Research Article

On Controlled Rectangular Metric Spaces and an Application

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In this paper, we introduce the notion of controlled rectangular metric spaces as a generalization of rectangular metric spaces and rectangular b -metric spaces. Further, we establish some related fixed point results. Our main results extend many existing ones in the literature. The obtained results are also illustrated with the help of an example. In the last section, we apply our results to a common real-life problem in a general form by getting a solution for the Fredholm integral equation in the setting of controlled rectangular metric spaces.

1. Introduction

The fixed point theory is a growing and exciting field of mathematics with a variety of variant applications in mathematical sciences, proposing newer applications in discrete dynamics and super fractals. The fixed point theory is a fundamental tool to various theoretical and applied fields, such as variational and linear inequalities, the approximation theory, nonlinear analysis, integral and differential equations and inclusions, the dynamic systems theory, mathematics of fractals, mathematical economics (game theory, equilibrium problems, and optimization problems), and mathematical modeling; see [1–3]. In particular, fixed point techniques have been applied in such diverse fields; see [4, 5]. There are particular real-life problems, whose statements are fairly easy to understand, which can be argued using some versions of fixed point theorems; see [6, 7].

The notion of extended b -metric spaces was introduced by Kamran et al. [9] as a generalization of metric spaces and b -metric spaces [10, 11]. This metric type space has been generalized in several directions (for instance, controlled metric spaces [12], double controlled metric spaces [13], and others [14–19]). In a different perception, Branciari [20] proposed rectangular metric spaces. In the same order, Asim et al. [21] included a control function to initiate the concept of extended rectangular b -metric spaces as a generalization of rectangular b -metric spaces [22]. In [23], Mlaiki et al. introduced controlled rectangular b -metric spaces, which generalize rectangular metric spaces and rectangular b -metric spaces.

In this paper, our goal is to introduce the notion of controlled rectangular metric spaces, which is different from controlled rectangular b -metric spaces, and generalize rectangular metric spaces as well as rectangular b -metric spaces. Further, we prove some fixed point

results on such spaces as a generalization of many preexisting results in the literature. Also, we give examples for the justification of our results. In the last, as an application, we give an existence theorem for the Fredholm integral equation in the setting of controlled rectangular metric spaces.

2. Preliminaries

In this section, we collect some basic concepts related to our main results.

Definition 1 [22]. A mapping $d_\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [0, \infty)$ on a nonempty set $\mathcal{M}$ is called a rectangular b -metric space, if there exists a constant $s \geq 1$ such that for all $m_1, m_2 \in \mathcal{M}$ and all distinct $\mu_1, \mu_2 \in \mathcal{M}$ different from m_1 and m_2 , the following axioms are satisfied:

- (i) $d_\zeta(m_1, m_2) = 0$ iff $m_1 = m_2$
- (ii) $d_\zeta(m_1, m_2) = d_\zeta(m_2, m_1)$
- (iii) $d_\zeta(m_1, m_2) \leq s[d_\zeta(m_1, \mu_1) + d_\zeta(\mu_1, \mu_2) + d_\zeta(\mu_2, m_2)]$

In this case, the pair $(\mathcal{M}, d_\zeta)$ is called a rectangular b -metric space.

Definition 2 [9]. Let $\mathcal{M}$ be a nonempty set and $\zeta : \mathcal{M} \times \mathcal{M} \rightarrow [1, \infty)$ be a mapping. Then, a mapping $d_\zeta : \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty)$ is called an extended b -metric, if for all $m_1, m_2, m_3 \in \mathcal{M}$, it satisfies the following axioms:

- (i) $d_\zeta(m_1, m_2) = 0$ iff $m_1 = m_2$
- (ii) $d_\zeta(m_1, m_2) = d_\zeta(m_2, m_1)$
- (iii) $d_\zeta(m_1, m_3) \leq \zeta(m_1, m_3)[d_\zeta(m_1, m_2) + d_\zeta(m_2, m_3)]$

The pair $(\mathcal{M}, d_\zeta)$ is called an extended b -metric space.

Definition 3 [21]. A mapping $d_\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [0, \infty)$ on a nonempty set $\mathcal{M}$ is called an extended rectangular b -metric space, if for all $m_1, m_2 \in \mathcal{M}$ and all distinct $\mu_1, \mu_2 \in \mathcal{M}$ different from m_1 and m_2 , the following axioms are satisfied:

- (i) $d_\zeta(m_1, m_2) = 0$ iff $m_1 = m_2$
- (ii) $d_\zeta(m_1, m_2) = d_\zeta(m_2, m_1)$
- (iii) $d_\zeta(m_1, m_2) \leq \zeta(m_1, m_2)[d_\zeta(m_1, \mu_1) + d_\zeta(\mu_1, \mu_2) + d_\zeta(\mu_2, m_2)]$

where $\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [1, \infty)$ is a mapping. In this case, the pair $(\mathcal{M}, d_\zeta)$ is called an extended rectangular b -metric space.

Note that the topology of rectangular metric spaces need not be Hausdorff. For more examples, see the

papers of Sarma et al. [24] and Samet [25]. The topological structure of rectangular metric spaces is not compatible with the topology of classic metric spaces; see Example 7 in the paper of Suzuki [26]. In the same direction, extended rectangular b -metric spaces cannot be Hausdorff.

Definition 4 [23]. A mapping $d_\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [0, \infty)$ on a nonempty set $\mathcal{M}$ is called a controlled rectangular b -metric space, if for all distinct $m_1, m_2, \mu_1, \mu_2 \in \mathcal{M}$, the following axioms are satisfied:

- (i) $d_\zeta(m_1, m_2) = 0$ iff $m_1 = m_2$
- (ii) $d_\zeta(m_1, m_2) = d_\zeta(m_2, m_1)$
- (iii) $d_\zeta(m_1, m_2) \leq \zeta(m_1, m_2, \mu_1, \mu_2)[d_\zeta(m_1, \mu_1) + d_\zeta(\mu_1, \mu_2) + d_\zeta(\mu_2, m_2)]$

where $\zeta : \mathcal{M}^4 \longrightarrow [1, \infty)$ is a mapping. In this case, the pair $(\mathcal{M}, d_\zeta)$ is called a controlled rectangular b -metric space.

As a generalization of metric spaces, Mlaiki et al. in [12] introduced the concept of controlled metric spaces as follows.

