Research Article

\( k \)-Fractional Variants of Hermite-Mercer-Type Inequalities via \( s \)-Convexity with Applications

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This article is aimed at studying novel generalizations of Hermite-Mercer-type inequalities within the Riemann-Liouville \( k \)-fractional integral operators by employing \( s \)-convex functions. Two new auxiliary results are derived to govern the novel fractional variants of Hadamard-Mercer-type inequalities for differentiable mapping \( \Psi \) whose derivatives in the absolute values are convex. Moreover, the results also indicate new lemmas for \( \Psi' \), \( \Psi'' \), and \( \Psi''' \) and new bounds for the Hadamard-Mercer-type inequalities via the well-known Hölder’s inequality. As an application viewpoint, certain estimates in respect of special functions and special means of real numbers are also illustrated to demonstrate the applicability and effectiveness of the suggested scheme.

1. Introduction

Recently, two fundamental notions have been introduced in pure and applied analysis having potential utilities in every field and are known as convexity and concavity. Interestingly, the convexity theory is attributed to Jensen. Several monographs and articles have played a prominent role in the developments, speculations, and modifications of convexity in different directions such as \( \eta \)-convexity, harmonic convexity, \( h \)-convexity, and strong convexity. Moreover, a strong connection has been developed between diverse kinds of convex functions and inequality theory. Their fertile applications in optimization theory, functional analysis, physics, and statistical theory have made it a much fascinating subject, and hence, it is assumed as an incorporative subject between combinatorics, orthogonal polynomials, hypergeometric functions, quantum theory, and linear programming. This is the major motivation behind the keen investigation and progress of the integral inequalities in the literature [1, 2].

Let \( 0 < \zeta_1 \leq \zeta_2 \leq \cdots \leq \zeta_n \) and let \( \rho = (\rho_1, \rho_2, \cdots, \rho_n) \) be non-negative weights such that \( \sum_{j=1}^{n} \rho_j = 1 \). The famous Jensen’s inequality (see [1]) in the literature states that if \( \Psi \) is a convex function on the interval \( [\theta_1, \theta_2] \), then

\[
\Psi\left(\sum_{j=1}^{n} \rho_j \zeta_j\right) \leq \sum_{j=1}^{n} \rho_j \Psi(\zeta_j),
\]

(1)

for all \( \zeta_j \in [\theta_1, \theta_2] \), \( \rho_j \in [0, 1] \), and \( j = 1, 2, \cdots, n \). It is one of the key inequalities that helps to extract bounds for useful distances in information theory (see [3, 4]).

In 2003, a new variant of Jensen’s inequality was introduced by Mercer [5].

If \( \Psi \) is a convex function on \( [\theta_1, \theta_2] \), then

\[
\Psi\left(\theta_1 + \theta_2 - \sum_{j=1}^{n} \rho_j \zeta_j\right) \leq \Psi(\theta_1) + \Psi(\theta_2) - \sum_{j=1}^{n} \rho_j \Psi(\zeta_j),
\]

(2)

for all \( \zeta_j \in [\theta_1, \theta_2] \), \( \rho_j \in [0, 1] \), and \( j = 1, 2, \cdots, n \).

Matkovic and Pečarić proposed several generalizations on Jensen-Mercer operator inequalities [6]. Later on,
Niezgoda [7] has provided several extensions to higher dimensions for Mercer-type inequalities. Recently, the Jensen-Mercer-type inequality has made a significant contribution to inequality theory due to its prominent characteristics.

In the present study, we consider $s \in (0, \infty)$, the class of Breckner $s$-convex functions (which for $0 < s < 1$ were called in [8, 9]), $s$-convex in the second sense. In [10], Dragomir and Fitzpatrick introduce the concept of a real-valued Breckner $s$-convex function $\Psi$ on a convex subset $C$ of a linear space $V$ as

$$\Psi(\rho_1 \zeta_1 + \rho_2 \zeta_2) \leq \rho_1^s \Psi(\zeta_1) + (1 - \rho_1)^s \Psi(\zeta_2),$$  (3)

whenever $\rho_1, \rho_2 \geq 0$ with $\rho_1 + \rho_2 = 1$ and $\zeta_1, \zeta_2 \in C$. For $s = 1$, it reduces to the usual notion of convexity. As a result, he generalizes Jensen’s inequality (1) as

$$\Psi \left( \sum_{j=1}^{n} \rho_j \zeta_j \right) \leq \sum_{j=1}^{n} \rho_j^s \Psi(\zeta_j),$$  (4)

whenever $\rho_j \geq 0$, $\zeta_j \in C$, and $\sum_{j=1}^{n} \rho_j = 1$.

In [9], the class of $s$-convex functions in the first and second senses is introduced along with their significant properties.

Definition 1. Let $s \in (0, 1]$, a real-valued function $\Psi$ on an interval $I = (0, \infty)$ is $s$-convex in the second sense provided that (3) holds for all $\zeta_1, \zeta_2 \in I$ and $\rho_1, \rho_2 \geq 0$ with $\rho_1 + \rho_2 = 1$.

They denote this class of function by $(\Psi \in K^s_2)$. Moreover, they proved that the class $(\Psi \in K^s_2)$ is stronger than convexity in the first and original sense for $0 < s < 1$. Several properties of $s$-convex functions in both senses are presented in a comprehensive manner with supporting examples. It is interesting to see that if $0 < s < 1$ and $\Psi \in K^s_2$, then $\Psi$ is nonnegative. This result may not hold in general when the function is convex (i.e., $s = 1$). Also, the situation is more interesting when $f(0) = 0$.

Viewing this literature, we intend to extend the Jensen-Mercer inequality for Breckner $s$-convex functions. For this, we use the ideology of Mercer’s concept [5] and give the following important lemma.

Lemma 2. If $\Psi$ is a real-valued Breckner $s$-convex function on the interval $[\theta_1, \theta_2] \subset \mathbb{R}^+$ and $s > 0$ such that $\rho_1, \rho_2 \geq 0$, $\rho_1 + \rho_2 = 1$, and $\rho_1^s + \rho_2^s \leq 1$, then for any finite positive increasing sequence $\{\xi_n\}_{n=1}^{\infty} \subset [\theta_1, \theta_2]$, we have

$$\Psi(\theta_1 + \theta_2 - \xi_k) \leq \Psi(\theta_1) + \Psi(\theta_2) - \Psi(\xi_k),$$  (5)

for all $1 \leq k \leq n$.

Proof. Let $y_k = \theta_1 + \theta_2 - \xi_k$. Then, $\theta_1 + \theta_2 = y_k + \xi_k$, so the pairs $\theta_1, \theta_2$ and $y_k, \xi_k$ possess the same midpoint. Since that is the case, there exist $\rho_1, \rho_2 \in [0, 1]$, with $\rho_1 + \rho_2 = 1$ such that $\xi_k = \rho_1 \theta_1 + \rho_2 \theta_2$ and $y_k = \rho_1 \theta_1 + \rho_1 \theta_2$. Therefore, employing (3) and the assumed condition, we get

$$\Psi(y_k) = \Psi(\rho_1 \theta_1 + \rho_2 \theta_2) \leq \rho_1^s \Psi(\theta_1) + \rho_2^s \Psi(\theta_2)$$

$$\leq (1 - \rho_1^s) \Psi(\theta_1) + (1 - \rho_2^s) \Psi(\theta_2) = \Psi(\theta_1) + \Psi(\theta_2) - \rho_1^s \Psi(\theta_1) - \rho_2^s \Psi(\theta_2)$$

$$\leq \Psi(\theta_1) + \Psi(\theta_2) - \rho_1^s \Psi(\theta_1) - \rho_2^s \Psi(\theta_2)$$

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$$\leq \Psi(\theta_1) + \Psi(\theta_2) - \rho_1^s \Psi(\theta_1) - \rho_2^s \Psi(\theta_2)$$

which completes the proof.

Now, we give the result for the Jensen-Mercer inequality in the Breckner $s$-sense.

