi_{1,\zeta+1}(2t) + 8t^{2\zeta-1}\Xi_{1,2\zeta}(2t); \end{aligned} \tag{76}$$

if we set $\zeta = 1$ and Equations (13) and (14), then the exact solutions of Equation (70) are presented by

$$\psi(x_1, x_2, t) = \left(x_1^2 - x_2^2\right)e^{2t}. \tag{77}$$

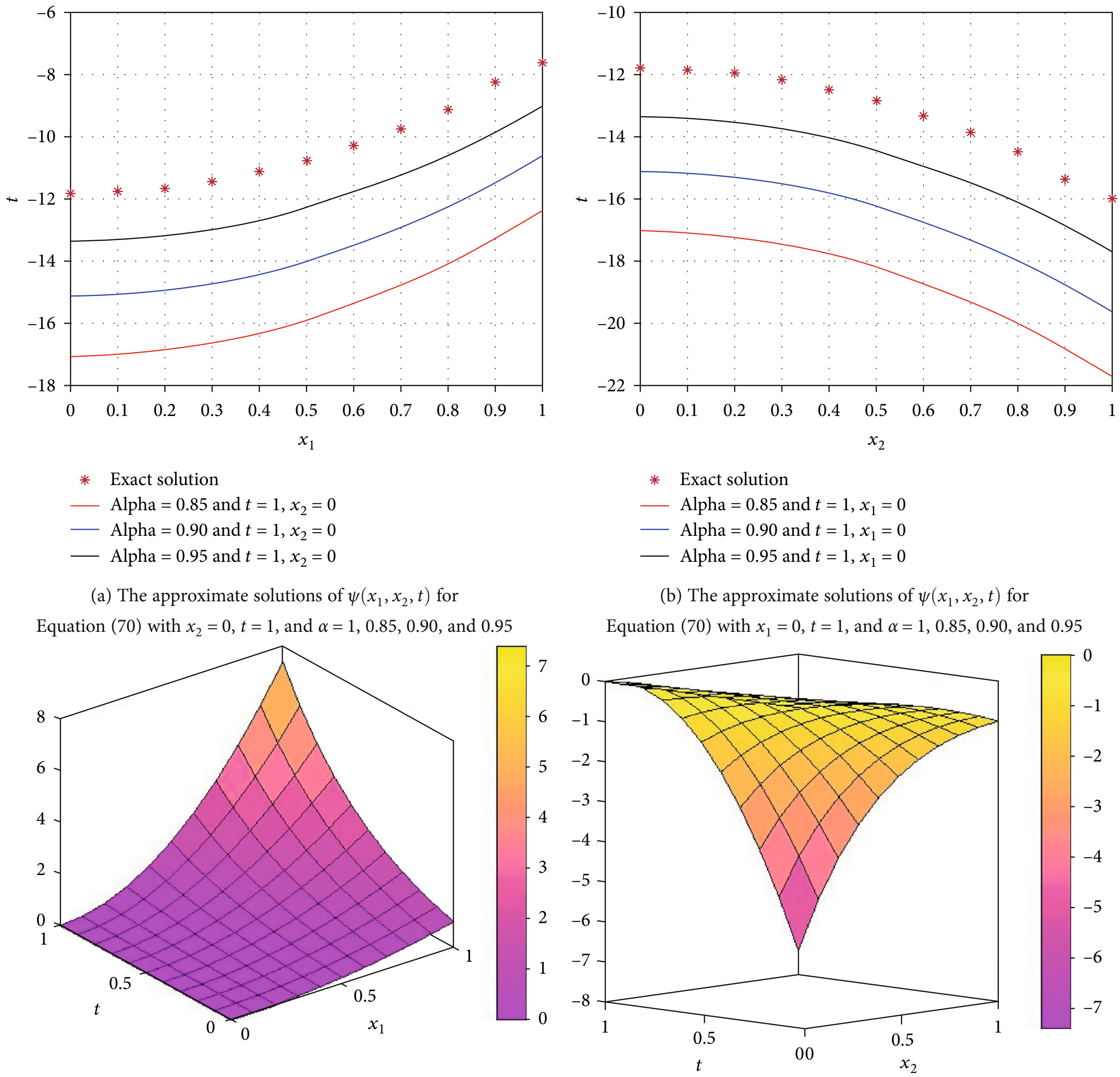
Figures 2(a) and 2(b) show the approximate and exact solutions of Equation (70); at $t = 1$ and $\alpha = 1$, we get the exact solution of Equation (70). For the different values of α such as $\alpha = 0.85$, $\alpha = 0.90$, and $\alpha = 0.95$, we obtain the approximate solution. Figures 2(c) and 2(d) represent the surface of the function $\psi(x_1, x_2, t)$.

5. Singular 2-D Coupled Pseudoparabolic Equation and 3-DLADM

The aim of this part is devoted to establishing the solution of the coupled singular $2 + 1$-D pseudoparabolic equation by applying 3-DLADM.

