Hindawi
Journal of Function Spaces
Volume 2021, Article ID 5567628, 12 pages
https://doi.org/10.1155/2021/5567628

Research Article

Map Ideal of Type the Domain of r-Cesàro Matrix in the Variable Exponent $\ell_{t(.)}$ and Its Eigenvalue Distributions

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Received 10 February 2021; Revised 2 April 2021; Accepted 19 April 2021; Published 28 April 2021

Academic Editor: Ciqiang Zhuo

In this article, we define a new sequence space generated by the domain of r-Cesàro matrix in Nakano sequence space. Some geometric and topological properties of this sequence space, the multiplication maps defined on it, and the eigenvalue distributions of map ideal with s-numbers that belong to this sequence space have been examined.

1. Introduction

The vector spaces $\ell_{t(.)}$ are contained in the variable exponent spaces $L_{t(.)}$. Regarding the 2nd half of the twentieth century, it used to be fulfilled that these variable exponent spaces constituted the proper framework for the mathematical components of numerous issues for which the classical Lebesgue spaces have been inadequate. The relevancy of these spaces and their homes made them a famous and environment friendly device in the remedy of a range of situations; these days, the region of $L_{t(.)}(\Omega)$ spaces is a prolific subject of lookup with ramifications achieving into very numerous mathematical specialties [1]. Learning about the variable exponent Lebesgue spaces $L_{t(.)}$ obtained in addition impetus from the mathematical description of the hydrodynamics of non-Newtonian fluids [2, 3]. Applications of non-Newtonian fluids additionally known as electrorheological vary from their use in army science to civil engineering and orthopedics. By $\mathscr{C}^{\mathrm{N}}$, ℓ_∞, ℓ_r, and c_0, we suggest the spaces of each, bounded, r-absolutely summable and null sequences of complex numbers $N = \{0, 1, 2, \cdots\}$. We evidence the space of all, finite rank, approximable and compact bounded linear maps from a Banach space $\mathscr{P}$ into a Banach space $\mathscr{Q}$ by $\mathbb{B}(\mathscr{P}, \mathscr{Q})$, $\mathbb{F}(\mathscr{P}, \mathscr{Q})$, $\mathscr{A}(\mathscr{P}, \mathscr{Q})$, and $\mathscr{K}(\mathscr{P}, \mathscr{Q})$, and if $\mathscr{P} = \mathscr{Q}$, we mark $\mathbb{B}(\mathscr{P})$, $\mathbb{F}(\mathscr{P})$, $\mathscr{A}(\mathscr{P})$, and $\mathscr{K}(\mathscr{P})$, respectively (see [4, 5]). The ideal of all, finite rank, approximable and compact maps is denoted by $\mathbb{B}$, $\mathbb{F}$, $\mathscr{A}$, and $\mathscr{K}$. We designate $e_l = (0, 0, \cdots, 1, 0, 0, \cdots)$, as 1 presents at the l^{th} coordinate, with $l \in N$.

Lemma 1 [5]. *Pick up $U \in \mathbb{B}(\mathscr{P}, \mathscr{Q})$. Assume $U \notin \mathscr{A}(\mathscr{P}, \mathscr{Q})$; then, there are maps $X \in \mathbb{B}(\mathscr{P})$ and $Y \in \mathbb{B}(\mathscr{Q})$ so that $YUXe_l = e_l$, for every $l \in N$.*

Definition 2 [5]. A Banach space $\mathscr{V}$ is named simple if the algebra $\mathbb{B}(\mathscr{V})$ includes one and only one nontrivial closed ideal.

Theorem 3 [5]. *Let $\mathscr{V}$ be an infinite dimensional Banach space; then,*

$$\mathbb{F}(\mathscr{V}) \subsetneq \mathscr{A}(\mathscr{V}) \subsetneq \mathscr{K}(\mathscr{V}) \subsetneq \mathbb{B}(\mathscr{V}). \tag{1}$$

Definition 4 [6]. A map $U \in \mathbb{B}(\mathscr{V})$ is entitled Fredholm if $\dim\,(\mathrm{Range}(U))^c < \infty$, $\dim\,(\ker\,(U)) < \infty$, and $\mathrm{Range}(U)$ is closed, where $(\mathrm{Range}(U))^c$ mentions the complement of $\mathrm{Range}(U)$.