Definition 5 [12]. Let $\mathcal{M}$ be a nonempty set and $\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [1, \infty)$. Then, a mapping $d_\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [0, \infty)$ is called a controlled metric, if for all $m_1, m_2, m_3 \in \mathcal{M}$, it satisfies the following axioms:

- (i) $d_\zeta(m_1, m_2) = 0$ iff $m_1 = m_2$
- (ii) $d_\zeta(m_1, m_2) = d_\zeta(m_2, m_1)$
- (iii) $d_\zeta(m_1, m_3) \leq \zeta(m_1, m_2)d_\zeta(m_1, m_2) + \zeta(m_2, m_3)d_\zeta(m_2, m_3)$

The pair $(\mathcal{M}, d_\zeta)$ is called a controlled metric space.

Note that Definition 5 generalizes b -metric spaces and is different from Definition 2.

Example 1 [12]. Let $\mathcal{M} = \{1, 2, \dots\}$. Define $d_\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [0, \infty)$ as

$$d_\zeta(x, y) = \begin{cases} 0, & \text{if } m_1 = m_2, \\ \frac{1}{m_1}, & \text{if } m_1 \text{ is even and } m_2 \text{ is odd,} \\ \frac{1}{m_2}, & \text{if } m_1 \text{ is odd and } m_2 \text{ is even,} \\ 1, & \text{otherwise.} \end{cases} \quad (1)$$

Hence, $(\mathcal{M}, d_\zeta)$ is a controlled metric space, where ζ

$: \mathcal{M} \times \mathcal{M} \longrightarrow [1, \infty)$ is defined as

$$\zeta(m_1, m_2) = \begin{cases} m_1, & \text{if } m_1 \text{ is even and } m_2 \text{ is odd,} \\ m_2, & \text{if } m_1 \text{ is odd and } m_2 \text{ is even,} \\ 1, & \text{otherwise.} \end{cases} \quad (2)$$

3. Main Results

In this section, we introduce the notion of controlled rectangular metric spaces. Also, we establish some fixed point results.

Definition 6. A mapping $d_\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [0, \infty)$ on a non-empty set $\mathcal{M}$ is called a controlled rectangular metric space, if for all $m_1, m_2 \in \mathcal{M}$ and all distinct $\mu_1, \mu_2 \in \mathcal{M}$ different from m_1 and m_2 , the following axioms are satisfied:

- (i) $d_\zeta(m_1, m_2) = 0$ iff $m_1 = m_2$
- (ii) $d_\zeta(m_1, m_2) = d_\zeta(m_2, m_1)$
- (iii) $d_\zeta(m_1, m_2) \leq \zeta(m_1, \mu_1)d_\zeta(m_1, \mu_1) + \zeta(\mu_1, \mu_2)d_\zeta(\mu_1, \mu_2) + \zeta(\mu_2, m_2)d_\zeta(\mu_2, m_2)$

where $\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [1, \infty)$ is a mapping. In this case, the pair $(\mathcal{M}, d_\zeta)$ is called a controlled rectangular metric space.

Remark 7.

- (i) Every rectangular metric space and rectangular b -metric is a controlled rectangular metric space
- (ii) Clearly, Definition 6 is different from Definition 4
- (iii) Every controlled metric space is a controlled rectangular metric space, but its converse is not true in general. See the following example

Example 2. Let $\mathcal{M} = [0, \infty)$. Define a mapping $d_\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [0, \infty)$ by

$$d_\zeta(m_1, m_2) = \begin{cases} 0, & \text{if } m_1 = m_2, \\ \frac{1}{m_1}, & \text{if } m_1 \geq 1 \text{ and } m_2 \in [0, 1), \\ \frac{1}{m_2}, & \text{if } m_2 \geq 1 \text{ and } m_1 \in [0, 1), \\ 1, & \text{otherwise.} \end{cases} \quad (3)$$

Then, d_ζ is a controlled rectangular metric space, where $\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [1, \infty)$ is a mapping defined as

$$\zeta(m_1, m_2) = \begin{cases} m_1, & \text{if } m_1, m_2 \geq 1, \\ 1, & \text{otherwise.} \end{cases} \quad (4)$$

Clearly, $(\mathcal{M}, d_\zeta)$ is not a controlled metric space if we take $m_1, m_3 \geq 1$ and $m_2 \in [0, 1)$. Then, $d_\zeta(m_1, m_3) = 1$, d_ζ

$(m_1, m_2) = 1/m_1$, $d_\zeta(m_2, m_3) = 1/m_3$, $\zeta(m_1, m_2) = 1$, and $\zeta(m_2, m_3) = 1$. Here, the triangle inequality is not satisfied:

$$\begin{aligned} d_\zeta(m_1, m_3) &= 1 > \zeta(m_1, m_2)d_\zeta(m_1, m_2) \\ &\quad + \zeta(m_2, m_3)d_\zeta(m_2, m_3) = \frac{1}{m_1} + \frac{1}{m_3}. \end{aligned} \quad (5)$$

Example 3. Let $\mathcal{M} = \{1, 2, 3, 4\}$. Define $d_\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [0, \infty)$ as

$$\begin{aligned} d_\zeta(1, 1) &= d_\zeta(2, 2) = d_\zeta(3, 3) = d_\zeta(4, 4) = 0, \\ d_\zeta(1, 2) &= d_\zeta(2, 1) = d_\zeta(2, 3) = d_\zeta(3, 2) = d_\zeta(3, 4) = d_\zeta(4, 3) \\ &= d_\zeta(1, 3) = d_\zeta(3, 1) = 80, \\ d_\zeta(1, 4) &= d_\zeta(4, 1) = 1000, \\ d_\zeta(2, 4) &= d_\zeta(4, 2) = 450. \end{aligned} \quad (6)$$

Then, $(\mathcal{M}, d_\zeta)$ is a controlled rectangular metric space with $\zeta : \mathcal{M} \times \mathcal{M} \longrightarrow [1, \infty)$ defined as $\zeta(m_1, m_2) = \max\{m_1, m_2\} + 2$, for all $m_1, m_2 \in \mathcal{M}$. However, $(\mathcal{M}, d_\zeta)$ is not a rectangular metric space; for instance, notice

$$d_\zeta(1, 4) = 1000 > d_\zeta(1, 2) + d_\zeta(2, 3) + d_\zeta(3, 4) = 240. \quad (7)$$

The concepts of convergence, Cauchyness, and completeness can simply be generalized in terms of controlled rectangular metric spaces.