The coupled singular 2-D pseudoparabolic equation of fractional order is given by

$$\begin{aligned} D_t^\zeta\psi - \frac{1}{x_1}\left(x_1\frac{\partial}{\partial x_1}\psi\right)_{x_1} - \frac{1}{x_2}\left(x_2\frac{\partial}{\partial x_2}\psi\right)_{x_2} \\ - \frac{1}{x_1}\left(x_1\frac{\partial}{\partial x_1}\psi\right)_{x_1t} - \frac{\mu}{x_1}\psi_{x_1} + \lambda\varphi = f(x_1, x_2, t), \end{aligned} \tag{78}$$



(a) The approximate solutions of $\psi(x_1, x_2, t)$ for Equation (70) with $x_2 = 0$, $t = 1$, and $\alpha = 1, 0.85, 0.90$, and 0.95

(b) The approximate solutions of $\psi(x_1, x_2, t)$ for Equation (70) with $x_1 = 0$, $t = 1$, and $\alpha = 1, 0.85, 0.90$, and 0.95

(c) The surface shows $\psi(x_1, x_2, t)$ for Equation (70) with $x_2 = 0$

(d) The surface shows $\psi(x_1, x_2, t)$ for Equation (70) with $x_1 = 0$

Figure 2

$$\begin{aligned} D_t^{\zeta}\varphi &- \frac{1}{x_1}\left(x_1\frac{\partial}{\partial x_1}\varphi\right)_{x_1} - \frac{1}{x_2}\left(x_2\frac{\partial}{\partial x_2}\varphi\right)_{x_2} \\ &- \frac{1}{x_1}\left(x_1\frac{\partial}{\partial x_1}\varphi\right)_{x_1 t} - \frac{\gamma}{x_1}\varphi_{x_1} + \lambda\psi = g(x_1, x_2, t), \end{aligned} \tag{79}$$

subject to $\psi(x_1, x_2, 0) = f_1(x_1, x_2), \varphi(x_1, x_2, 0) = g_1(x_1, x_2),$ (80)

where $f(x_1, x_2, t)$, $g(x_1, x_2, t)$, $f_1(x_1, x_2)$, and $g_1(x_1, x_2)$ are given functions and λ is the coupling parameter. One can obtain the solution of Equation (79) by using 3-DLADM. This method contains the following steps.

(1) Multiplying both sides of Equation (79) by $x_1 x_2$ leads to the following equation:

$$\begin{aligned} x_1 x_2 D_t^{\beta}\psi &- x_2\left(x_1\frac{\partial}{\partial x_1}\psi\right)_{x_1} - x_1\left(x_2\frac{\partial}{\partial x_2}\psi\right)_{x_2} \\ &- x_2\left(x_1\frac{\partial}{\partial x_1}\psi\right)_{x_1 t} - \mu x_2\psi_{x_1} + \lambda x_1 x_2\varphi \\ &= x_1 x_2 f(x_1, x_2, t), x_1 x_2 D_t^{\beta}\varphi - x_2\left(x_1\frac{\partial}{\partial x_1}\varphi\right)_{x_1} \\ &- x_1\left(x_2\frac{\partial}{\partial x_2}\varphi\right)_{x_2} - x_2\left(x_1\frac{\partial}{\partial x_1}\varphi\right)_{x_1 t} \\ &- \gamma x_2\varphi_{x_1} + \lambda x_1 x_2\psi = x_1 x_2 g(x_1, x_2, t). \end{aligned} \tag{81}$$

(2) We apply 3-DLT and both sides of Equation (81) and 2-DLT for Equation (80); we get $L_3[x_1x_2D_t^{\zeta}\psi] = L_3[x_2(x_1(\partial/\partial x_1)\psi)_{x_1} + x_1(x_2(\partial/\partial x_2)\psi)_{x_2} + x_2(x_1(\partial/\ \partial x_1)\psi)_{x_1t}] + L_3[\mu x_2\psi_{x_1} - \lambda x_1x_2\varphi] + L_3[x_1x_2f(x_1, x_2, t)]$,

$$L_3\left[x_1x_2D_t^{\zeta}\varphi\right] = L_3\left[x_2\left(x_1\frac{\partial}{\partial x_1}\varphi\right)_{x_1} + x_1\left(x_2\frac{\partial}{\partial x_2}\varphi\right)_{x_2} + x_2\left(x_1\frac{\partial}{\partial x_1}\varphi\right)_{x_1t}\right] + L_3\left[\gamma x_2\varphi_{x_1} - \lambda x_1x_2\psi\right] + L_3[x_1x_2g(x_1, x_2, t)], \tag{82}$$