Theorem 3. Let $\rho_1, \rho_2, \ldots, \rho_n$ be positive real numbers $n \geq 2$ such that $\sum_{k=1}^{n} \rho_k = 1$ and $\sum_{k=1}^{n} \rho_k^s \leq 1$. If $\Psi$ is a real-valued Breckner $s$-convex function on $[\theta_1, \theta_2] \subset \mathbb{R}^+$, then for any finite positive increasing sequence $\{\xi_n\}_{n=1}^{\infty} \subset [\theta_1, \theta_2]$, we have

$$\Psi \left( \frac{\theta_1 + \theta_2 - \sum_{k=1}^{n} \rho_k \xi_k}{2} \right) \leq \frac{1}{\theta_1 - \theta_2} \int_{\theta_1}^{\theta_2} \Psi(\lambda) d\lambda \leq \frac{\Psi(\theta_1) + \Psi(\theta_2)}{2},$$  (6)

provided that if a mapping $\Psi : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on $I$ and $\theta_1, \theta_2 \in I$.

Fractional-order calculus deals with more general behavior than integer-order calculus, and it not only provides new mathematical methods for practical systems but also has been applied into various fields due to its accurate description in many active fields, such as fraction-order memristive chaotic circuit, fractional-orderrelaxation-oscillation model, mathematical biology, and economics (see [12]).

The theory of Riemann-Liouville $k$-fractional integrals is a pertinent extension of Riemann-Liouville fractional integrals. It is important to note that if $k \neq 1$, the properties of Riemann-Liouville $k$-fractional integrals are quite dissimilar.
from those of general fractional integrals. For this, the Riemann-Liouville \( k \)-fractional integrals have agitated the interest of many researchers. Now, we demonstrate some essential concepts of \( k \)-fractional calculus for the investigation of our results.

**Definition 5** (see [13]). Diaz and Pariguan have defined the \( k \)-gamma function \( \Gamma_k(\cdot) \), a generalization of the classical gamma function, which is given by the following formula:

\[
\Gamma_k(a) = \lim_{m \to \infty} \frac{m!k^m}{(mk)^{a-1}}, \quad k > 0, a \in \mathbb{C} \setminus k\mathbb{Z}^-, \tag{9}
\]

where \( (a)_m \) is the Pochhammer \( k \)-symbol given by

\[
(a)_m = (a + k)(a + 2k) \cdots (a + (m-1)k). \tag{10}
\]

It is shown that the Mellin transform of the exponential function \( e^{\lambda k t} \) is the \( k \)-gamma function clearly given by

\[
\Gamma_k(a) = \int_0^\infty e^{\lambda k t} \lambda^{a-1} d\lambda, \tag{11}
\]

for \( \text{Re}(a) > 0 \) with \( a \Gamma_k(a) = \Gamma_k(a+k) \), where \( \Gamma_k(\cdot) \) stands for the \( k \)-gamma function.

Many researchers have generalized the classical and fractional operators by introducing a parameter \( k > 0 \) about a decade ago. Mubeen et al. [14] used special \( k \)-function theory in fractional calculus for the first time in the literature in the form of Riemann-Liouville \( k \)-fractional integrals.

**Definition 6** (see [15]). Let \( \Psi \in L_1(\theta_1, \theta_2) \). The Riemann-Liouville \( k \)-fractional integrals of \( \Psi \) with \( k > 0 \), \( k \int_{\theta_1}^\theta \Psi \) and \( k \int_{\theta_2}^\theta \Psi \) of order \( \alpha > 0 \) with \( \theta_1 \geq 0 \) are defined by

\[
k\int_{\theta_1}^\theta \Psi(t) dt = \frac{1}{k \Gamma(\alpha)} \int_{\theta_1}^{\theta} (\zeta - \lambda)^{(\alpha-1)} \lambda^{k-1} \Psi(\lambda) d\lambda, \quad \zeta > \theta_1, \tag{12}
\]

\[
k\int_{\theta_2}^\theta \Psi(t) dt = \frac{1}{k \Gamma(\alpha)} \int_{\theta_2}^{\theta} (\lambda - \zeta)^{(\alpha-1)} \lambda^{k-1} \Psi(\lambda) d\lambda, \quad \zeta < \theta_2,
\]

respectively.

**Remark 7.** If \( k = 1 \), then Riemann-Liouville \( k \)-fractional integrals reduce to classical Riemann-Liouville fractional integrals. And if \( \alpha = 1 \) and \( k = 1 \), the fractional integral reduces to the classical integral.

Many \( k \)-fractional operators, their properties, related identities, and inequalities are proved during the past years. For instance, see [16, 17] and references therein.

### 2. Main Results

This section contains several new generalizations of Hermite-Hadamard-Mercer-type inequalities for \( s \)-convex functions in the second sense (Breckner sense) via \( k \)-fractional calculus theory.

Throughout the paper, we assumed the following assumptions:

- \( A_1 \): let \( \zeta_1, \zeta_2 \in [\theta_1, \theta_2] \subseteq \mathbb{R}^+ \) with \( \zeta_1 < \zeta_2 \), \( \alpha, k > 0 \), and for some fixed \( s \in (0, 1] \), \( \lambda \in [0, 1] \) and \( \Gamma_k(\cdot) \) is the \( k \)-gamma function.
- \( A_2 \): for \( \lambda \in [0, 1] \), \( ((1 + \lambda)/2)s + ((1 - \lambda)/2)s \leq 1 \), whenever we use the definition of the Jensen-Mercer inequality for the \( s \)-convex function.

**Theorem 8.** Let \( \Psi : [\theta_1, \theta_2] \to \mathbb{R} \) be the \( s \)-convex function such that \( \Psi \in L_1(\theta_1, \theta_2) \) along with assumptions \( A_1 \) and \( A_2 \). Then, the following Riemann-Liouville \( k \)-fractional integral inequalities hold:

\[
\frac{1}{2^{(a+k)-s}} \Psi \left( \theta_1 + \theta_2 - \frac{\zeta_1 + \zeta_2}{2} \right) \leq \frac{\Gamma_k(\alpha+k)}{(\zeta_2 - \zeta_1)^{a+k}} \times \left\{ k^{\alpha} f_{\theta_2}^{\alpha} \Psi \left( \theta_1 + \theta_2 - \frac{\zeta_1 + \zeta_2}{2} \right) + k^{\alpha} f_{\theta_2}^{\alpha} \Psi \left( \frac{\zeta_1 + \zeta_2}{2} \right) \right\} \leq \frac{\Psi(\theta_1) + \Psi(\theta_2)}{2^{(a+k)-s}} \tag{13}
\]

\[
\Psi \left( \theta_1 + \theta_2 - \frac{\zeta_1 + \zeta_2}{2} \right) \leq \left| \Psi(\theta_1) + \Psi(\theta_2) \right| - \frac{2^{(a+k)-s} \Gamma_k(\alpha+k)}{(\zeta_2 - \zeta_1)^{a+k}} \times \left\{ k^{\alpha} f_{\zeta_1}^{\alpha} \Psi \left( \frac{\zeta_1 + \zeta_2}{2} \right) + k^{\alpha} f_{\zeta_2}^{\alpha} \Psi \left( \frac{\zeta_1 + \zeta_2}{2} \right) \right\} \leq \Psi(\theta_1) + \Psi(\theta_2) - \Psi \left( \frac{\zeta_1 + \zeta_2}{2} \right) \tag{14}
\]

where \( U(\alpha, k, s, \lambda) = \int_0^1 \lambda^{(a+k)-1}(1 + \lambda)^s d\lambda \) and \( B(\cdot, \cdot) \) is the beta function.