Definition 8. Let $(\mathcal{M}, d_\zeta)$ be a controlled rectangular metric space. Then,

- (i) A sequence $\{m_n\}$ in $(\mathcal{M}, d_\zeta)$ is said to be convergent to $m \in \mathcal{M}$, if $\lim_{n \rightarrow \infty} d_\zeta(m_n, m) = 0$
- (ii) A sequence $\{m_n\}$ in $(\mathcal{M}, d_\zeta)$ is called a Cauchy sequence, if $\lim_{n, r \rightarrow \infty} d_\zeta(m_n, m_r) = 0$
- (iii) $(\mathcal{M}, d_\zeta)$ is called a complete controlled rectangular metric space, if every Cauchy sequence in $\mathcal{M}$ is convergent to some point of $\mathcal{M}$

Definition 9. Let $(\mathcal{M}, d_\zeta)$ be a controlled rectangular metric space. Let $m \in \mathcal{M}$ and $\tau > 0$. Then,

- (i) The open ball $B(m, \tau)$ is define as

$$B(m, \tau) = \{m_1 \in \mathcal{M}, d_\zeta(m, m_1) < \tau\}. \quad (8)$$

- (ii) The mapping $f : \mathcal{M} \longrightarrow \mathcal{M}$ is called continuous at $m \in \mathcal{M}$, if for $v > 0$, there is $v > 0$ such that $f(B(m, v))$

$v)) \subseteq B(\mathbf{f}(m), v)$. Thus, if $\mathbf{f}$ is continuous at m , then for any sequence $\{m_n\}$ converging to m , we have $\lim_{n \rightarrow \infty} \mathbf{f}m_n = \mathbf{f}m$.

Note that a rectangular b -metric space is not continuous in general, and it is the same for controlled rectangular metric spaces.

Lemma 10. Let $(\mathcal{M}, d_\zeta)$ be a controlled rectangular metric space and $\{m_n\}$ be a Cauchy sequence in $\mathcal{M}$ such that $m_n \neq m_r$, whenever $n \neq r$. If $\lim_{n,r \rightarrow \infty} \zeta(m_n, m_r) < \infty$ for all $m_n, m_r \in \mathcal{M}$, then $\{m_n\}$ has a unique limit.

Proof. Suppose that a sequence $\{m_n\}$ in $\mathcal{M}$ has two limit points $\mu, v \in \mathcal{M}$, that is, $\lim_{n \rightarrow \infty} m_n = \mu$ and $\lim_{n \rightarrow \infty} m_n = v$. $\{m_n\}$ is a Cauchy sequence for $m_n \neq m_r$, whenever $n \neq r$. Hence, from condition (iii) of Definition 6, we have

$$\begin{aligned} d_\zeta(\mu, v) &\leq \zeta(\mu, m_n) d_\zeta(\mu, m_n) + \zeta(m_n, m_r) d_\zeta(m_n, m_r) \\ &\quad + \zeta(m_r, v) d_\zeta(m_r, v) \longrightarrow 0 \quad \text{as } n, r \longrightarrow \infty. \end{aligned} \quad (9)$$

This implies that

$$d_\zeta(\mu, v) = 0. \quad (10)$$

Hence, $\{m_n\}$ has a unique limit point in $\mathcal{M}$.

Definition 11. Let $(\mathcal{M}, d_\zeta)$ be a controlled rectangular metric space. Then,

(i) For a mapping $\mathbf{f} : \mathcal{M} \longrightarrow \mathcal{M}$, we define

$$\begin{aligned} \mathcal{O}(m, n) &= \{m, \mathbf{f}m, \dots, \mathbf{f}^n m\}, \\ \mathcal{O}(m, \infty) &= \{m, \mathbf{f}m, \dots, \mathbf{f}^n m, \dots\}, \end{aligned} \quad (11)$$

where $m \in \mathcal{M}$ and $n \in \mathbb{N}$. The $\mathcal{O}(m, \infty)$ is called an orbit of $\mathbf{f}$.

(ii) A mapping $\mathbf{f} : \mathcal{M} \longrightarrow \mathcal{M}$ is called $\mathbf{f}$ -orbitally continuous, if $\lim_{k \rightarrow \infty} \mathbf{f}_k^n m = m$ implies $\lim_{k \rightarrow \infty} \mathbf{f}(\mathbf{f}_k^n m) = \mathbf{f}m$, where $m \in \mathcal{M}$.

(iii) A mapping $\mathbf{f} : \mathcal{M} \longrightarrow \mathcal{M}$ is called $\mathbf{f}$ -orbitally complete, if every Cauchy sequence in $\mathcal{O}(m, \infty)$ is convergent in $\mathcal{M}$.

Our main result is similar to the Banach contraction principle in the setting of controlled rectangular metric spaces. Throughout this section, for a mapping $\mathbf{f} : \mathcal{M} \longrightarrow \mathcal{M}$ and $m \in \mathcal{M}$, we consider an orbit $\mathcal{O}(m, \infty)$.

Theorem 12. Let $\mathbf{f} : \mathcal{M} \longrightarrow \mathcal{M}$ be a mapping on a controlled rectangular metric space $(\mathcal{M}, d_\zeta)$. Suppose that the following axioms hold:

(i) For all $m_1, m_2 \in \mathcal{M}$, we have

$$d_\zeta(\mathbf{f}m_1, \mathbf{f}m_2) \leq \lambda d_\zeta(m_1, m_2), \quad (12)$$

where $\lambda \in [0, 1)$.

(ii) $\sup_{q \geq 1} \lim_{i \rightarrow \infty} \zeta(m_i, m_q) (\zeta(m_{i+1}, m_{i+2}) / \zeta(m_{i-1}, m_i)) \lambda < 1$, for any $m_n \in \mathcal{M}$.

(iii) $(\mathcal{M}, d_\zeta)$ is $\mathbf{f}$ -orbitally complete

(iv) $\mathbf{f}$ is orbitally continuous

(v) For each $m \in \mathcal{M}$, $\lim_{n \rightarrow \infty} \zeta(m_n, \mu)$ and $\lim_{n \rightarrow \infty} \zeta(\mu, m_n)$ exist and are finite

Then, $\mathbf{f}$ has a unique fixed point in $\mathcal{M}$.