on using Theorem 8 and 2-DLT for condition, we obtain

$$\begin{aligned}\frac{\partial^2}{\partial p_1\partial p_2}\Psi(p_1, p_2, s) &= \frac{1}{s^{\zeta}}\frac{\partial^2}{\partial p_1\partial p_2}L_{x_1}L_{x_2}[f_1(x_1, x_2)] + \frac{1}{s^{\zeta}}\frac{\partial^2}{\partial p_1\partial p_2}(L_3[f(x_1, x_2, t)]) + \frac{1}{s^{\zeta}}L_3\\ &\quad\times\left[x_2\left(x_1\frac{\partial}{\partial x_1}\psi\right)_{x_1} + x_1\left(x_2\frac{\partial}{\partial x_2}\psi\right)_{x_2} + x_2\left(x_1\frac{\partial}{\partial x_1}\psi\right)_{x_1t}\right]\\ &\quad + \frac{1}{s^{\zeta}}L_3\left[\mu x_2\psi_{x_1} - \lambda x_1x_2\varphi\right], \frac{\partial^2}{\partial p_1\partial p_2}\Phi(p_1, p_2, s)\\ &= \frac{1}{s^{\zeta}}\frac{\partial^2}{\partial p_1\partial p_2}L_{x_1}L_{x_2}[g_1(x_1, x_2)] + \frac{1}{s^{\zeta}}\frac{\partial^2}{\partial p_1\partial p_2}(L_3[g(x_1, x_2, t)])\\ &\quad + \frac{1}{s^{\zeta}}L_3\left[x_2\left(x_1\frac{\partial}{\partial x_1}\varphi\right)_{x_1} + x_1\left(x_2\frac{\partial}{\partial x_2}\varphi\right)_{x_2} + x_2\left(x_1\frac{\partial}{\partial x_1}\varphi\right)_{x_1t}\right]\\ &\quad + \frac{1}{s^{\zeta}}L_3\left[\gamma x_2\varphi_{x_1} - \lambda x_1x_2\psi\right].\end{aligned} \tag{83}$$

(3) By integrating Equation (83) from 0 to p_1 and 0 to p_2 with respect to p_1 and p_2, respectively, we have

$$\begin{aligned}\Psi(p_1, p_2, s) &= \frac{1}{s^{\zeta}}\int_0^{p_1}\int_0^{p_2}\left(\frac{\partial^2}{\partial p_1\partial p_2}L_{x_1}L_{x_2}[f_1(x_1, x_2)]\right)dp_1dp_2\\ &\quad + \frac{1}{s^{\zeta}}\int_0^{p_1}\int_0^{p_2}\left(\frac{\partial^2}{\partial p_1\partial p_2}(L_3[f(x_1, x_2, t)])\right)dp_1dp_2\\ &\quad + \frac{1}{s^{\zeta}}\int_0^{p_1}\int_0^{p_2}L_3[\phi]dp_1dp_2,\end{aligned} \tag{84}$$

$$\begin{aligned}\Phi(p_1, p_2, s) &= \frac{1}{s^{\zeta}}\int_0^{p_1}\int_0^{p_2}\left(\frac{\partial^2}{\partial p_1\partial p_2}L_{x_1}L_{x_2}[g_1(x_1, x_2)]\right)dp_1dp_2\\ &\quad + \frac{1}{s^{\zeta}}\int_0^{p_1}\int_0^{p_2}\left(\frac{\partial^2}{\partial p_1\partial p_2}(L_3[g(x_1, x_2, t)])\right)dp_1dp_2\\ &\quad + \frac{1}{s^{\zeta}}\int_0^{p_1}\int_0^{p_2}L_3[\chi]dp_1dp_2.\end{aligned} \tag{85}$$

Finally, by using the inverse of 3-DLT, we can evaluate $\psi(x_1, x_2, t)$ and $\varphi(x_1, x_2, t)$ as follows:

$$\begin{aligned}\psi(x_1, x_2, t) &= f_1(x_1, x_2) + L_3^{-1}\left[\frac{1}{s^{\zeta}}\int_0^{p_1}\int_0^{p_2}\left(\frac{\partial^2}{\partial p_1\partial p_2}(L_3[f(x_1, x_2, t)])\right)dp_1dp_2\right]\\ &\quad + L_3^{-1}\left[\frac{1}{s^{\zeta}}\int_0^{p_1}\int_0^{p_2}L_3[\phi]dp_1dp_2\right],\end{aligned} \tag{86}$$

$$\begin{aligned}\varphi(x_1, x_2, t) &= g_1(x_1, x_2) + L_3^{-1}\left[\frac{1}{s^{\zeta}}\int_0^{p_1}\int_0^{p_2}\left(\frac{\partial^2}{\partial p_1\partial p_2}(L_3[g(x_1, x_2, t)])\right)dp_1dp_2\right]\\ &\quad + L_3^{-1}\left[\frac{1}{s^{\zeta}}\int_0^{p_1}\int_0^{p_2}L_3[\chi]dp_1dp_2\right].\end{aligned} \tag{87}$$

The 3-DLADM decomposes the unknown functions $\psi(x_1, x_2, t)$ and $\varphi(x_1, x_2, t)$ by the infinite series of components as

$$\begin{aligned}\psi(x_1, x_2, t) &= \sum_{m=0}^{\infty}\psi_m(x_1, x_2, t),\\ \varphi(x_1, x_2, t) &= \sum_{m=0}^{\infty}\varphi_m(x_1, x_2, t).\end{aligned} \tag{88}$$