Definition 5 [7]. A subclass $\mathbb{W}$ of $\mathbb{B}$ is named a map ideal if every component $\mathbb{W}(\mathscr{P}, \mathscr{Q}) = \mathbb{W} \cap \mathbb{B}(\mathscr{P}, \mathscr{Q})$ executes the next setup:

(i) $I_\Omega \in \mathbb{W}$, if Ω illustrates Banach space of one dimension

(ii) $\mathbb{W}(\mathscr{P}, \mathscr{Q})$ is a linear space on $\mathscr{C}$

(iii) Suppose $X \in \mathbb{B}(\mathscr{P}_0, \mathscr{P})$, $Y \in \mathbb{W}(\mathscr{P}, \mathscr{Q})$, and $Z \in \mathbb{B}(\mathscr{Q}, \mathscr{Q}_0)$, then $ZYX \in \mathbb{W}(\mathscr{P}_0, \mathscr{Q}_0)$, where $\mathscr{P}_0$ and $\mathscr{Q}_0$ are normed spaces

Faried and Bakery [8] made current the notion of prequasi ideal which is added established than the quasi ideal.

Definition 6. A function $\Psi : \mathbb{W} \longrightarrow [0,\infty)$ is named a prequasi norm on the map ideal $\mathbb{W}$ if the next setting encompasses the following:

(1) For each $X \in \mathbb{W}(\mathscr{P}, \mathscr{Q})$, $\Psi(X) \geq 0$ and $\Psi(X) = 0 \Leftrightarrow X = 0$

(2) We have $E_0 \geq 1$ so as to $\Psi(\kappa X) \leq E_0 |\kappa| \Psi(X)$, for all $X \in \mathbb{W}(\mathscr{P}, \mathscr{Q})$ and $\kappa \in \mathscr{C}$

(3) We have $G_0 \geq 1$ for $\Psi(Z_1 + Z_2) \leq G_0[\Psi(Z_1) + \Psi(Z_2)]$, for all $Z_1, Z_2 \in \mathbb{W}(\mathscr{P}, \mathscr{Q})$

(4) We have $D_0 \geq 1$ if $X \in \mathbb{B}(\mathscr{P}_0, \mathscr{P})$, $Y \in \mathbb{W}(\mathscr{P}, \mathscr{Q})$, and $Z \in \mathbb{B}(\mathscr{Q}, \mathscr{Q}_0)$; then, $\Psi(ZYX) \leq D_0 \|Z\| \Psi(Y) \|X\|$

Theorem 7 [8]. *Ψ is prequasi norm on the map ideal $\mathbb{W}$, whenever Ψ is a quasinorm on the map ideal $\mathbb{W}$.*

Definition 8 [9]. An s-number function is a map detailed on $\mathbb{B}(\mathscr{P}, \mathscr{Q})$ which sort to every map $X \in \mathbb{B}(\mathscr{P}, \mathscr{Q})$ a nonnegative scalar sequence $(s_l(X))_{l=0}^{\infty}$ overbearing that the next setting encompasses the following:

(a) $\|X\| = s_0(X) \geq s_1(X) \geq s_2(X) \geq \cdots \geq 0$, for every $X \in \mathbb{B}(\mathscr{P}, \mathscr{Q})$

(b) $s_{l+a-1}(X_1 + X_2) \leq s_l(X_1) + s_a(X_2)$, for each $X_1, X_2 \in \mathbb{B}(\mathscr{P}, \mathscr{Q})$ and $l, a \in N$

(c) Ideal property: $s_a(ZYX) \leq \|Z\| s_a(Y) \|X\|$, for all $X \in \mathbb{B}(\mathscr{P}_0, \mathscr{P})$, $Y \in \mathbb{B}(\mathscr{P}, \mathscr{Q})$, and $Z \in \mathbb{B}(\mathscr{Q}, \mathscr{Q}_0)$, where $\mathscr{P}_0$ and $\mathscr{Q}_0$ are discretionary Banach spaces

(d) For $G \in \mathbb{B}(\mathscr{P}, \mathscr{Q})$ and $\gamma \in \mathscr{C}$, one has $s_a(\gamma G) = |\gamma| s_a(G)$

(e) Rank property: assume rank $(X) \leq a$, then $s_a(X) = 0$, for each $X \in \mathbb{B}(\mathscr{P}, \mathscr{Q})$