Proof. Consider an arbitrary point $m_0 \in \mathcal{M}$, and define an iterative sequence $\{m_n\}$ over m_0 as follows:

$$m_1 = \mathbf{f}m_0, m_2 = \mathbf{f}m_1 = \mathbf{f}(\mathbf{f}m_0) = \mathbf{f}^2 m_0, \dots, m_n = \mathbf{f}^n m_0, \dots \quad (13)$$

From equation (12), we have

$$d_\zeta(m_1, m_2) = d_\zeta(\mathbf{f}m_0, \mathbf{f}^2 m_0) \leq \lambda d_\zeta(m_0, \mathbf{f}m_0) = d_\zeta(m_0, m_1). \quad (14)$$

Recursively, we have

$$\begin{aligned} d_\zeta(m_n, m_{n+1}) &= d_\zeta(\mathbf{f}^n m_0, \mathbf{f}^{n+1} m_0) \leq \lambda d_\zeta(\mathbf{f}^{n-1} m_0, \mathbf{f}^n m_0) \\ &\leq \lambda^2 d_\zeta(\mathbf{f}^{n-2} m_0, \mathbf{f}^{n-1} m_0) \dots \\ &\leq \lambda^n d_\zeta(m_0, m_1). \end{aligned} \quad (15)$$

That is,

$$d_\zeta(m_n, m_{n+1}) \leq \lambda^n d_\zeta(m_0, m_1). \quad (16)$$

By taking limit $n \longrightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} d_\zeta(m_n, m_{n+1}) = 0, \quad (17)$$

$$\lim_{n \rightarrow \infty} d_\zeta(m_{n+1}, m_{n+2}) = 0. \quad (18)$$

Next, we show that $\{m_n\}$ is a Cauchy sequence. For this, we will take the following two cases.

Case 1. Let ρ be odd, that is, $\rho = 2r + 1$, where $r \geq 1$. Then, from condition (iii) of Definition 6 and equation (16) for $n + \rho > n$, we have

$$\begin{aligned}
 d_\zeta(m_n, m_{n+2r+1}) &\leq \zeta(m_n, m_{n+1})d_\zeta(m_n, m_{n+1}) + \zeta \\
 &\cdot (m_{n+1}, m_{n+2})d_\zeta(m_{n+1}, m_{n+2}) + \zeta(m_{n+2}, m_{n+2r+1}) \\
 &\times d_\zeta(m_{n+2}, m_{n+2r+1}) \leq [\zeta(m_n, m_{n+1})\lambda^n + \zeta \\
 &\cdot (m_{n+1}, m_{n+2})\lambda^{n+1}]d_\zeta(m_0, m_1) + \zeta(m_{n+2}, m_{n+2r+1})d_\zeta \\
 &\cdot (m_{n+2}, m_{n+2r+1}) \leq [\zeta(m_n, m_{n+1})\lambda^n + \zeta \\
 &\cdot (m_{n+1}, m_{n+2})\lambda^{n+1}]d_\zeta(m_0, m_1) + \zeta(m_{n+2}, m_{n+2r+1})\zeta \\
 &\cdot (m_{n+2}, m_{n+3}) \times d_\zeta(m_{n+2}, m_{n+3}) + \zeta(m_{n+2}, m_{n+2r+1})\zeta \\
 &\cdot (m_{n+3}, m_{n+4})d_\zeta(m_{n+3}, m_{n+4}) + \zeta(m_{n+2}, m_{n+2r+1}) \times \zeta \\
 &\cdot (m_{n+4}, m_{n+2r+1})d_\zeta(m_{n+4}, m_{n+2r+1}) \leq d_\zeta \\
 &\cdot (m_0, m_1)[\zeta(m_n, m_{n+1})\lambda^n + \zeta(m_{n+2}, m_{n+3})\lambda^{n+1}] \\
 &+ d_\zeta(m_0, m_1) \times [\zeta(m_{n+2}, m_{n+2r+1})\zeta(m_{n+2}, m_{n+3})\lambda^{n+2} \\
 &+ \zeta(m_{n+2}, m_{n+2r+1})\zeta(m_{n+3}, m_{n+4})\lambda^{n+3}] + \dots + d_\zeta \\
 &\cdot (m_0, m_1)\zeta(m_{n+2}, m_{n+2r+1}) \dots \zeta(m_{n+2r-2}, m_{n+2n-1})\lambda^{n+2r-2} \\
 &+ \zeta(m_{n+2}, m_{n+2r+1}) \dots \zeta(m_{n+2r-1}, m_{n+2r})\lambda^{n+2r-1} \\
 &+ \zeta(m_{n+2}, m_{n+2r+1}) \dots \zeta(m_{n+2r}, m_{n+2r+1})\lambda^{n+2r}d_\zeta(m_0, m_1) \\
 &\leq d_\zeta(m_0, m_1)[\zeta(m_n, m_{n+1})\lambda^n + \zeta(m_{n+2}, m_{n+3})\lambda^{n+1}] \\
 &+ \sum_{i=1}^{r-1} \prod_{j=1}^i \zeta(m_{n+2j}, m_{n+2r+1}) \times \zeta(m_{n+2i}, m_{n+2i+1})\lambda^{n+2i}d_\zeta \\
 &\cdot (m_0, m_1) + \sum_{i=1}^{r-1} \prod_{j=1}^i \zeta(m_{n+2j+1}, m_{n+2r+1})\zeta(m_{n+2i+1}, m_{n+2i+2}) \\
 &\times \lambda^{n+2i+1}d_\zeta(m_0, m_1) + \prod_{i=1}^{r-1} \zeta(m_{n+2j}, m_{n+2r+1})\lambda^{n+2r}d_\zeta(m_0, m_1).
 \end{aligned} \tag{19}$$

As

$$\begin{aligned}
 &\sum_{i=1}^{r-1} \prod_{j=1}^i \zeta(m_{n+2j}, m_{n+2r+1})\zeta(m_{n+2i}, m_{n+2i+1})\lambda^{n+2i} \\
 &\leq \sum_{i=1}^{r-1} \prod_{j=1}^i \zeta(m_{2j}, m_{n+2r+1})\zeta(m_{2i}, m_{n+2i+1})\lambda^{2i},
 \end{aligned} \tag{20}$$

$$\begin{aligned}
 &\sum_{i=1}^{r-1} \prod_{j=1}^i \zeta(m_{n+2j+1}, m_{n+2r+1})\zeta(m_{n+2i+1}, m_{n+2i+2})\lambda^{n+2i+1} \\
 &\leq \sum_{i=1}^{r-1} \prod_{j=1}^i \zeta(m_{2j+1}, m_{n+2r+1})\zeta(m_{2i+1}, m_{n+2i+2})\lambda^{2i+1},
 \end{aligned} \tag{21}$$

therefore, we obtain

$$\begin{aligned}
 d_\zeta(m_n, m_{n+2r+1}) &\leq d_\zeta(m_0, m_1)[\zeta(m_n, m_{n+1})\lambda^n \\
 &+ \zeta(m_{n+2}, m_{n+3})\lambda^{n+1}] + \sum_{i=1}^{r-1} \prod_{j=1}^i \zeta(m_{2j}, m_{n+2r+1}) \\
 &\times \zeta(m_{2i}, m_{2i+1})\lambda^{2i}d_\zeta(m_0, m_1) \\
 &+ \sum_{i=1}^{r-1} \prod_{j=1}^i \zeta(m_{2j+1}, m_{n+2r+1})\zeta(m_{2i+1}, m_{2i+2}) \\
 &\times \lambda^{2i+1}d_\zeta(m_0, m_1) + \prod_{i=1}^{r-1} \zeta(m_{n+2j}, m_{n+2r+1})\lambda^{n+2r}d_\zeta(m_0, m_1).
 \end{aligned} \tag{22}$$