**Proof.** By employing the definition of the \( s \)-convex function \( \Psi \), we get

\[
\frac{\psi(\theta_1 + \theta_2 - \frac{u+v}{2})}{2} = \frac{\psi(\theta_1 + \theta_2 - u + \theta_1 + \theta_2 - v)}{2} \leq \frac{1}{2} (\psi(\theta_1 + \theta_2 - u) + \psi(\theta_1 + \theta_2 - v))(\forall u, v \in [\theta_1, \theta_2]). \tag{15}
\]

By change of variables \( u = ((1 + \lambda)/2) \zeta_1 + ((1 - \lambda)/2) \zeta_2 \) and \( v = ((1 - \lambda)/2) \zeta_1 + ((1 + \lambda)/2) \zeta_2, \lambda \in [0, 1], \) we get
\[ 2^a \Psi \left( \theta_1 + \theta_2 - \frac{\xi_1 + \xi_2}{2} \right) \leq \left[ \Psi \left( \theta_1 + \theta_2 - \left( \frac{1 + \lambda}{2} \xi_1 + \frac{1 - \lambda}{2} \xi_2 \right) \right) + \Psi \left( \theta_1 + \theta_2 - \left( \frac{1 - \lambda}{2} \xi_1 + \frac{1 + \lambda}{2} \xi_2 \right) \right) \right]. \]

Now, multiplying the above inequality by \( \lambda^{(\alpha k)^{-1}} \) and then integrating w.r.t. \( \lambda \) over \([0, 1]\) yield

\[
2^a \Psi \left( \theta_1 + \theta_2 - \frac{\xi_1 + \xi_2}{2} \right) \int_0^1 \lambda^{(\alpha k)^{-1}} d\lambda \leq \int_0^1 \lambda^{(\alpha k)^{-1}} \cdot \left[ \Psi \left( \theta_1 + \theta_2 - \left( \frac{1 + \lambda}{2} \xi_1 + \frac{1 - \lambda}{2} \xi_2 \right) \right) + \Psi \left( \theta_1 + \theta_2 - \left( \frac{1 - \lambda}{2} \xi_1 + \frac{1 + \lambda}{2} \xi_2 \right) \right] d\lambda.
\]

By change of variable, we have

\[
\frac{1}{2^{(\alpha k)^{-1}}} \cdot \Psi \left( \theta_1 + \theta_2 - \frac{\xi_1 + \xi_2}{2} \right) \leq \frac{\Gamma_k(\alpha + k)}{(\xi_2 - \xi_1)^{\alpha k}} \times \left\{ k^{\alpha} \Psi \left( \theta_1 + \theta_2 - \frac{\xi_1 + \xi_2}{2} \right) + k^{\alpha} \Psi \left( \theta_1 + \theta_2 - \frac{\xi_1 + \xi_2}{2} \right) \right\},
\]

and so the first inequality of (13) is proved.

Now, for the proof of the second inequality of (13), we first note that if \( \Psi \) is an \( s \)-convex function, then for \( \lambda \in [0, 1] \), it gives

\[
\Psi \left( \theta_1 + \theta_2 - \left( \frac{1 + \lambda}{2} \xi_1 + \frac{1 - \lambda}{2} \xi_2 \right) \right) \leq s\Psi(\theta_1) + \Psi(\theta_2) - \left[ \left( \frac{1 + \lambda}{2} \right)^s \Psi(\xi_1) + \left( \frac{1 - \lambda}{2} \right)^s \Psi(\xi_2) \right].
\]

\[
\Psi \left( \theta_1 + \theta_2 - \left( \frac{1 - \lambda}{2} \xi_1 + \frac{1 + \lambda}{2} \xi_2 \right) \right) \leq s\Psi(\theta_1) + \Psi(\theta_2) - \left[ \left( \frac{1 - \lambda}{2} \right)^s \Psi(\xi_1) + \left( \frac{1 + \lambda}{2} \right)^s \Psi(\xi_2) \right].
\]

By adding the inequalities of (19) and (20), we have

\[
\Psi \left( \theta_1 + \theta_2 - \left( \frac{1 + \lambda}{2} \xi_1 + \frac{1 - \lambda}{2} \xi_2 \right) \right) + \Psi \left( \theta_1 + \theta_2 - \left( \frac{1 - \lambda}{2} \xi_1 + \frac{1 + \lambda}{2} \xi_2 \right) \right) \leq 2(\Psi(\theta_1) + \Psi(\theta_2)) - \left[ \left( \frac{1 - \lambda}{2} \right)^s \Psi(\xi_1) + \left( \frac{1 + \lambda}{2} \right)^s \Psi(\xi_2) \right].
\]

Now, multiplying the above inequality by \( \lambda^{(\alpha k)^{-1}} \) and then integrating w.r.t. \( \lambda \) over \([0, 1]\), we get

\[
\int_0^1 \lambda^{(\alpha k)^{-1}} \cdot \left[ \Psi \left( \theta_1 + \theta_2 - \left( \frac{1 + \lambda}{2} \xi_1 + \frac{1 - \lambda}{2} \xi_2 \right) \right) + \Psi \left( \theta_1 + \theta_2 - \left( \frac{1 - \lambda}{2} \xi_1 + \frac{1 + \lambda}{2} \xi_2 \right) \right] d\lambda \leq \frac{2}{\alpha k} (\Psi(\theta_1) + \Psi(\theta_2)).
\]

Consequently, we get

\[
\frac{2^{\alpha k} k^\alpha f_k(\alpha, \theta, \xi_1, \xi_2)}{(\xi_2 - \xi_1)^{\alpha k}} \times \left\{ k^{\alpha} \Psi \left( \theta_1 + \theta_2 - \frac{\xi_1 + \xi_2}{2} \right) + k^{\alpha} \Psi \left( \theta_1 + \theta_2 - \frac{\xi_1 + \xi_2}{2} \right) \right\} \leq \frac{2}{\alpha k} (\Psi(\theta_1) + \Psi(\theta_2)) - \frac{1}{2^2} (U(\alpha, k, s, \lambda) + B(q, s, \lambda)).
\]

Combining (18) and (23), one can get (13). In order to prove (14), we employ the Jensen-Mercer inequality for the \( s \)-convex function \( \Psi \) in the second sense; then, for \( \lambda \in [0, 1] \), it yields

\[
\Psi \left( \theta_1 + \theta_2 - \frac{\xi_1 + \xi_2}{2} \right) \leq s\Psi(\theta_1) + \Psi(\theta_2) - \left[ \left( \frac{1 - \lambda}{2} \right)^s \Psi(\xi_1) + \left( \frac{1 + \lambda}{2} \right)^s \Psi(\xi_2) \right].
\]

Now, by change of variables \( u = (1 + \lambda)/2 \xi_1 + ((1 - \lambda)/2) \xi_2 \) and \( v = ((1 - \lambda)/2) \xi_1 + ((1 + \lambda)/2) \xi_2 \), \( \forall \xi_1, \xi_2 \in [\theta_1, \theta_2] \) and \( \lambda \in [0, 1] \) in (25), we have

\[
\Psi \left( \theta_1 + \theta_2 - \frac{\xi_1 + \xi_2}{2} \right) \leq s\Psi(\theta_1) + \Psi(\theta_2) - \left[ \left( \frac{1 - \lambda}{2} \right)^s \Psi(\xi_1) + \left( \frac{1 + \lambda}{2} \right)^s \Psi(\xi_2) \right].
\]
Multiplying by $\lambda^{(ak)}$ and then integrating w.r.t. $\lambda$ over $[0,1]$ gives

$$\frac{2^k}{\alpha} \Psi \left( \theta_1 + \theta_2 - \frac{\zeta_1 + \zeta_2}{2} \right) \leq \frac{2^k}{\alpha} [\Psi(\theta_1) + \Psi'(\theta_2)]$$

$$- \int_0^1 \lambda^{(ak)} \left[ \Psi \left( \frac{1 + \lambda}{2} \zeta_1 + \frac{1 - \lambda}{2} \zeta_2 \right) + \Psi \left( \frac{1 - \lambda}{2} \zeta_1 + \frac{1 + \lambda}{2} \zeta_2 \right) \right] d\lambda.$$  \hspace{1cm} (27)

It follows that

$$\frac{2^k}{\alpha} \Psi \left( \theta_1 + \theta_2 - \frac{\zeta_1 + \zeta_2}{2} \right) \leq \frac{2^k}{\alpha} (\Psi(\theta_1) + \Psi'(\theta_2))$$

$$- \frac{2^{(ak)-1}}{(\zeta_2 - \zeta_1)^{(ak)}} \left\{ \int k^{\alpha} \Psi \left( \frac{\zeta_1 + \zeta_2}{2} \right) \right\}$$

$$+ \frac{k^{\alpha}}{(\zeta_2 - \zeta_1)^{(ak)}} \left\{ \Psi \left( \theta_1 + \theta_2 - \frac{\zeta_1 + \zeta_2}{2} \right) \right\}$$

$$\leq \left[ \Psi(\theta_1) + \Psi'(\theta_2) \right] - \frac{2^{(ak)-1}}{(\zeta_2 - \zeta_1)^{(ak)}}$$

$$\times \left\{ \int k^{\alpha} \Psi \left( \frac{\zeta_1 + \zeta_2}{2} \right) + \frac{k^{\alpha}}{(\zeta_2 - \zeta_1)^{(ak)}} \right\}.$$  \hspace{1cm} (28)

and so the first inequality of (14) is proved.