By substituting Equation (88) into Equations (86) and (87), we get

$$\begin{aligned}\sum_{m=0}^{\infty}\psi_m(x_1, x_2, t) &= f_1(x_1, x_2) + L_3^{-1}\\ &\quad\cdot\left[\frac{1}{s^{\zeta}}\int_0^{p_1}\int_0^{p_2}\left(\frac{\partial^2}{\partial p_1\partial p_2}(L_3[f(x_1, x_2, t)])\right)dp_1dp_2\right]\\ &\quad + L_3^{-1}\left[\frac{1}{s^{\zeta}}\int_0^{p_1}\int_0^{p_2}L_3\left[\sum_{m=0}^{\infty}\phi_m\right]dp_1dp_2\right],\end{aligned} \tag{89}$$

$$\begin{aligned}\sum_{m=0}^{\infty}\varphi_m(x_1, x_2, t) &= g_1(x_1, x_2) + L_3^{-1}\\ &\quad\cdot\left[\frac{1}{s^{\zeta}}\int_0^{p_1}\int_0^{p_2}\left(\frac{\partial^2}{\partial p_1\partial p_2}(L_3[g(x_1, x_2, t)])\right)dp_1dp_2\right]\\ &\quad + L_3^{-1}\left[\frac{1}{s^{\zeta}}\int_0^{p_1}\int_0^{p_2}L_3\left[\sum_{m=0}^{\infty}\chi_m\right]dp_1dp_2\right],\end{aligned} \tag{90}$$

where

$$\begin{aligned}\phi_m &= x_2\frac{\partial}{\partial x_1}\left(x_1\psi_{mx_1}\right) + x_1\frac{\partial}{\partial x_2}\left(x_2\psi_{mx_2}\right)\\ &\quad + x_2\frac{\partial^2}{\partial x_1\partial t}\left(x_1\psi_{mx_1}\right) + \mu x_2\psi_{mx_1} - \lambda x_1x_2\varphi_m,\end{aligned} \tag{91}$$

$$\begin{aligned}\chi_m &= x_2 \frac{\partial}{\partial x_1}\left(x_1 \varphi_{m x_1}\right) + x_1 \frac{\partial}{\partial x_2}\left(x_2 \varphi_{m x_2}\right) \\ &\quad + x_2 \frac{\partial^2}{\partial x_1 \partial t}\left(x_1 \varphi_{m x_1}\right) + \mu x_2 \varphi_{m x_1} - \lambda x_1 x_2 \psi_m.\end{aligned} \tag{92}$$

Our method recommends that the zeroth components ψ_0 and φ_0 are determined by the initial conditions and nonhomogeneous parts:

$$\psi_0 = f_1(x_1, x_2) + L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(\frac{\partial^2}{\partial p_1 \partial p_2}\left(L_3[f(x_1, x_2, t)]\right)\right) dp_1 dp_2\right], \tag{93}$$

$$\varphi_0 = g_1(x_1, x_2) + L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2}\left(\frac{\partial^2}{\partial p_1 \partial p_2}\left(L_3[g(x_1, x_2, t)]\right)\right) dp_1 dp_2\right]. \tag{94}$$

The remainder terms are given by

$$\psi_{m+1} = L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2} L_3\left[\sum_{n=0}^{\infty} \phi_m\right] dp_1 dp_2\right], \tag{95}$$

$$\varphi_{n+1} = L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2} L_3\left[\sum_{n=0}^{\infty} \chi_m\right] dp_1 dp_2\right]. \tag{96}$$

In order to check the applicability of our method for solving the fractional coupled pseudoparabolic equation, the next example has been considered.

Example 15. Time fractional coupled pseudoparabolic equations are given by

$$\begin{aligned}D_t^\zeta \psi &= \frac{1}{x_1}\left(x_1 \frac{\partial}{\partial x_1}\psi\right)_{x_1} + \frac{1}{x_2}\left(x_2 \frac{\partial}{\partial x_2}\psi\right)_{x_2} \\ &\quad + \frac{1}{x_1}\left(x_1 \frac{\partial}{\partial x_1}\psi\right)_{x_1 t} + \frac{2}{x_1}\psi_{x_1} - \frac{2}{x_2}\psi_{x_2} - 2\varphi,\end{aligned} \tag{97}$$

$$\begin{aligned}D_t^\zeta \varphi &= \frac{1}{x_1}\left(x_1 \frac{\partial}{\partial x_1}\varphi\right)_{x_1} + \frac{1}{x_2}\left(x_2 \frac{\partial}{\partial x_2}\varphi\right)_{x_2} \\ &\quad + \frac{1}{x_1}\left(x_1 \frac{\partial}{\partial x_1}\varphi\right)_{x_1 t} + \frac{2}{x_1}\varphi_{x_1} - \frac{2}{x_2}\varphi_{x_2} - 2\psi,\end{aligned} \tag{98}$$

where

$$0 \le x_1, x_2, t < \infty, 1,\ 0 < \zeta \le 1, \tag{99}$$

with initial condition

$$\begin{aligned}\psi(x_1, x_2, 0) &= x_1^2 - x_2^2, \\ \varphi(x_1, x_2, 0) &= x_1^2 - x_2^2.\end{aligned} \tag{100}$$