Since $\sup_{q \geq 1} \lim_{i \rightarrow \infty} \zeta(m_i, m_q)(\zeta(m_{i+1}, m_{i+2})/\zeta(m_{i-1}, m_i))\lambda < 1$, the series

$$\begin{aligned}
 &\sum_{i=1}^{\infty} \prod_{j=1}^i \zeta(m_{2j}, m_{n+2r+1})\zeta(m_{2i}, m_{2i+1})\lambda^{2i}, \\
 &\sum_{i=1}^{\infty} \prod_{j=1}^i \zeta(m_{2j+1}, m_{n+2r+1})\zeta(m_{2i+1}, m_{2i+2})\lambda^{2i+1},
 \end{aligned} \tag{23}$$

converge by the ratio test. Let

$$\mathcal{S} = \sum_{i=1}^{\infty} \prod_{j=1}^i \zeta(m_{2j}, m_{n+2r+1})\zeta(m_{2i}, m_{2i+1})\lambda^{2i}, \tag{24}$$

$$\mathcal{S}_n = \sum_{i=1}^n \prod_{j=1}^i \zeta(m_{2j}, m_{n+2r+1})\zeta(m_{2i}, m_{2i+1})\lambda^{2i},$$

$$\mathcal{S}' = \sum_{i=1}^{\infty} \prod_{j=1}^i \zeta(m_{2j+1}, m_{n+2r+1})\zeta(m_{2i+1}, m_{2i+2})\lambda^{2i+1},$$

$$\mathcal{S}'_n = \sum_{i=1}^n \prod_{j=1}^i \zeta(m_{2j+1}, m_{n+2r+1})\zeta(m_{2i+1}, m_{2i+2})\lambda^{2i+1}. \tag{25}$$

Then, equation (22) takes the following form:

$$\begin{aligned}
 d_\zeta(m_n, m_{n+2r+1}) &\leq d_\zeta(m_0, m_1)[\zeta(m_n, m_{n+1})\lambda^n \\
 &+ \zeta(m_{n+2}, m_{n+3})\lambda^{n+1}] + d_\zeta(m_0, m_1)[\mathcal{S}_{r-1} - \mathcal{S}_{n+1}]d_\zeta \\
 &\cdot (m_0, m_1) + d_\zeta(m_0, m_1)[\mathcal{S}'_{r-1} - \mathcal{S}'_{n+1}]d_\zeta(m_0, m_1) \\
 &+ \prod_{i=1}^{r-1} \zeta(m_{n+2j}, m_{n+2r+1})\lambda^{n+2r}d_\zeta(m_0, m_1).
 \end{aligned} \tag{26}$$

By taking limit $n \rightarrow \infty$ in equation (26), we get

$$\lim_{n \rightarrow \infty} d_\zeta(m_n, m_{n+2r+1}) = 0. \tag{27}$$

Case 2. Let ρ be even, that is, $\rho = 2r$, where $r \geq 1$. Then, from condition (iii) of Definition 6 and equation (16) for $n + \rho > n$, we have

$$\begin{aligned}
 d_\zeta(m_n, m_{n+2r}) &\leq \zeta(m_n, m_{n+1})d_\zeta(m_n, m_{n+1}) \\
 &+ \zeta(m_{n+1}, m_{n+2})d_\zeta(m_{n+1}, m_{n+2}) + \zeta(m_{n+2}, m_{n+2r}) \\
 &\times d_\zeta(m_{n+2}, m_{n+2r}) \leq [\zeta(m_n, m_{n+1})\lambda^n \\
 &+ \zeta(m_{n+1}, m_{n+2})\lambda^{n+1}]d_\zeta(m_0, m_1) \\
 &+ \zeta(m_{n+2}, m_{n+2r})d_\zeta(m_{n+2}, m_{n+2r}) \leq [\zeta(m_n, m_{n+1})\lambda^n \\
 &+ \zeta(m_{n+1}, m_{n+2})\lambda^{n+1}]d_\zeta(m_0, m_1) \\
 &+ \zeta(m_{n+2}, m_{n+2r})\zeta(m_{n+2}, m_{n+3}) \times d_\zeta(m_{n+2}, m_{n+3}) \\
 &+ \zeta(m_{n+2}, m_{n+2r})\zeta(m_{n+3}, m_{n+4})d_\zeta(m_{n+3}, m_{n+4}) \\
 &+ \zeta(m_{n+2}, m_{n+2r}) \times \zeta(m_{n+4}, m_{n+2r})d_\zeta(m_{n+4}, m_{n+2r}) \\
 &\leq d_\zeta(m_0, m_1)[\zeta(m_n, m_{n+1})\lambda^n + \zeta(m_{n+2}, m_{n+3})\lambda^{n+1}] \\
 &+ d_\zeta(m_0, m_1) \times [\zeta(m_{n+2}, m_{n+2r})\zeta(m_{n+2}, m_{n+3})\lambda^{n+2} \\
 &+ \zeta(m_{n+2}, m_{n+2r})\zeta(m_{n+3}, m_{n+4})\lambda^{n+3}] \\
 &+ \dots + d_\zeta(m_0, m_1)\zeta(m_{n+2}, m_{n+2r}) \dots \zeta \\
 &\cdot (m_{n+2r-4}, m_{n+2r-3})\lambda^{n+2r-4} + \zeta(m_{n+2}, m_{n+2r}) \dots \zeta \\
 &\cdot (m_{n+2r-3}, m_{n+2r-2})\lambda^{n+2r-3} + \zeta(m_{n+2}, m_{n+2r}) \dots \zeta \\
 &\cdot (m_{n+2r-2}, m_{n+2r})\lambda^{n+2r-2}d_\zeta(m_0, m_2) \leq d_\zeta \\
 &\cdot (m_0, m_1)[\zeta(m_n, m_{n+1})\lambda^n + \zeta(m_{n+2}, m_{n+3})\lambda^{n+1}] \\
 &+ \sum_{i=1}^{r-2} \prod_{j=1}^i \zeta(m_{n+2j}, m_{n+2r}) \times \zeta(m_{n+2i}, m_{n+2i+1})\lambda^{n+2i}d_\zeta \\
 &\cdot (m_0, m_1) + \sum_{i=1}^{r-2} \prod_{j=1}^i \zeta(m_{n+2j+1}, m_{n+2r})\zeta \\
 &\cdot (m_{n+2i+1}, m_{n+2i+2}) \times \lambda^{n+2i+1}d_\zeta(m_0, m_1) \\
 &+ \prod_{i=1}^{r-1} \zeta(m_{n+2j}, m_{n+2r})\lambda^{n+2r-2}d_\zeta(m_0, m_2).
 \end{aligned} \tag{28}$$