Now, for the proof of the second inequality of (14), we first note that if $\Psi$ is an s-convex function, then for $\theta \in [0,1]$, it gives

$$\Psi \left( \frac{\zeta_1 + \zeta_2}{2} \right) \leq \frac{\Psi((1 + \lambda)/2) \zeta_1 + (1 - \lambda)/2 \zeta_2 + (1 - \lambda)/2 \zeta_1 + (1 + \lambda)/2 \zeta_2}{2}$$

$$\leq \frac{(1 + \lambda)/2) \zeta_1 + (1 - \lambda)/2 \zeta_2 + (1 - \lambda)/2 \zeta_1 + (1 + \lambda)/2 \zeta_2}{2}.$$  \hspace{1cm} (29)

Multiplying by $\lambda^{(ak)}$ and then integrating w.r.t. $\lambda$ over $[0,1]$ gives

$$\frac{2^k}{\alpha} \Psi \left( \frac{\zeta_1 + \zeta_2}{2} \right) \leq \frac{2^k}{\alpha} \left\{ \int \lambda^{(ak)} \right\}$$

$$+ \frac{2^{(ak)-1}}{(\zeta_2 - \zeta_1)^{(ak)}} \left\{ \int k^{\alpha} \Psi \left( \frac{\zeta_1 + \zeta_2}{2} \right) \right\}$$

$$\leq \frac{2^{(ak)-1}}{(\zeta_2 - \zeta_1)^{(ak)}} \left\{ \int k^{\alpha} \Psi \left( \frac{\zeta_1 + \zeta_2}{2} \right) \right\}.$$  \hspace{1cm} (30)

Therefore, we have

$$- \frac{2^{(ak)-1}}{(\zeta_2 - \zeta_1)^{(ak)}} \int k^{\alpha} \Psi \left( \frac{\zeta_1 + \zeta_2}{2} \right)$$

$$+ \frac{k^{\alpha}}{(\zeta_2 - \zeta_1)^{(ak)}} \left\{ \Psi \left( \theta_1 + \theta_2 - \frac{\zeta_1 + \zeta_2}{2} \right) \right\} \leq -\Psi \left( \frac{\zeta_1 + \zeta_2}{2} \right).$$  \hspace{1cm} (31)

Adding $\Psi(\theta_1) + \Psi'(\theta_2)$ to both sides in (31), we get the second inequality of (14).

Remark 9. Under the assumption of Theorem 8 for inequality (13) with $s = \alpha = k = 1$, one has

$$\Psi \left( \theta_1 + \theta_2 - \frac{\zeta_1 + \zeta_2}{2} \right) \leq \frac{1}{\zeta_2 - \zeta_1} \int_0^{\zeta_1} \Psi(\theta_1 + \theta_2 - \lambda) d\lambda \leq \Psi(\theta_1) + \Psi'(\theta_2).$$  \hspace{1cm} (32)

The inequality (32) is proposed by Kian and Moslehian in [18].

Remark 10. If we choose $s = k = 1$ in Theorem 8, we get Theorem 2 in [19].

3. New Identities and Related Results via Riemann-Liouville $k$-Fractional Integrals

Lemma 11. Let $\Psi: \mathbb{[}0,1\mathbb{]} \rightarrow \mathbb{R}$ be a differentiable mapping on $[\theta_1, \theta_2]$ with $\theta_1 < \theta_2$, along with assumption $A_1$. If $\Psi' \in L[\theta_1, \theta_2]$, then the following equality for Riemann-Liouville $k$-fractional integrals holds:

$$\int \frac{k^\alpha}{(\zeta_2 - \zeta_1)^{(ak)}} \left\{ \int \Psi \left( \theta_1 + \theta_2 - \frac{\zeta_1 + \zeta_2}{2} \right) \right\}$$

$$- \frac{2^{(ak)-1}}{(\zeta_2 - \zeta_1)^{(ak)}}$$

$$\times \left\{ \int k^\alpha \Psi \left( \frac{\zeta_1 + \zeta_2}{2} \right) \right\}$$

$$\leq \frac{(1 + s)/2) \zeta_1 + (1 - s)/2 \zeta_2 + (1 - s)/2 \zeta_1 + (1 + s)/2 \zeta_2}{2}.$$  \hspace{1cm} (33)

Proof. It suffices to write that

$$I = \frac{\zeta_2 - \zeta_1}{4} \left\{ I_1 - I_2 \right\},$$  \hspace{1cm} (34)

where

$$I_1 = \int_0^{\zeta_1} \frac{d\lambda}{(\zeta_2 - \zeta_1)^{(ak)}} \left( \theta_1 + \theta_2 - \frac{1 + \lambda}{2} \zeta_1 + \frac{1 - \lambda}{2} \zeta_2 \right)$$

$$= \frac{2\Psi'\left( \theta_1 + \theta_2 - \frac{\zeta_1 + \zeta_2}{2} \right)}{(\zeta_2 - \zeta_1)^{(ak)}}$$

$$\times \left( w - \left( \frac{\theta_1 - \theta_2 - \frac{1 + \lambda}{2} \zeta_1 + \frac{1 - \lambda}{2} \zeta_2}{2} \right) \right) \left( \frac{\theta_1 - \theta_2 - \frac{1 + \lambda}{2} \zeta_1 + \frac{1 - \lambda}{2} \zeta_2}{2} \right)$$

$$- \frac{2^{(ak)-1}}{(\zeta_2 - \zeta_1)^{(ak)}}$$

$$\times \left\{ \int k^\alpha \Psi \left( \frac{\theta_1 - \theta_2 - \frac{1 + \lambda}{2} \zeta_1 + \frac{1 - \lambda}{2} \zeta_2}{2} \right) \right\}.$$  \hspace{1cm} (35)
Analogously,
\[
I_2 = \int_0^1 \lambda^{\alpha k} \Psi'(\theta_1 + \theta_2) = -\frac{1 - \lambda}{2} \frac{\lambda}{\xi_1} + \frac{1 + \lambda}{2} \frac{\lambda}{\xi_2}) d\lambda
\]
\[
= -\frac{2}{\alpha k + 1}(a k) \int_0^1 \frac{\lambda^{\alpha k + 1}(a k)}{\xi_1^{\alpha k + 1}} \frac{\lambda^{\alpha k + 1}(a k)}{\xi_2^{\alpha k + 1}}
\times \left\{ \frac{1}{\xi_1^{\alpha k + 1}} \frac{1}{\xi_2^{\alpha k + 1}} \left( \frac{\lambda^{\alpha k + 1}(a k)}{\xi_1^{\alpha k + 1}} - \omega \right) \right\} d\lambda \]
\[
= -\frac{2}{\alpha k + 1}(a k) \int_0^1 \frac{\lambda^{\alpha k + 1}(a k)}{\xi_1^{\alpha k + 1}} \frac{\lambda^{\alpha k + 1}(a k)}{\xi_2^{\alpha k + 1}}
\times \left\{ \frac{1}{\xi_1^{\alpha k + 1}} \frac{1}{\xi_2^{\alpha k + 1}} \left( \frac{\lambda^{\alpha k + 1}(a k)}{\xi_1^{\alpha k + 1}} - \omega \right) \right\} d\lambda \]
\[
= \frac{1}{\alpha k + 1}(a k) \int_0^1 \frac{\lambda^{\alpha k + 1}(a k)}{\xi_1^{\alpha k + 1}} \frac{\lambda^{\alpha k + 1}(a k)}{\xi_2^{\alpha k + 1}}
\times \left\{ \frac{1}{\xi_1^{\alpha k + 1}} \frac{1}{\xi_2^{\alpha k + 1}} \left( \frac{\lambda^{\alpha k + 1}(a k)}{\xi_1^{\alpha k + 1}} - \omega \right) \right\} d\lambda \]
(36)

Combining (35) and (36) with (34), we get (33).