Using the 3-DLADM procedure Equations (94), (95), and (96), we obtain the following components:

$$\begin{aligned}\psi_0 &= x_1^2 - x_2^2, \\ \varphi_0 &= x_1^2 - x_2^2,\end{aligned} \tag{101}$$

at $m = 0$,

$$\begin{aligned}\psi_1 &= L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2} L_3\left[x_2\left(x_1 \frac{\partial}{\partial x_1}\psi_0\right)_{x_1} + x_1\left(x_2 \frac{\partial}{\partial x_2}\psi_0\right)_{x_2}\right.\right. \\ &\quad \left.\left. + x_2\left(x_1 \frac{\partial}{\partial x_1}\psi_0\right)_{x_1 t}\right] dp_1 dp_2\right] \\ &\quad + L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2} L_3\left[2x_2\psi_{0x_1} - 2x_1\psi_{0x_2} - 2x_1x_2\varphi_0\right] dp_1 dp_2\right] \\ &= L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2} L_3\left[8x_1x_2 - 2x_1^3x_2 + 2x_1x_2^3\right] dp_1 dp_2\right] \\ &= L_3^{-1}\left[\frac{8}{p_1p_2s^{\zeta+1}} - \frac{4}{p_1^3p_2s^{\zeta+1}} + \frac{4}{p_1p_2^3s^{\zeta+1}}\right],\end{aligned}$$

$$\psi_1 = \frac{8t^\zeta}{\Gamma(\zeta+1)} - \frac{2t^\zeta x_1^2}{\Gamma(\zeta+1)} + \frac{2t^\zeta x_2^2}{\Gamma(\zeta+1)}, \tag{102}$$

$$\begin{aligned}\varphi_1 &= L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2} L_3\left[x_2\left(x_1 \frac{\partial}{\partial x_1}\varphi_0\right)_{x_1} + x_1\left(x_2 \frac{\partial}{\partial x_2}\varphi_0\right)_{x_2}\right.\right. \\ &\quad \left.\left. + x_2\left(x_1 \frac{\partial}{\partial x_1}\varphi_0\right)_{x_1 t}\right] dp_1 dp_2\right] \\ &\quad + L_3^{-1}\left[\frac{1}{s^\zeta}\int_0^{p_1}\int_0^{p_2} L_3\left[2x_2\varphi_{0x_1} - 2x_1\varphi_{0x_2} - 2x_1x_2\psi_0\right] dp_1 dp_2\right] \\ &= L_3^{-1}\left[\frac{8}{p_1p_2s^{\zeta+1}} - \frac{4}{p_1^3p_2s^{\zeta+1}} + \frac{4}{p_1p_2^3s^{\zeta+1}}\right],\end{aligned}$$

$$\varphi_1 = \frac{8t^\zeta}{\Gamma(\zeta+1)} - \frac{2t^\zeta x_1^2}{\Gamma(\zeta+1)} + \frac{2t^\zeta x_2^2}{\Gamma(\zeta+1)}; \tag{103}$$

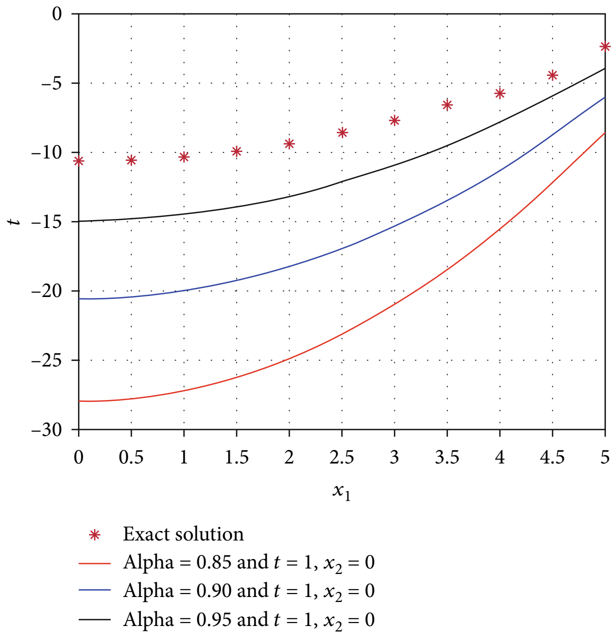
by the same way, at $m = 1$, we have

$$\psi_2 = -\frac{8t^{2^\zeta-1}}{\Gamma(2^\zeta)} - \frac{32t^{2^\zeta}}{\Gamma(2^\zeta+1)} + \frac{4t^{2^\zeta}x_1^2}{\Gamma(2^\zeta+1)} - \frac{4t^{2^\zeta}x_2^2}{\Gamma(2^\zeta+1)}, \tag{104}$$