As

$$\begin{aligned}
 &\sum_{i=1}^{r-2} \prod_{j=1}^i \zeta(m_{n+2j}, m_{n+2r})\zeta(m_{n+2i}, m_{n+2i+1})\lambda^{n+2i} \\
 &\leq \sum_{i=1}^{r-2} \prod_{j=1}^i \zeta(m_{2j}, m_{n+2r})\zeta(m_{2i}, m_{2i+1})\lambda^{2i}, \\
 &\sum_{i=1}^{r-2} \prod_{j=1}^i \zeta(m_{n+2j+1}, m_{n+2r+1})\zeta(m_{n+2i+1}, m_{n+2i+2})\lambda^{n+2i+1} \\
 &\leq \sum_{i=1}^{r-2} \prod_{j=1}^i \zeta(m_{2j+1}, m_{n+2r+1})\zeta(m_{2i+1}, m_{2i+2})\lambda^{2i+1},
 \end{aligned} \tag{29}$$

$$\tag{30}$$

therefore, we obtain

$$\begin{aligned}
 d_\zeta(m_n, m_{n+2r}) &\leq d_\zeta(m_0, m_1)[\zeta(m_n, m_{n+1})\lambda^n \\
 &+ \zeta(m_{n+2}, m_{n+3})\lambda^{n+1}] + \sum_{i=1}^{r-2} \prod_{j=1}^i \zeta(m_{2j}, m_{n+2r}) \\
 &\times \zeta(m_{2i}, m_{2i+1})\lambda^{2i}d_\zeta(m_0, m_1) \\
 &+ \sum_{i=1}^{r-2} \prod_{j=1}^i \zeta(m_{2j+1}, m_{n+2r})\zeta(m_{2i+1}, m_{2i+2}) \\
 &\times \lambda^{2i+1}d_\zeta(m_0, m_1) + \prod_{i=1}^{r-1} \zeta \\
 &\cdot (m_{n+2j}, m_{n+2r})\lambda^{n+2r-2}d_\zeta(m_0, m_2).
 \end{aligned} \tag{31}$$

Since $\sup_{q \geq 1} \lim_{i \rightarrow \infty} \zeta(m_i, m_q)(\zeta(m_{i+1}, m_{i+2})/\zeta(m_{i-1}, m_i))\lambda < 1$, the series

$$\begin{aligned}
 &\sum_{i=1}^{\infty} \prod_{j=1}^i \zeta(m_{2j}, m_{n+2r})\zeta(m_{2i}, m_{2i+1})\lambda^{2i}, \\
 &\sum_{i=1}^{\infty} \prod_{j=1}^i \zeta(m_{2j+1}, m_{n+2r})\zeta(m_{2i+1}, m_{2i+2})\lambda^{2i+1},
 \end{aligned} \tag{32}$$

converge by the ratio test. Let

$$\mathcal{S} = \sum_{i=1}^{\infty} \prod_{j=1}^i \zeta(m_{2j}, m_{n+2r})\zeta(m_{2i}, m_{2i+1})\lambda^{2i}, \tag{33}$$

$$\mathcal{S}_n = \sum_{i=1}^n \prod_{j=1}^i \zeta(m_{2j}, m_{n+2r})\zeta(m_{2i}, m_{2i+1})\lambda^{2i},$$

$$\mathcal{S}' = \sum_{i=1}^{\infty} \prod_{j=1}^i \zeta(m_{2j+1}, m_{n+2r})\zeta(m_{2i+1}, m_{2i+2})\lambda^{2i+1}, \tag{34}$$

$$\mathcal{S}'_n = \sum_{i=1}^n \prod_{j=1}^i \zeta(m_{2j+1}, m_{n+2r})\zeta(m_{2i+1}, m_{2i+2})\lambda^{2i+1}.$$

Then, equation (31) takes the following form:

$$\begin{aligned}
 d_\zeta(m_n, m_{n+2r}) &\leq d_\zeta(m_0, m_1)[\zeta(m_n, m_{n+1})\lambda^n \\
 &+ \zeta(m_{n+2}, m_{n+3})\lambda^{n+1}] + d_\zeta(m_0, m_1)[\mathcal{S}_{r-1} - \mathcal{S}_{n+1}] \\
 &+ d_\zeta(m_0, m_1)[\mathcal{S}'_{r-1} - \mathcal{S}'_{n+1}] \\
 &+ \prod_{i=1}^{r-1} \zeta(m_{n+2j}, m_{n+2r})\lambda^{n+2r-2}d_\zeta(m_0, m_1).
 \end{aligned} \tag{35}$$

By taking limit $n \rightarrow \infty$ in equation (35), we get

$$\lim_{n \rightarrow \infty} d_\zeta(m_n, m_{n+2r}) = 0. \tag{36}$$

Hence, in both cases, $\lim_{n \rightarrow \infty} d_\zeta(m_n, m_{n+\rho}) = 0$,

which shows that $\{m_n\}$ is a Cauchy sequence. As $\mathcal{M}$ is $\mathbf{f}$ -orbitally complete, so there exists $\mu \in \mathcal{M}$ such that $\lim_{n \rightarrow \infty} m_n = \mu$. Next, we show that μ is a fixed point of $\mathbf{f}$. As $\mathbf{f}$ is orbitally continuous, so we have

$$d_\zeta(\mu, \mathbf{f}\mu) \leq \zeta(\mu, m_n)d_\zeta(\mu, m_n) + \zeta(m_n, m_{n+1})d_\zeta(m_n, m_{n+1}) + \zeta(m_{n+1}, \mathbf{f}\mu)d_\zeta(m_{n+1}, \mathbf{f}\mu). \quad (37)$$

Since for each $m \in \mathcal{M}$, $\lim_{n \rightarrow \infty} \zeta(m_n, m)$ and $\lim_{n \rightarrow \infty} \zeta(m, m_n)$ exist and are finite, so by taking limit $n \rightarrow \infty$ and using equation (17), we get

$$\lim_{n \rightarrow \infty} d_\zeta(\mu, \mathbf{f}\mu) = 0. \quad (38)$$

Therefore, $\mathbf{f}\mu = \mu$. Hence, μ is a fixed point of $\mathbf{f}$. In view of Lemma 10, m is the unique fixed point of $\mathbf{f}$.