**Corollary 12.** For \( \alpha = k = 1 \) in Lemma 11, we acquire

\[
\Psi'(\theta_1 + \theta_2 - \xi_1 + i(\theta_1 + \theta_2 - \xi_2)) \frac{1}{\xi_1 - \xi_2} \frac{\lambda^{\alpha \theta_2 - \xi_2}}{\xi_1^{\alpha \theta_2 - \xi_2}} \Psi'(u) d\mu
\]
\[
= \frac{\zeta_2 - \zeta_1}{4} \int_0^1 \lambda \Psi'(\theta_1 + \theta_2 - \frac{1 + \lambda}{2} \frac{\lambda}{\xi_1} + \frac{1 + \lambda}{2} \frac{\lambda}{\xi_2}) \frac{1}{\xi_1 - \xi_2} \frac{\lambda^{\alpha \theta_2 - \xi_2}}{\xi_1^{\alpha \theta_2 - \xi_2}} \Psi'(u) d\mu
\]
(37)

**Remark 13.** Taking \( \alpha = k = 1 \) with \( \xi_1 = \theta_1 \) and \( \xi_2 = \theta_2 \) in Lemma 11, we get Lemma 2.1 in [20] and the following equality holds:

\[
\frac{1}{2} \frac{\lambda^{\alpha \theta_2 - \xi_2}}{\xi_1^{\alpha \theta_2 - \xi_2}} \frac{\lambda^{\alpha \theta_2 - \xi_2}}{\xi_1^{\alpha \theta_2 - \xi_2}} \Psi'(u) d\mu
\]
(38)

**Theorem 14.** Suppose that \( \Psi : [\theta_1, \theta_2] \rightarrow \mathbb{R} \) is a differentiable mapping on \( \theta_1, \theta_2 \) with \( \theta_1 < \theta_2 \) and \( \Psi' \in L[\theta_1, \theta_2] \) along with assumptions \( A_1 \) and \( A_2 \). If \( |\Psi'| \) is an s-convex function on \( \theta_1, \theta_2 \), then the following inequality for Riemann-Liouville k-fractional integrals holds:

\[
\left| \frac{\Psi'(\theta_1 + \theta_2 - \xi_1) + \Psi'(\theta_1 + \theta_2 - \xi_2)}{2} - \frac{2 \left( a k + 1 \right) \Gamma_k(\alpha + k)}{\xi_1^{\alpha k + 1}} \right| d\mu
\]
\[
= \frac{2}{\alpha k + 1}(a k) \int_0^1 \frac{\lambda^{\alpha k + 1}(a k)}{\xi_1^{\alpha k + 1}} \frac{\lambda^{\alpha k + 1}(a k)}{\xi_2^{\alpha k + 1}}
\times \left\{ \frac{1}{\xi_1^{\alpha k + 1}} \frac{1}{\xi_2^{\alpha k + 1}} \left( \frac{\lambda^{\alpha k + 1}(a k)}{\xi_1^{\alpha k + 1}} - \omega \right) \right\} d\lambda \]
(39)

**Proof.** By using Lemma 11 and the Jensen-Mercer inequality and the s-convexity of \(|\Psi'| \), we have

\[
\left| \frac{\Psi'(\theta_1 + \theta_2 - \xi_1) + \Psi'(\theta_1 + \theta_2 - \xi_2)}{2} - \frac{2 \left( a k + 1 \right) \Gamma_k(\alpha + k)}{\xi_1^{\alpha k + 1}} \right|
\]
\[
= \frac{2}{\alpha k + 1}(a k) \int_0^1 \frac{\lambda^{\alpha k + 1}(a k)}{\xi_1^{\alpha k + 1}} \frac{\lambda^{\alpha k + 1}(a k)}{\xi_2^{\alpha k + 1}}
\times \left\{ \frac{1}{\xi_1^{\alpha k + 1}} \frac{1}{\xi_2^{\alpha k + 1}} \left( \frac{\lambda^{\alpha k + 1}(a k)}{\xi_1^{\alpha k + 1}} - \omega \right) \right\} d\lambda \]
(40)

After simple computations, we get the required result of (39).

**Corollary 15.** For \( s = \alpha = k = 1 \) in Theorem 14, we get

\[
\left| \frac{\Psi'(\theta_1 + \theta_2 - \xi_1) + \Psi'(\theta_1 + \theta_2 - \xi_2)}{2} - \frac{2 \left( a k + 1 \right) \Gamma_k(\alpha + k)}{\xi_1^{\alpha k + 1}} \right|
\]
\[
= \frac{2}{\alpha k + 1}(a k) \int_0^1 \frac{\lambda^{\alpha k + 1}(a k)}{\xi_1^{\alpha k + 1}} \frac{\lambda^{\alpha k + 1}(a k)}{\xi_2^{\alpha k + 1}}
\times \left\{ \frac{1}{\xi_1^{\alpha k + 1}} \frac{1}{\xi_2^{\alpha k + 1}} \left( \frac{\lambda^{\alpha k + 1}(a k)}{\xi_1^{\alpha k + 1}} - \omega \right) \right\} d\lambda \]
(41)

**Remark 16.** Taking \( s = \alpha = k = 1 \) with \( \xi_1 = \theta_1 \) and \( \xi_2 = \theta_2 \) in Theorem 14, we recapture Theorem 2.2 in [21]:

\[
\left| \frac{\Psi'(\theta_1 + \theta_2 - \xi_1) + \Psi'(\theta_1 + \theta_2 - \xi_2)}{2} - \frac{2 \left( a k + 1 \right) \Gamma_k(\alpha + k)}{\xi_1^{\alpha k + 1}} \right|
\]
\[
= \frac{2}{\alpha k + 1}(a k) \int_0^1 \frac{\lambda^{\alpha k + 1}(a k)}{\xi_1^{\alpha k + 1}} \frac{\lambda^{\alpha k + 1}(a k)}{\xi_2^{\alpha k + 1}}
\times \left\{ \frac{1}{\xi_1^{\alpha k + 1}} \frac{1}{\xi_2^{\alpha k + 1}} \left( \frac{\lambda^{\alpha k + 1}(a k)}{\xi_1^{\alpha k + 1}} - \omega \right) \right\} d\lambda \]
(42)

**Remark 17.** If we choose \( s = k = 1 \) in Theorem 14, we get Theorem 4 in [19].

**Theorem 18.** Suppose that \( \Psi : [\theta_1, \theta_2] \rightarrow \mathbb{R} \) is a differentiable function on \( \theta_1, \theta_2 \) with \( \theta_1 < \theta_2 \) along with assumptions \( A_1 \) and \( A_2 \). If \( |\Psi'| \) is an s-convex function on \( \theta_1, \theta_2 \), then the following inequality for Riemann-Liouville k-fractional integrals holds:
Proof. Employing Lemma 11 and the Jensen-Mercer inequality with noted Hölder’s inequality and utilizing the s-convexity of $|\Psi'|^q$, we have

$$
\left(\frac{\Psi'(\theta_1 + \theta_2 - \zeta_1) + \Psi'(\theta_1 + \theta_2 - \zeta_2)}{2} - \frac{2^{(a(k)-1)} \Gamma_k(\alpha + k)}{(\zeta_2 - \zeta_1)^{ak}} \right) \times \left\{ \begin{array}{l}
\frac{\zeta_1 - \zeta_2}{4} \int_0^{1/2} |\Psi'\left(\theta_1 + \theta_2 - \frac{1 + \lambda}{2}, \theta_1 + \frac{1 - \lambda}{2} \right) | \, d\lambda \\
\times \left( \int_0^{1/2} |\Psi'(\theta_1 + \theta_2 - \frac{1 - \lambda}{2}, \theta_1 + \frac{1 + \lambda}{2} \) | \, d\lambda \right)^{1/q}
\end{array} \right\}
$$

This completes the proof.