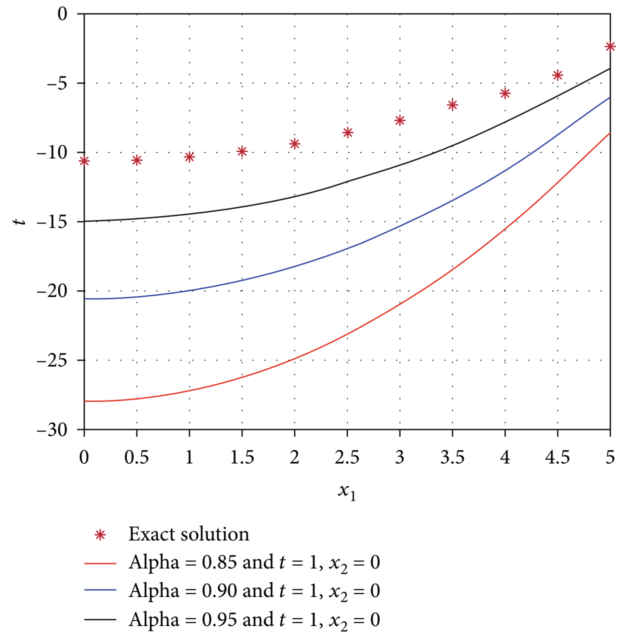
$$\varphi_2 = -\frac{8t^{2^\zeta-1}}{\Gamma(2^\zeta)} - \frac{32t^{2^\zeta}}{\Gamma(2^\zeta+1)} + \frac{4t^{2^\zeta}x_1^2}{\Gamma(2^\zeta+1)} - \frac{4t^{2^\zeta}x_2^2}{\Gamma(2^\zeta+1)}; \tag{105}$$

at $n = 2$,

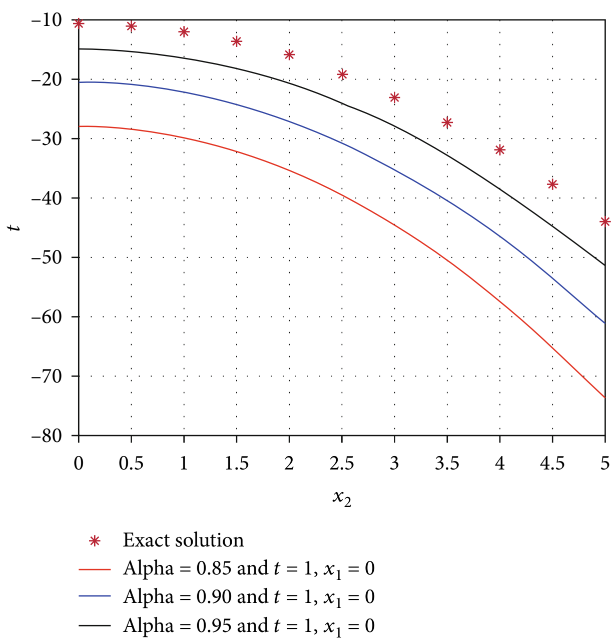
$$\psi_3 = \frac{32t^{3^\zeta-1}}{\Gamma(3^\zeta)} + \frac{96t^{3^\zeta}}{\Gamma(3^\zeta+1)} - \frac{8t^{3^\zeta}x_1^2}{\Gamma(3^\zeta+1)} + \frac{8t^{3^\zeta}x_2^2}{\Gamma(3^\zeta+1)}, \tag{106}$$



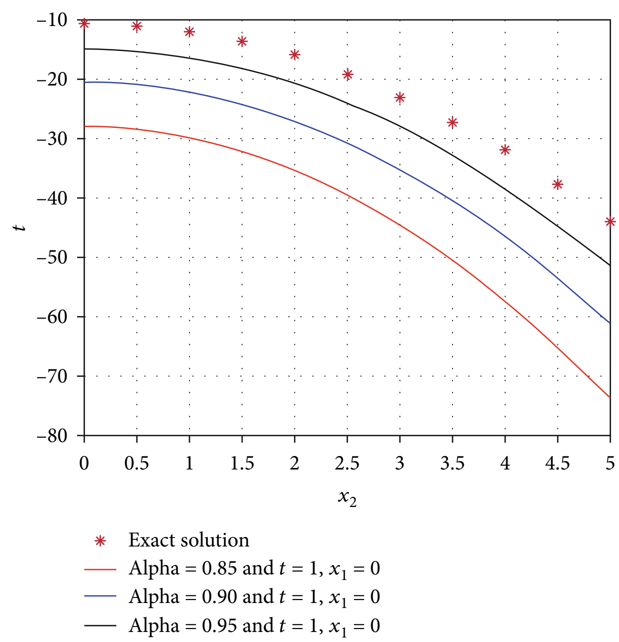
(a) The approximate solutions of $\phi(x_1, x_2, t)$ for Equation (98) with $x_2 = 0$, $t = 1$, and $\alpha = 1, 0.85, 0.90$, and 0.95



(b) The approximate solutions of $\psi(x_1, x_2, t)$ for Equation (98) with $x_2 = 0$, $t = 1$, and $\alpha = 1, 0.85, 0.90$, and 0.95