Example 4. Let $X = [1, 2]$. Define $d_\zeta : \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty)$ by $d_\zeta(m_1, m_2) = (m_1 - m_2)^2$. Then, $(\mathcal{M}, d_\zeta)$ is a complete controlled rectangular metric space with $\zeta : \mathcal{M} \times \mathcal{M} \rightarrow [1, \infty)$ defined as $\zeta(m_1, m_2) = 3m_1 + 2m_2 + 5$. Define a mapping $\mathbf{f} : \mathcal{M} \rightarrow \mathcal{M}$ by

$$\mathbf{f}m = \frac{m}{4}. \quad (39)$$

Clearly, all the axioms of Theorem 12 are satisfied, and hence, $m = 0$ is a fixed point of $\mathbf{f}$.

Corollary 13. Let $\mathbf{f} : \mathcal{M} \rightarrow \mathcal{M}$ be a mapping on a complete controlled rectangular metric space $(\mathcal{M}, d_\zeta)$. Suppose that the following axioms hold:

(i) For all $m_1, m_2 \in \mathcal{M}$, we have

$$d_\zeta(\mathbf{f}m_1, \mathbf{f}m_2) \leq \lambda d_\zeta(m_1, m_2), \quad \lambda \in [0, 1). \quad (40)$$

(ii) $\sup_{q \geq 1} \lim_{i \rightarrow \infty} \zeta(m_i, m_q)(\zeta(m_{i+1}, m_{i+2})/\zeta(m_{i-1}, m_i))\lambda < 1$, for any $m_n \in \mathcal{M}$

(iii) $\mathbf{f}$ is continuous

Then, $\mathbf{f}$ has a unique fixed point.

Remark 14.

(i) By putting $\zeta(m_1, m_2) = s$, for all $m_1, m_2 \in \mathcal{M}$ in Theorem 12, we get Theorem 2.1 of George et al. [22]

(ii) By putting $\zeta(m_1, m_2) = 1$, for all $m_1, m_2 \in \mathcal{M}$ in Theorem 12, we get the following corollary in view of Das and Dey [27]

Corollary 15. Let $\mathbf{f} : \mathcal{M} \rightarrow \mathcal{M}$ be a mapping on a rectangular metric space $(\mathcal{M}, d_\zeta)$. Suppose that the following axioms hold:

(i) For all $m_1, m_2 \in \mathcal{M}$, $d_\zeta(\mathbf{f}m_1, \mathbf{f}m_2) \leq \lambda d_\zeta(m_1, m_2)$, where $\lambda \in [0, 1)$

(ii) $(\mathcal{M}, d_\zeta)$ is $\mathbf{f}$ -orbitally complete

(iii) $\mathbf{f}$ is orbitally continuous

Then, $\mathbf{f}$ has a unique fixed point.

Theorem 16. Let $\mathbf{f} : \mathcal{M} \rightarrow \mathcal{M}$ be a mapping on complete controlled rectangular metric space $(\mathcal{M}, d_\zeta)$, which satisfies the following axioms:

(i) For all $m_1, m_2 \in \mathcal{M}$, we have

$$d_\zeta(\mathbf{f}m_1, \mathbf{f}m_2) \leq \lambda [d_\zeta(m_1, \mathbf{f}m_1) + d_\zeta(m_2, \mathbf{f}m_2)], \quad (41)$$

where $\lambda \in [0, 1/2)$.

(ii) $\sup_{q \geq 1} \lim_{i \rightarrow \infty} \zeta(m_i, m_q)(\zeta(m_{i+1}, m_{i+2})/\zeta(m_{i-1}, m_i))\lambda < 1$, for any $m_n \in \mathcal{M}$, where $\lambda \neq 1/\zeta(m_1, m_2)$ for each $m_1, m_2 \in \mathcal{M}$

(iii) For each $m \in \mathcal{M}$, $\lim_{n \rightarrow \infty} \zeta(m_n, m_{n+1}) \leq 1$, $\lim_{n \rightarrow \infty} \zeta(m_n, \mu)$, and $\lim_{n \rightarrow \infty} \zeta(\mu, m_n)$ exist and are finite

Then, $\mathbf{f}$ has a unique fixed point in $\mathcal{M}$.

Proof. Let us take an arbitrary element $m_0 \in \mathcal{M}$ and choose $m_1 = \mathbf{f}m_0$ and $m_2 = \mathbf{f}m_1$. Then, from equation (41), we obtain

$$d_\zeta(m_1, m_2) = d_\zeta(\mathbf{f}m_0, \mathbf{f}m_1) \leq \lambda [d_\zeta(m_0, \mathbf{f}m_0) + d_\zeta(m_1, \mathbf{f}m_1)] = \lambda [d_\zeta(m_0, m_1) + d_\zeta(m_1, m_2)]. \quad (42)$$

This implies that

$$d_\zeta(m_1, m_2) \leq \frac{\lambda}{1-\lambda} d_\zeta(m_0, m_1), \quad (43)$$

where $\omega = \lambda/(1-\lambda) < 1$, as $\lambda < 1/2$. By recursively applying equation (41), we obtain

$$d_\zeta(m_n, m_{n+1}) \leq \omega^n d_\zeta(m_0, m_1). \quad (44)$$

Thus, by taking the limit in equation (44), we have

$$\lim_{n \rightarrow \infty} d_\zeta(m_n, m_{n+1}) = 0. \quad (45)$$

Again from equation (41), we have

$$\begin{aligned} d_{\zeta}(m_n, m_{n+2}) &= d_{\zeta}(\mathbf{f}m_{n-1}, \mathbf{f}m_{n+1}) \leq \lambda [d_{\zeta}(m_{n-1}, \mathbf{f}m_{n-1}) \\ &\quad + d_{\zeta}(m_{n+1}, \mathbf{f}m_{n+1})] = \lambda [d_{\zeta}(m_{n-1}, m_n) \\ &\quad + d_{\zeta}(m_{n+1}, m_{n+2})]. \end{aligned} \quad (46)$$