Theorem 19. Suppose that $\Psi : [\theta_1, \theta_2] \rightarrow \mathbb{R}$ is a differentiable function on $(\theta_1, \theta_2)$ with $\theta_1 < \theta_2$ along with assumptions A1 and A2. If $|\Psi'| \in L[\theta_1, \theta_2]$ and $|\Psi'|^q$ is an s-convex function on $[\theta_1, \theta_2]$, where $(1/r) + (1/q) = 1$, $r \geq 1$, with $q = r/(r-1)$, then the following inequality for Riemann-Liouville k-fractional integrals holds:

$$
\left(\frac{\Psi'(\theta_1 + \theta_2 - \zeta_1) + \Psi'(\theta_1 + \theta_2 - \zeta_2)}{2} - \frac{2^{(a(k)-1)} \Gamma_k(\alpha + k)}{(\zeta_2 - \zeta_1)^{ak}} \right) \times \left\{ \begin{array}{l}
\frac{\zeta_1 - \zeta_2}{4} \int_0^{1/2} |\Psi'\left(\theta_1 + \theta_2 - \frac{1 + \lambda}{2}, \theta_1 + \frac{1 - \lambda}{2} \right) | \, d\lambda \\
\times \left( \int_0^{1/2} |\Psi'(\theta_1 + \theta_2 - \frac{1 - \lambda}{2}, \theta_1 + \frac{1 + \lambda}{2} \) | \, d\lambda \right)^{1/q}
\end{array} \right\}
$$

Proof. For $r \geq 1$, taking into account Lemma 11 and the Jensen-Mercer inequality with the noted power-mean inequality and utilizing the s-convexity of $|\Psi'|^q$, we have

$$
\left(\frac{\Psi'(\theta_1 + \theta_2 - \zeta_1) + \Psi'(\theta_1 + \theta_2 - \zeta_2)}{2} - \frac{2^{(a(k)-1)} \Gamma_k(\alpha + k)}{(\zeta_2 - \zeta_1)^{ak}} \right) \times \left\{ \begin{array}{l}
\frac{\zeta_1 - \zeta_2}{4} \int_0^{1/2} |\Psi'\left(\theta_1 + \theta_2 - \frac{1 + \lambda}{2}, \theta_1 + \frac{1 - \lambda}{2} \right) | \, d\lambda \\
\times \left( \int_0^{1/2} |\Psi'(\theta_1 + \theta_2 - \frac{1 - \lambda}{2}, \theta_1 + \frac{1 + \lambda}{2} \) | \, d\lambda \right)^{1/q}
\end{array} \right\}
$$

(44)
\begin{equation}
\frac{2^{(a+k)}\Gamma_k(a+k)}{\zeta_2 - \zeta_1} \left( \int_{(\theta_1, \theta_2, \zeta_1, \zeta_2)} f^a \int_{(\theta_1, \theta_2, \zeta_1, \zeta_2)} f^a \right) \psi(\theta_1 + \theta_2 - \zeta_1, \zeta_2) - \psi(\theta_1 + \theta_2 - \zeta_1, \zeta_2)
\end{equation}

**Proof.** It suffices to write that

\begin{equation}
I = \frac{(\zeta_2 - \zeta_1)^2}{8((a/k) + 1)} (I_1 + I_2),
\end{equation}

where

\begin{align*}
I_1 &= \int_0^1 (1 - \lambda)^{(a+k)+1} \psi^n \left( \frac{1}{1 - \lambda} \frac{1}{\zeta_1 + \zeta_2} \right) \psi \left( \frac{1}{1 - \lambda} \frac{1}{\zeta_1 + \zeta_2} \right) d\lambda \\
&= -\frac{2}{(\zeta_2 - \zeta_1)} \psi \psi \left( \frac{1}{1 - \lambda} \frac{1}{\zeta_1 + \zeta_2} \right) \psi \left( \frac{1}{1 - \lambda} \frac{1}{\zeta_1 + \zeta_2} \right) d\lambda
\end{align*}

and similarly, we can find

\begin{align*}
I_2 &= \int_0^1 (1 - \lambda)^{(a+k)+1} \psi^n \left( \frac{1}{1 - \lambda} \frac{1}{\zeta_1 + \zeta_2} \right) \psi \left( \frac{1}{1 - \lambda} \frac{1}{\zeta_1 + \zeta_2} \right) d\lambda
\end{align*}

Combining (49) and (50) with (48), we get the identity (47).

**Remark 21.** In Lemma 20, taking \( k = 1 \), with \( \zeta_1 = \theta_1 \) and \( \zeta_2 = \theta_2 \), recaptures Lemma 1 in [22].

**Remark 22.** For \( \alpha = k = 1 \), with \( \zeta_1 = \theta_1 \) and \( \zeta_2 = \theta_2 \), in Lemma 20, it reduces to Lemma 2 proved in [22].
Theorem 23. Suppose that $\Psi : [\theta_1, \theta_2] \rightarrow \mathbb{R}$ is a differentiable mapping on $(\theta_1, \theta_2)$ with $\theta_1 < \theta_2$ and $\Psi' \in L[\theta_1, \theta_2]$ along with assumptions $A_1$ and $A_2$. If $|\Psi'|^q$ is an $s$-convex function on $[\theta_1, \theta_2]$, then the following inequality holds:

$$\frac{2^{(a+k)}-1}{(\zeta - \zeta)^{(a+k)}} \left\{ k \int_{[\theta_1, \theta_2]} \Psi' \left( \theta_1 + \theta_2 - \zeta \right) \right\} - \Psi \left( \theta_1 + \theta_2 - \frac{\zeta + \zeta}{2} \right) \leq \frac{8((a/k) + 1)}{\zeta - \zeta} \left\{ \int_{0}^{1} \left( 1 - \lambda \right)^{(a+k)+1} \left( 1 + \lambda \right) \right\} d\lambda \text{ for }\zeta \neq \zeta . \tag{51}$$

where $U_1(\alpha, k, s, \lambda) = \int_{0}^{1} (1 - \lambda)^{(a+k)+1} (1 + \lambda)^{s} d\lambda$.

**Proof.** By using Lemma 20 with the Jensen-Mercer inequality and the s-convexity of $|\Psi'|^q$, we have

$$\frac{2^{(a+k)}-1}{(\zeta - \zeta)^{(a+k)}} \left\{ k \int_{[\theta_1, \theta_2]} \Psi' \left( \theta_1 + \theta_2 - \zeta \right) \right\} - \Psi \left( \theta_1 + \theta_2 - \frac{\zeta + \zeta}{2} \right) \leq \frac{8((a/k) + 1)}{\zeta - \zeta} \left\{ \int_{0}^{1} \left( 1 - \lambda \right)^{(a+k)+1} \left( 1 + \lambda \right) \right\} d\lambda \text{ for }\zeta \neq \zeta . \tag{52}$$

After simplifications, we get the required result.

**Remark 24.** For choosing $k = 1$ with $\zeta_1 = \theta_1$ and $\zeta_2 = \theta_2$ in Theorem 23, we will get Theorem 2 proved in [22].

Theorem 25. Suppose that $\Psi : [\theta_1, \theta_2] \rightarrow \mathbb{R}$ is a differentiable function on I along with assumptions $A_1$ and $A_2$. If $\Psi' \in L[\theta_1, \theta_2]$ and $|\Psi'|^q$ is an $s$-convex function, where $(1/q) + (1/r) = 1, q > 1$, then the following inequality holds:

$$\frac{2^{(a+k)}-1}{(\zeta - \zeta)^{(a+k)}} \left\{ k \int_{[\theta_1, \theta_2]} \Psi' \left( \theta_1 + \theta_2 - \zeta \right) \right\} - \Psi \left( \theta_1 + \theta_2 - \frac{\zeta + \zeta}{2} \right) \leq \frac{8((a/k) + 1)}{\zeta - \zeta} \left\{ \int_{0}^{1} \left( 1 - \lambda \right)^{(a+k)+1} \right\} d\lambda \text{ for }\zeta \neq \zeta . \tag{53}$$

Proof. By using Lemma 20 and the well-known Hölder’s inequality and the Jensen-Mercer inequality along with the fact that $|\Psi'|^q$ is an $s$-convex function, we have

$$\frac{2^{(a+k)}-1}{(\zeta - \zeta)^{(a+k)}} \left\{ k \int_{[\theta_1, \theta_2]} \Psi' \left( \theta_1 + \theta_2 - \zeta \right) \right\} - \Psi \left( \theta_1 + \theta_2 - \frac{\zeta + \zeta}{2} \right) \leq \frac{8((a/k) + 1)}{\zeta - \zeta} \left\{ \int_{0}^{1} \left( 1 - \lambda \right)^{(a+k)+1} \right\} d\lambda \text{ for }\zeta \neq \zeta . \tag{54}$$
\[
\left( \int_0^1 |\Psi'(\theta_1) + \Psi'(\theta_2)| \, d\lambda \right)^{\frac{1}{q}} \leq \frac{1}{2} \left( \frac{1 - \lambda}{2} \right)^{\frac{1}{q}} |\Psi'(\theta_1)|^q + \frac{1}{2} \left( \frac{1 - \lambda}{2} \right)^{\frac{1}{q}} |\Psi'(\theta_2)|^q + \frac{2^{s+1} - 1}{2^s(s+1)} |\Psi''(\theta_1)|^q + \frac{1}{2^s(s+1)} |\Psi''(\theta_2)|^q \right)^{\frac{1}{q}}.
\]

By direct computations, we get the required result.