(f) Norming property: $s_{l \geq a}(I_a) = 0$ or $s_{l<a}(I_a) = 1$, where I_a mirrors the unit map on the a-dimensional Hilbert space ℓ_2^a

In an assorted illustration of s-numbers, we intimate the next setting:

(1) The a-th Kolmogorov number, demonstrated by $d_a(X)$, is indicated by

$$d_a(X) = \inf_{\dim J \leq a} \sup_{\|f\| \leq 1} \inf_{g \in J} \|Xf - g\|. \tag{2}$$

(2) The a-th approximation number, established by $\alpha_a(X)$, is denoted by

$$\alpha_a(X) = \inf \{\|X - Y\| : Y \in \mathbb{B}(\mathscr{P}, \mathscr{Q}) \text{ and rank } (Y) \leq a\}. \tag{3}$$

Notations 9 [8].

$$\begin{aligned}
&\mathbb{B}_{\mathscr{V}}^{s} := \{\mathbb{B}_{\mathscr{V}}^{s}(\mathscr{P}, \mathscr{Q}) \,;\, \mathscr{P} \text{ and } \mathscr{Q} \text{ are Banach Spaces}\}, \text{where } \mathbb{B}_{\mathscr{V}}^{s}(\mathscr{P}, \mathscr{Q}) := \left\{X \in \mathbb{B}(\mathscr{P}, \mathscr{Q}) : (s_a(X))_{a=0}^{\infty} \in \mathscr{V}\right\},\\
&\mathbb{B}_{\mathscr{V}}^{\alpha} := \{\mathbb{B}_{\mathscr{V}}^{\alpha}(\mathscr{P}, \mathscr{Q}) \,;\, \mathscr{P} \text{ and } \mathscr{Q} \text{ are Banach Spaces}\}, \text{where } \mathbb{B}_{\mathscr{V}}^{\alpha}(\mathscr{P}, \mathscr{Q}) := \left\{X \in \mathbb{B}(\mathscr{P}, \mathscr{Q}) : (\alpha_a(X))_{a=0}^{\infty} \in \mathscr{V}\right\},\\
&\mathbb{B}_{\mathscr{V}}^{d} := \left\{\mathbb{B}_{\mathscr{V}}^{d}(\mathscr{P}, \mathscr{Q})\, \mathscr{P} \text{ and } \mathscr{Q} \text{ are Banach Spaces}\right\}, \text{where } \mathbb{B}_{\mathscr{V}}^{d}(\mathscr{P}, \mathscr{Q}) := \left\{X \in \mathbb{B}(\mathscr{P}, \mathscr{Q}) : (d_a(X))_{a=0}^{\infty} \in \mathscr{V}\right\}.
\end{aligned} \tag{4}$$

The ideals and multiplication mappings possess extensive grazing of mathematics in functional analysis, namely, in the theory of fixed point, eigenvalue distributions theorem, and geometric structure of Banach spaces. A few of map ideals in the class of Banach spaces or Hilbert spaces are evident by inconsistent scalar sequence spaces. For representative the ideal of compact maps is evident by the space c_0 and $d_a(X)$, for $X \in \mathbb{B}(\mathscr{P}, \mathscr{Q})$. Pietsch [5] approved the quasi-ideals $\mathbb{B}_{\ell_b}^{\alpha}$, for $0 < b < \infty$. He investigated that the ideals of nuclear maps and of Hilbert Schmidt maps between Hilbert spaces are explored by ℓ_1 and ℓ_2, respectively. He examined that $\mathbb{F}(\ell_b)$ are dense in $\mathbb{B}(\ell_b)$, and the algebra $\mathbb{B}(\ell_b)$, where $(1 \leq b < \infty)$, constructed simple Banach space. Pietsch [10] approved that $\mathbb{B}_{\ell_b}^{\alpha}$, for $0 < b < \infty$, is small. Makarov and Faried [11] examined that for each infinite dimensional Banach space $\mathscr{P}$, $\mathscr{Q}$ and $r > b > 0$, then $\mathbb{B}_{\ell_b}^{\alpha}(\mathscr{P}, \mathscr{Q}) Þ \mathbb{B}_{\ell_r}^{\alpha}(\mathscr{P}, \mathscr{Q}) \mathscr{U} \mathbb{B}(\mathscr{P}, \mathscr{Q})$. Yaying et al. [12] defined and examined the sequence space, χ_r^t, whose r-Cesàro matrix is in ℓ_t, with $r \in (0, 1]$ and $1 \leq t$