By using equation (45), we obtain

$$\lim_{n \rightarrow \infty} d_{\zeta}(m_n, m_{n+2}) = 0. \quad (47)$$

Now, we will show that $\{m_n\}$ is a Cauchy sequence. By following the same procedure as in the proof of Theorem 12 and using equations (45) and (47), we conclude that $\{m_n\}$ is a Cauchy sequence. As $\mathcal{M}$ is complete, so there exists $\mu \in \mathcal{M}$ such that

$$\lim_{n \rightarrow \infty} d_{\zeta}(m_n, \mu) = 0. \quad (48)$$

Next, we show that μ is a fixed point of $\mathbf{f}$. From condition (iii) of Definition 6, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} d_{\zeta}(\mu, \mathbf{f}\mu) &\leq \zeta(\mu, m_n) d_{\zeta}(\mu, m_n) + \zeta(m_n, m_{n+1}) d_{\zeta}(m_n, m_{n+1}) \\ &\quad + \zeta(m_{n+1}, \mathbf{f}\mu) d_{\zeta}(m_{n+1}, \mathbf{f}\mu) \leq \zeta(\mu, m_n) d_{\zeta}(\mu, m_n) \\ &\quad + \zeta(m_n, m_{n+1}) d_{\zeta}(m_n, m_{n+1}) + \zeta(m_{n+1}, \mathbf{f}\mu) d_{\zeta} \\ &\quad \cdot (\mathbf{f}m_n, \mathbf{f}\mu) \leq \zeta(\mu, m_n) d_{\zeta}(\mu, m_n) \\ &\quad + \zeta(m_n, m_{n+1}) d_{\zeta}(m_n, m_{n+1}) \\ &\quad + \zeta(m_{n+1}, \mathbf{f}\mu) \lambda [d_{\zeta}(m_n, \mathbf{f}m_n) + d_{\zeta}(\mu, \mathbf{f}\mu)]. \end{aligned} \quad (49)$$

$\lambda \neq 1/\zeta(m_1, m_2)$ for each $m_1, m_2 \in \mathcal{M}$. This implies that

$$\begin{aligned} d_{\zeta}(\mu, \mathbf{f}\mu) &\leq \frac{\zeta(\mu, m_n)}{(1 - \lambda\zeta(m_{n+1}, \mathbf{f}\mu))} d_{\zeta}(\mu, m_n) \\ &\quad + \frac{\zeta(m_n, m_{n+1}) + \lambda\zeta(m_{n+1}, \mathbf{f}\mu)}{(1 - \lambda\zeta(m_{n+1}, \mathbf{f}\mu))} d_{\zeta}(m_n, m_{n+1}). \end{aligned} \quad (50)$$

For each $m \in \mathcal{M}$, $\lim_{n \rightarrow \infty} \zeta(m_n, m_{n+1}) \leq 1$, $\lim_{n \rightarrow \infty} \zeta(m_n, m)$, and $\lim_{n \rightarrow \infty} \zeta(m, m_n)$ exist and are finite. Therefore, by taking limit $n \rightarrow \infty$ in equation (50) and using equations (47) and (48), we obtain

$$d_{\zeta}(\mu, \mathbf{f}\mu) = 0, \quad (51)$$

which implies that $\mathbf{f}\mu = \mu$. For the uniqueness, let ν be another fixed point of $\mathbf{f}$ and $\mu \neq \nu$. From equation (41), we obtain

$$\begin{aligned} d_{\zeta}(\mu, \nu) &= d_{\zeta}(\mathbf{f}\mu, \mathbf{f}\nu) \leq \lambda [d_{\zeta}(\mu, \mathbf{f}\mu) + d_{\zeta}(\nu, \mathbf{f}\nu)] \\ &= \lambda [d_{\zeta}(\mu, \mu) + d_{\zeta}(\nu, \nu)], \end{aligned} \quad (52)$$

where $d_{\zeta}(\mu, \mu) = 0$ and $d_{\zeta}(\nu, \nu) = 0$. Hence, from the above inequality, we obtain $d_{\zeta}(\mu, \nu) = 0$, that is, $\mu = \nu$, and μ is a unique fixed point of $\mathbf{f}$.

4. Application

In this section, we will apply Corollary 13 to prove the existence and uniqueness of a solution for the following Fredholm integral equation:

$$m(t) = \int_a^b \tau(t, r, m(r)) dr + v(t), \quad \text{for } t, r \in [a, b], \quad (53)$$

where $v : [a, b] \rightarrow \mathbb{R}$ and $\tau : [a, b] \times [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ both are continuous functions. Let $\mathcal{M} = \mathbb{C}([a, b])$ be the space of all continuous real-valued functions defined on the closed interval $[a, b]$. Consider

$$d_{\zeta}(m_1, m_2) = \sup_{t \in [a, b]} |m_1(t) - m_2(t)|^2. \quad (54)$$

Clearly, $(\mathcal{M}, d_{\zeta})$ is a complete controlled rectangular metric space with $\zeta : \mathcal{M} \times \mathcal{M} \rightarrow [1, \infty)$ defined as $\zeta(m_1, m_2) = 3|m_1(t)| + 2|m_2(t)| + 5$. Next, we will prove our result as follows.

Theorem 17. For all $m_1, m_2 \in \mathcal{M}$ and $t, r \in [a, b]$, the following condition holds:

$$|\tau(t, r, m_1(r)) - \tau(t, r, m_2(r))| \leq \frac{1}{2(b-a)} |m_1(r) - m_2(r)|. \quad (55)$$

Then, the integral equation (53) has a unique solution.

Proof. Define $\mathbf{f} : \mathcal{M} \rightarrow \mathcal{M}$ by

$$\mathbf{f}m(t) = \int_a^b \tau(t, r, m(r)) dr + v(t), \quad \text{for } t, r \in [a, b], \quad (56)$$

where $v : [a, b] \rightarrow \mathbb{R}$ and $\tau : [a, b] \times [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ both are continuous functions. Clearly, γ is a fixed point of $\mathbf{f}$, if and only if γ is a solution of the integral equation (53). For all $m_1, m_2 \in \mathcal{M}$, we have

$$\begin{aligned} |\mathbf{f}m_1(t) - \mathbf{f}m_2(t)|^2 &= \left| \int_a^b [\tau(t, r, m_1(r)) - \tau(t, r, m_2(r))] dr \right|^2 \\ &\leq \frac{1}{4(b-a)^2} \sup_{s \in [a, b]} |m_1(s) - m_2(s)|^2 \left(\int_a^b ds \right)^2 \\ &\leq \frac{1}{4} d_{\zeta}(m_1, m_2). \end{aligned} \quad (57)$$

It implies that

$$d_{\zeta}(\mathbf{f}m_1, \mathbf{f}m_2) \leq \frac{1}{4}d_{\zeta}(m_1, m_2), \quad (58)$$

where $\lambda = 1/4 \in (0, 1)$. Thus, all the conditions of Corollary 13 are satisfied. Hence, $\mathbf{f}$ has a unique fixed point; that is, the Fredholm integral equation (53) has a solution.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally in writing this article. All authors read and approved the final manuscript.

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