**Remark 26.** For choosing \( k = 1 \) with \( \zeta_1 = \theta_1 \) and \( \zeta_2 = \theta_2 \) in Theorem 25, we get Theorem 3 proved in [22].

**Corollary 27.** For \( \alpha = k = 1 \) in Theorem 25, we get

\[
\left[ \frac{1}{\zeta_2 - \zeta_1} \Psi(u) \, du - \Psi \left( \theta_1 + \theta_2 - \frac{\zeta_1 + \zeta_2}{2} \right) \right]
\leq \frac{(\zeta_2 - \zeta_1)^2}{16} \left( \frac{1}{2^r + 1} \right)^{\frac{1}{r}} \left\{ \left( |\Psi'(\theta_1)|^q + |\Psi'(\theta_2)|^q \right)^{\frac{1}{q}} \right. \\
+ \frac{2^{s+1} - 1}{2^s(s+1)} |\Psi''(\theta_1)|^q + \frac{1}{2^s(s+1)} |\Psi''(\theta_2)|^q \left. \right\}^{\frac{1}{q}}.
\]

Finally, we state our results for third-order differentiable functions \( \Psi''' \).

**Lemma 28.** Let \( \Psi : [\theta_1, \theta_2] \rightarrow \mathbb{R} \) be a differentiable mapping on \( (\theta_1, \theta_2) \) with \( \theta_1 < \theta_2 \) along with assumption \( A_1 \). If \( \Psi''' \in L[\theta_1, \theta_2] \), then the following equality for Riemann-Liouville \( k \)-fractional integrals holds:
and similarly, we can find

\[
I_2 = \int_0^1 \left( 1 - \lambda \right)^{(a+k)\gamma} \Psi^{\gamma} \left( \theta_1 + \theta_2 - \frac{1 - \lambda}{2} \zeta_1 + \frac{1 + \lambda}{2} \zeta_2 \right) d\lambda \\
= \frac{2}{\zeta_2 - \zeta_1} \Psi' \left( \theta_1 + \theta_2 - \frac{\zeta_1 + \zeta_2}{2} \right) \frac{4(a(k) + 2)}{(\zeta_2 - \zeta_1)^2} \Psi' \\
\cdot \left( \theta_1 + \theta_2 - \frac{\zeta_1 + \zeta_2}{2} \right) \frac{8(a(k) + 2)}{(\zeta_2 - \zeta_1)^2} \Psi' \\
\cdot \left( \theta_1 + \theta_2 - \frac{\zeta_1 + \zeta_2}{2} \right) \frac{2(a(k) + 2\zeta_1)}{(\zeta_2 - \zeta_1)^4} \Psi' \\
\cdot \left( \theta_1 + \theta_2 - \frac{\zeta_1 + \zeta_2}{2} \right) \frac{1}{(\zeta_2 - \zeta_1)^4} \Psi \left( \theta_1 + \theta_2 - \zeta_1 \right) \tag{59}
\]

Replacing the values of the integrals \( I_1 \) and \( I_2 \) in (57), we get the identity (56).

Remark 29. In Lemma 28, choosing \( k = 1 \) with \( \zeta_1 = \theta_1 \) and \( \zeta_2 = \theta_2 \), it recaptures Lemma 3.1 proved in [23].

Remark 30. For \( k = \alpha = 1 \) with \( \zeta_1 = \theta_1 \) and \( \zeta_2 = \theta_2 \) in Lemma 28, it reduces to Lemma 2.1 proved in [24].

**Theorem 31.** Suppose that \( \Psi : (\theta, \theta_2) \rightarrow \mathbb{R} \) is a three times differentiable mapping on \( (\theta_1, \theta_2) \) with \( \theta_1 < \theta_2 \) and \( \Psi \in L[\theta_1, \theta_2] \) along with assumptions \( A_1 \) and \( A_2 \). If \( |\Psi^{\gamma}| \) is an s-convex function on \( [\theta_1, \theta_2] \), then the following inequality for \( k \)-fractional integrals holds:

\[
\left| \int_0^1 (1 - \lambda)^{(a+k)\gamma} \Psi^{\gamma} \left( \theta_1 + \theta_2 - \frac{1 - \lambda}{2} \zeta_1 + \frac{1 + \lambda}{2} \zeta_2 \right) d\lambda \right| \\
\leq \frac{16}{(a(k) + 1)((a(k) + 2)k)} \left( \int_0^1 (1 - \lambda)^{(a+k)\gamma} \left| \Psi^{\gamma} \left( \theta_1 + \theta_2 - \frac{1 - \lambda}{2} \zeta_1 + \frac{1 + \lambda}{2} \zeta_2 \right) \right| d\lambda \right)^{1/2}
\]

**Remark 32.** Choosing \( k = s = 1 \) with \( \zeta_1 = \theta_1 \) and \( \zeta_2 = \theta_2 \) in Theorem 31, we will get Theorem 18 proved in [23].

**Theorem 33.** Suppose that \( \Psi : (\theta, \theta_2) \rightarrow \mathbb{R} \) is a three times differentiable function on \( I \) along with assumptions \( A_1 \) and \( A_2 \). If \( \Psi^{\gamma} \in L[\theta_1, \theta_2] \) and \( |\Psi^{\gamma}|^{q} \) is an s-convex function, where \( (1/r) + (1/q) = 1, q > 1 \), then the following \( k \)-fractional integral inequality holds:

\[
\left| \int_0^1 (1 - \lambda)^{(a+k)\gamma} \Psi^{\gamma} \left( \theta_1 + \theta_2 - \frac{1 - \lambda}{2} \zeta_1 + \frac{1 + \lambda}{2} \zeta_2 \right) d\lambda \right| \\
\leq \frac{16}{(a(k) + 1)((a(k) + 2)k)} \left( \int_0^1 (1 - \lambda)^{(a+k)\gamma} \left| \Psi^{\gamma} \left( \theta_1 + \theta_2 - \frac{1 - \lambda}{2} \zeta_1 + \frac{1 + \lambda}{2} \zeta_2 \right) \right| d\lambda \right)^{1/2}
\]

**Proof.** By using Lemma 28 and the Jensen-Mercer inequality and the s-convexity of \( |\Psi^{\gamma}| \), we have

\[
\left| \int_0^1 (1 - \lambda)^{(a+k)\gamma} \Psi^{\gamma} \left( \theta_1 + \theta_2 - \frac{1 - \lambda}{2} \zeta_1 + \frac{1 + \lambda}{2} \zeta_2 \right) d\lambda \right|
\]

where \( U_2(a, k, s, \lambda) = \int_0^1 (1 - \lambda)^{(a+k)\gamma} (1 + \lambda)^{\gamma} d\lambda \).

**Proof.** By using Lemma 28 with the Jensen-Mercer inequality and the well-known Hölder’s inequality on the fact that \( |\Psi^{\gamma}|^{q} \) is s-convex, we have

\[
\frac{2(a+k-\gamma)^{\frac{1}{a+k}}}{(\zeta_2 - \zeta_1)^{a+k}}
\]
\begin{align*}
&\times \left\{ k f_{\theta_1, \theta_2 - ((\zeta_1, \zeta_2)/2))}^* \right\}(\Psi(\theta_1 + \theta_2 - \zeta_1)) \\
&\times \left\{ (k f_{\theta_1, \theta_2 - ((\zeta_1, \zeta_2)/2)}^*) - \left( \Psi(\theta_1 + \theta_2 - \zeta_2) \right) \right\} \\
&- \frac{(\zeta_2 - \zeta_1)^2}{4((a/k) + 1)((a/k) + 2)} \\
&\times \left| \left( \Psi''(\theta_1 + \theta_2 - \zeta_1 - \zeta_2) - \Psi(\theta_1 + \theta_2 - \zeta_1 - \zeta_2) \right) \right|
\end{align*}

\begin{equation}
\leq \frac{(\zeta_2 - \zeta_1)^3}{16((a/k) + 1)((a/k) + 2)} \\
\times \left| \left( \Psi''(\theta_1 + \theta_2 - \zeta_1 - \zeta_2) - \Psi(\theta_1 + \theta_2 - \zeta_1 - \zeta_2) \right) \right|
\end{equation}

After some simplifications, we get the required results.