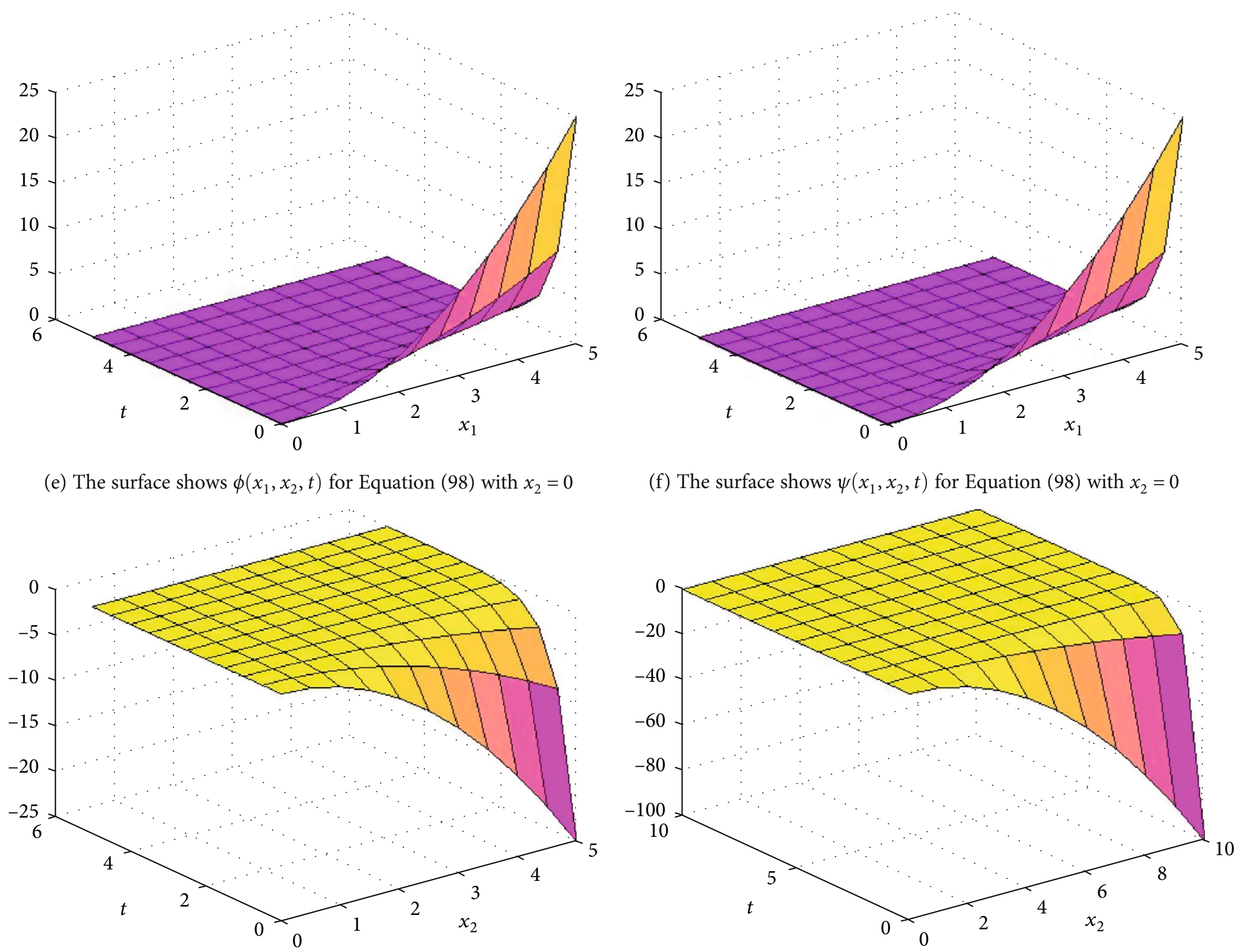


(c) The approximate solutions of $\phi(x_1, x_2, t)$ for Equation (98) with $x_1 = 0$, $t = 1$, and $\alpha = 1, 0.85, 0.90$, and 0.95



(d) The approximate solutions of $\psi(x_1, x_2, t)$ for Equation (98) with $x_1 = 0$, $t = 1$, and $\alpha = 1, 0.85, 0.90$, and 0.95

Figure 3: Continued.



(e) The surface shows $\phi(x_1, x_2, t)$ for Equation (98) with $x_2 = 0$

(f) The surface shows $\psi(x_1, x_2, t)$ for Equation (98) with $x_2 = 0$

(g) The surface shows $\phi(x_1, x_2, t)$ for Equation (98) with $x_1 = 0$

(h) The surface shows $\psi(x_1, x_2, t)$ for Equation (98) with $x_1 = 0$

Figure 3

$$\varphi_3 = \frac{32t^{3^\zeta-1}}{\Gamma(3^\zeta)} + \frac{96t^{3^\zeta}}{\Gamma(3^\zeta+1)} - \frac{8t^{3^\zeta}x_1^2}{\Gamma(3^\zeta+1)} + \frac{8t^{3^\zeta}x_2^2}{\Gamma(3^\zeta+1)}; \tag{107}$$

at $n = 3$,

$$\psi_4 = -\frac{96t^{4^\zeta-1}}{\Gamma(4^\zeta)} - \frac{256t^{4^\zeta}}{\Gamma(4^\zeta+1)} + \frac{16t^{4^\zeta}x_1^2}{\Gamma(4^\zeta+1)} - \frac{16t^{4^\zeta}x_2^2}{\Gamma(4^\zeta+1)}, \tag{108}$$

$$\varphi_4 = \frac{96t^{4^\zeta-1}}{\Gamma(4^\zeta)} - \frac{256t^{4^\zeta}}{\Gamma(4^\zeta+1)} + \frac{16t^{4^\zeta}x_1^2}{\Gamma(4^\zeta+1)} - \frac{16t^{4^\zeta}x_2^2}{\Gamma(4^\zeta+1)}, \tag{109}$$