$\leq \infty$. They explored the quasi Banach ideal of type χ_r^t, for $r \in (0,1]$ and $1 < t < \infty$. They establish its Schauder basis, $\alpha-$, $\beta-$, and $\gamma-$ duals, and found certain matrix classes connected with this sequence space. Basarir and Kara investigated the compact mappings on some Euler $B(m)$-difference sequence spaces [13], some difference sequence spaces of weighted means [14], the Riesz $B(m)$-difference sequence space [15], the B-difference sequence space derived by weighted mean [16], and the m^{th} order difference sequence space of generalized weighted mean [17]. Mursaleen and Noman [18, 19] introduced the compact mappings on some difference sequence spaces. The multiplication maps on Cesàro sequence spaces with the Luxemburg norm were studied by Komal et al. [20]. İlkhan et al. [21] examined the multiplication maps on Cesàro second-order function spaces. Recently, many authors in the literature have considered some nonabsolute-type sequence spaces and introduced recent high-quality papers, for example, Mursaleen and Noman [22] defined the sequence space ℓ_p^λ and ℓ_∞^λ of nonabsolute type and proved that the spaces ℓ_p^λ and ℓ_p^λ are linearly isomorphic for $0 < p \leq \infty$, ℓ_p^λ is a p-normed space and a BK-space in the cases for $0 < p < 1$ and $1 \leq p \leq \infty$, and formed the basis for the space ℓ_p^λ for $1 \leq p < \infty$. In [23], they examined the $\alpha-$, $\beta-$, and $\gamma-$ duals of ℓ_p^λ and ℓ_∞^λ of nonabsolute type, for $1 \leq p < \infty$. They characterized some related matrix classes and derived the characterizations of some other classes by means of a given basic lemma. On Cesàro summable sequences, Mursaleen and Başar [24] defined some spaces of double sequences whose Cesàro transforms are bounded, convergent in Pringsheim's sense, null in Pringsheim's sense, both convergent in Pringsheim's sense, and bounded, regularly convergent, and absolutely q-summable, respectively, and examined some topological properties of those sequence spaces. The addicted inequality will be run down in the development [25]. If $r_a \geq 1$ and $x_a, z_a \in \mathscr{C}$, with $a \in \mathrm{N}$, and $\hbar = \sup_a r_a$, then

$$|x_a + z_a|^{r_a} \leq 2^{\hbar-1}(|x_a|^{r_a} + |z_a|^{r_a}). \quad (5)$$

Suppose $r \in (0,1)$, $(t_l) \in R^{+N}$, where R^{+N} is the space of all sequences of positive reals, and $t_l \geq 1$, with $l \in N$, we define a new sequence space generated by the domain of r-Cesàro matrix in Nakano sequence space as

$$\left(\text{ces}_r^{(t)}\right)_v = \left\{f = (f_k) \in \mathscr{C}^{\mathrm{N}} : v(\rho f) < \infty, \text{for some } \rho > 0\right\}, \quad (6)$$

where $v(f) = \sum_{l=0}^{\infty} \left(|\sum_{z=0}^{l} r^z f_z| / [l+1]_r\right)^{t_l}$ and

$$[l]_r = \begin{cases} \dfrac{1-r^l}{1-r}, & r \neq 1, \\ l, & r = 1. \end{cases} \quad (7)$$

In case $(t_l) \in \ell_\infty$, we have

$$\left(\text{ces}_r^{(t)}\right)_v = \left\{f = (f_k) \in \mathscr{C}^{\mathrm{N}} : v(\rho f) < \infty, \text{for any } \rho > 0\right\}. \quad (8)$$

Remark 10.