**Remark 34.** Choosing \( k = s = 1 \) with \( \zeta_1 = \theta_1 \) and \( \zeta_2 = \theta_2 \) in Theorem 33, we will get Theorem 19 proved in [23].

### 4. Applications

#### 4.1. Some Applications of the Means

Let us consider the following special means for different values of \( a_1 \) and \( a_2 \):

1. The arithmetic mean:
   \begin{equation}
   A(a_1, a_2) = \frac{a_1 + a_2}{2}.
   \end{equation}

2. The geometric mean:
   \begin{equation}
   G(a_1, a_2) = (a_1 a_2)^{1/2}.
   \end{equation}

3. The harmonic mean:
   \begin{equation}
   H = \frac{2 a_1 a_2}{a_1 + a_2}.
   \end{equation}

**Proposition 35.** Suppose \( a_1, a_2 \in \mathbb{R}, 0 < a_1 < a_2, 0 \notin [a_1, a_2] \).

Then, for all \( r > 1 \), the following inequality holds:

\begin{align*}
&\left| \frac{1}{(\zeta_2 - \zeta_1)(s + 1)} \left[ (2 A(a_1, a_2) - a_1)^{r+1} - (2 A(a_1, a_2) - a_2)^{r+1} \right] \\
&\quad - (2 A(a_1, a_2))^r \right| \leq \frac{(\zeta_2 - \zeta_1)^2}{16 s(s - 1)} \left( \frac{1}{2r + 1} \right)^{1/r} \\
&\quad \times \left\{ \left( \Psi''(\theta_1 + \theta_2 - \zeta_1 - \zeta_2) - \Psi(\theta_1 + \theta_2 - \zeta_1 - \zeta_2) \right) \right\} \\
&\quad - \frac{1}{2(s + 1)} \left| a_2^{(s-2)q} \right|^q \left( 2^{r+1} - 1 \right) \left| a_1^{(s-2)q} \right|^q \\
&\quad + \left\{ \left( \Psi''(\theta_1 + \theta_2 - \zeta_1 - \zeta_2) - \Psi(\theta_1 + \theta_2 - \zeta_1 - \zeta_2) \right) \right\} \\
&\quad - \left( \frac{1}{2(s + 1)} \left| a_2^{(s-2)q} \right|^q + \left( 2^{r+1} - 1 \right) \left| a_1^{(s-2)q} \right|^q \right) \left( \frac{1}{2(s + 1)} \right)^{1/r}. \tag{67}
\end{align*}

**Proof.** The proof is an immediate consequence from Corollary 27 by selecting \( \Psi(x) = x^s \) and \( s \in (0, 1) \).
Proposition 36. Suppose $a_1, a_2 \in \mathbb{R}, s \in (0, 1), 0 < a_1 < a_2, 0 \neq [a_1, a_2]$. Then, for all $r > 1$, the following inequality holds:

$$\left| \frac{1}{(\zeta_2 - \zeta_1)(1-s)} \left( (2A(a_1, a_2) - a_1) - (2A(a_1, a_2) - a_2) \right) \right| \leq \frac{(\zeta_2 - \zeta_1)^2}{16} \left\{ \left| 2A(a_1, a_2) - a_1 \right|^{1/r} + \left| 2A(a_1, a_2) - a_2 \right|^{1/r} \right\} \times \left\{ \left| \phi^{(3)}(a_1) \right|^q + \left| \phi^{(3)}(a_2) \right|^q \right\}^{1/q}.$$

Proof. By employing the definition of the $\phi$-digamma function $\phi^{(3)}(\zeta)$, it is easy to notice that the $\gamma$-trigamma function $\zeta \mapsto \phi^{(3)}(\zeta)$ is completely monotonic on $(0, \infty)$. This ensures that the function $\phi^{(3)}(\zeta)$ is again completely monotonic on $(0, \infty)$ for each $\gamma \in (0, 1)$ and consequently is convex and non-negative (see [26], p. 167). Now, by applying Corollary 27, we extract that the inequality (71) is valid for $\gamma \in (0, 1)$.

As another application of inequality (71), we can deliver the following inequalities for the $\gamma$-trigamma and $\gamma$-polygamma functions and the analogue of harmonic numbers $H_{\gamma n}$ defined by

$$H_{\gamma n} = \sum_{k=0}^{\infty} \frac{\gamma^k}{1 - \gamma^k}, \quad n \in \mathbb{N}.$$  

(72)

So, from inequality (71), we use the equation

$$\phi^{(3)}(n + 1) = \phi^{(3)}(1) - \log \left( \gamma \right) H_{\gamma n}, \quad n \in \mathbb{N}.$$  

(73)

Consequently, we obtain the following result.

Corollary 38. Suppose $n \in \mathbb{N}, \gamma > 1, 0 < \gamma < 1$. Then, the following inequality holds:

$$\left| -\frac{\log \left( \gamma \right) H_{\gamma n}}{n} - \phi^{(3)}(A(1, n + 1)) \right| \leq \frac{n^{2}}{16} \left\{ \left| \phi^{(3)}(1) \right|^q + \left| \phi^{(3)}(n + 1) \right|^q \right\}^{1/q} \times \left\{ \left| \phi^{(3)}(1) \right|^q + \left| \phi^{(3)}(n + 1) \right|^q \right\}^{1/q} + \left\{ \left| \phi^{(3)}(1) \right|^q + \left| \phi^{(3)}(n + 1) \right|^q \right\}^{1/q}.$$

(74)

Proposition 39. Suppose $n$ is an integer and $\gamma > 1$. Then, the following inequality holds:

$$\frac{|H_{\gamma n} - \phi^{(3)}(A(1, n + 1))|}{n} \leq \frac{n^{2}}{16} \left( \frac{1}{2^{r+1}} \right)^{1/r} \times \left\{ \left| \phi^{(3)}(1) \right|^q + \left| \phi^{(3)}(n + 1) \right|^q \right\}^{1/q}.$$  

(75)

Proof. From inequality (74), when $\gamma \to 1$, we use the relation
\[ \lim_{q \rightarrow 1} \log (q) H_{mq} = \lim_{q \rightarrow 1} \left[ \left( \frac{\log(q)}{q - 1} \right) (q - 1) H_{mq} \right] = -\lim_{q \rightarrow 1} \sum_{k=1}^{n} \frac{1 - q}{1 - q^k} = -H_n. \] (76)

We obtain the required result.

**Remark 40.** By using the equation

\[ H_n = y + \varphi(n + 1), \] (77)

where \( y \) is the Euler-Mascheroni constant, the inequality (75) becomes

\[ \left| \frac{y + \varphi(n + 1)}{n} - \varphi'(A(1, n + 1)) \right| \leq \frac{n^2}{16} \left( \frac{1}{2r + 1} \right)^{1/2} \times \left\{ \left( \left| \left( \varphi^{(3)}(1) \right)^q + \left( \varphi^{(3)}(n + 1) \right)^q \right| \right) - \left( \frac{2^{q+1} - 1}{2^{q+1} + 1} \right) \left( \left| \varphi^{(3)}(1) \right|^q + \left| \varphi^{(3)}(n + 1) \right|^q \right) \right\}^{1/2} + \left\{ \left( \left| \varphi^{(3)}(1) \right|^q + \left| \varphi^{(3)}(n + 1) \right|^q \right) \right\}^{1/2}. \] (78)

5. Conclusion

In this paper, we have explored new \( k \)-fractional variants of Hermite-Mercer-type integral inequalities for \( s \)-convex functions. New results and novel connections are built for the left and right sides of Hermite-Hadamard-type inequalities for differentiable mappings whose derivatives in absolute values at certain powers are \( s \)-convex in the second sense. New integral identities for differentiable mappings are obtained, and related results are established. In the application viewpoint, our findings illustrate new generalizations with the connection of special function theory (special means of real numbers and \( q \)-digamma function) and harmonic numbers. It is quite open to think about Jensen-Hermite-Mercer variants for generalized integral operators having nonlocal and non-singular kernels by applying generalized convexities. However, it is not easy to extend such inequalities for other existing types of convexities. The suggested scheme is viable, effective, and computationally appealing in fractional differential equations, optimization theory, and other related areas of convexity.

**Data Availability**

All data required for this paper is included within this paper.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

**Authors’ Contributions**

All authors contributed equally to this paper.

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