thus for remaining elements. Using Equation (88), therefore, the approximate solutions are determined by

$$\begin{aligned}\psi(x_1, x_2, t) = {}& x_1^2 - x_2^2 - \frac{2t^\zeta x_1^2}{\Gamma(\zeta+1)} + \frac{2t^\zeta x_2^2}{\Gamma(\zeta+1)} + \frac{4t^{2^\zeta}x_1^2}{\Gamma(2^\zeta+1)} \\ & - \frac{4t^{2^\zeta}x_2^2}{\Gamma(2^\zeta+1)} - \frac{8t^{3^\zeta}x_1^2}{\Gamma(3^\zeta+1)} + \frac{8t^{3^\zeta}x_2^2}{\Gamma(3^\zeta+1)} \\ & + \frac{16t^{4^\zeta}x_1^2}{\Gamma(4^\zeta+1)} - \frac{16t^{4^\zeta}x_2^2}{\Gamma(4^\zeta+1)} + \frac{8t^\zeta}{\Gamma(\zeta+1)} \\ & - \frac{8t^{2^\zeta-1}}{\Gamma(2^\zeta)} - \frac{32t^{2^\zeta}}{\Gamma(2^\zeta+1)} + \frac{32t^{3^\zeta-1}}{\Gamma(3^\zeta)} + \frac{96t^{3^\zeta}}{\Gamma(3^\zeta+1)} \\ & - \frac{96t^{4^\zeta-1}}{\Gamma(4^\zeta)} - \frac{256t^{4^\zeta}}{\Gamma(4^\zeta+1)},\end{aligned} \tag{110}$$

$$
\begin{aligned}
\varphi(x_1,x_2,t) = {} & x_1^2 - x_2^2 - \frac{2t^{\zeta}x_1^2}{\Gamma(\zeta+1)} + \frac{2t^{\zeta}x_2^2}{\Gamma(\zeta+1)} + \frac{4t^{2^{\zeta}}x_1^2}{\Gamma(2^{\zeta}+1)} \\
& - \frac{4t^{2^{\zeta}}x_2^2}{\Gamma(2^{\zeta}+1)} - \frac{8t^{3^{\zeta}}x_1^2}{\Gamma(3^{\zeta}+1)} + \frac{8t^{3^{\zeta}}x_2^2}{\Gamma(3^{\zeta}+1)} + \frac{16t^{4^{\zeta}}x_1^2}{\Gamma(4^{\zeta}+1)} \\
& - \frac{16t^{4^{\zeta}}x_2^2}{\Gamma(4^{\zeta}+1)} + \frac{8t^{\zeta}}{\Gamma(\zeta+1)} - \frac{8t^{2^{\zeta}-1}}{\Gamma(2^{\zeta})} - \frac{32t^{2^{\zeta}}}{\Gamma(2^{\zeta}+1)} \\
& + \frac{32t^{3^{\zeta}-1}}{\Gamma(3^{\zeta})} + \frac{96t^{3^{\zeta}}}{\Gamma(3^{\zeta}+1)} - \frac{96t^{4^{\zeta}-1}}{\Gamma(4^{\zeta})} - \frac{256t^{4^{\zeta}}}{\Gamma(4^{\zeta}+1)}.
\end{aligned}
\tag{111}
$$

We set $\zeta = 1$, the fractional solution becomes

$$
\begin{aligned}
\psi(x_1,x_2,t) &= \psi_0 + \psi_1 + \psi_2 + \psi_3 + \cdots \\
&= \left(1 - (2t) + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} - \cdots\right)\left(x_1^2 - x_2^2\right), \\
\varphi(x_1,x_2,t) &= \varphi_0 + \varphi_1 + \varphi_2 + \varphi_3 + \cdots \\
&= \left(1 - (2t) + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} - \cdots\right)\left(x_1^2 - x_2^2\right).
\end{aligned}
\tag{112}
$$

Hence, the exact solution becomes

$$
\begin{aligned}
\psi(x_1,x_2,t) &= \left(x_1^2 - x_2^2\right)e^{-2t}, \\
\varphi(x_1,x_2,t) &= \left(x_1^2 - x_2^2\right)e^{-2t}.
\end{aligned}
\tag{113}
$$

Figures 3(a)–3(d) show the approximate and exact solutions of Equation (98); put $t = 1$ and $\alpha = 1$, we have the exact solution of Equation (98); by using different values of α such as $\alpha = 0.85$, $\alpha = 0.90$, and $\alpha = 0.95$, we get the approximate solution. The approximate solutions of the functions $\phi(x_1, x_2, t)$ and $\psi(x1_1, x_2), t$ are given by Figures 3(e)–3(h).

Conclusion 16. In the current study, a powerful method called the 3-DLADM was approved for finding approximate and exact solution of the time-fractional singular 2-D pseudo-parabolic equation. The suggested method is easier in its precept and active in solving linear and nonlinear singular two-dimensional pseudoparabolic equation. Therefore, we conclude that the 3-DLADM is very effective and more precise for any fractional order partial differential equations.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

The authors read and approved the final manuscript.

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