(1) When $r = 1$ and $t_l = t$, with $l \in \mathrm{N}$, then $\text{ces}_r^{(t)}$ is compressed to ces^t, introduced and studied by Ng and Lee [26]. Different types of Cesàro summable sequence spaces of nonabsolute type have been studied by many authors [27–31]

(2) If $t_l = t$, with $l \in \mathrm{N}$, $\text{ces}_r^{(t)}$ is truncated to χ_r^t studied by Yaying et al. [12]

The goal of this paper is efficient like so in Section 2 we offer the sufficient setting on any linear space of sequences $\mathscr{V}$, and we mark it a private sequence space ($\mathfrak{pss}$), so as to the class $\mathbb{B}_{\mathscr{V}}^s$ constructs a map ideal. We apply this theorem on $\text{ces}_r^{(t)}$. We define a subclass of the $\mathfrak{pss}$ which we will call a premodular $\mathfrak{pss}$ under the functional $v : \mathscr{V} \longrightarrow [0,\infty)$. We explain the sufficient conditions on $\text{ces}_r^{(t)}$ with definite functional v to become premodular $\mathfrak{pss}$. Which implies that $\text{ces}_r^{(t)}$ is a prequasi normed $\mathfrak{pss}$. In Section 3, we define a multiplication map on the prequasi normed $\mathfrak{pss}$, $(\text{ces}_r^{(t)})_v$, and give the necessity and sufficient setup on this sequence space such that the multiplication map is bounded, approximable, invertible, Fredholm, and closed range. In Section 4, firstly, we introduce the sufficient settings (not necessary) on the premodular $\mathfrak{pss}(\text{ces}_r^{(t)})_v$ so that $\mathbb{F}$ is dense in $\mathbb{B}^s_{(\text{ces}_r^{(t)})_v}$. This explains a negative answer of Rhoades [32] open problem about the linearity of s-type $(\text{ces}_r^{(t)})_v$ spaces. Secondly, we introduce the conditions on $(\text{ces}_r^{(t)})_v$ so that the components of prequasi ideal $\mathbb{B}^s_{\text{ces}_r^{(t)}}$ are complete and closed. Thirdly, we investigate the sufficient conditions on $(\text{ces}_r^{(t)})_v$ so as $\mathbb{B}^\alpha_{(\text{ces}_r^{(t)})_v}$ is precisely confined for altered powers. We explain the setup for which the prequasi ideal $\mathbb{B}^\alpha_{(\text{ces}_r^{(t)})_v}$ is minimum. Fourthly, we describe the setting for which the Banach prequasi ideal $\mathbb{B}^s_{(\text{ces}_r^{(t)})_v}$ is simple. Fifthly, we expound the sufficient setting on $(\text{ces}_r^{(t)})_v$ so as to the class of all bounded linear maps which sequence of eigenvalues in $(\text{ces}_r^{(t)})_v$ equals $\mathbb{B}^s_{(\text{ces}_r^{(t)})_v}$.

2. Linear Problem

In this section, we offer the enough setting on any linear space of sequences $\mathscr{V}$, and we mark it private sequence space ($\mathfrak{pss}$), so as the class $\mathbb{B}_{\mathscr{V}}^s$ creates a map ideal. We apply this setting on $\text{ces}_r^{(t)}$. We define a subclass of $\mathfrak{pss}$ under the functional $v : \mathscr{V} \longrightarrow [0,\infty)$, which we will call a premodular $\mathfrak{pss}$. We explain the enough setup on $\text{ces}_r^{(t)}$ with definite functional υ to become premodular $\mathfrak{pss}$, which implies that $ces_r^{(t)}$ is a prequasi normed $\mathfrak{pss}$.

Definition 11. The linear space of sequences $\mathscr{V}$ is named a $\mathfrak{pss}$, if it satisfies the following:

(1) $e_b \in \mathscr{V}$, with $b \in N$

(2) $\mathscr{V}$ is solid, i.e., for $f = (f_b) \in \mathscr{C}^{\mathrm{N}}$, $|g| = (|g_b|) \in \mathscr{V}$, and $|f_b| \le |g_b|$, over $b \in N$, then $|f| \in \mathscr{V}$

(3) $(|f_{[b/2]}|)_{b=0}^{\infty} \in \mathscr{V}$, while $[b/2]$ illustrates the integral part of $b/2$, if $(|f_b|)_{b=0}^{\infty} \in \mathscr{V}$

Theorem 12. *If the linear sequence space $\mathscr{V}$ is $\mathfrak{pss}$, then $\mathbb{B}_{\mathscr{V}}^{s}$ is a map ideal.*

Proof. Assume the linear sequence space $\mathscr{V}$ is $\mathfrak{pss}$.

(i) Suppose $X \in \mathbb{F}(\mathscr{P}, \mathscr{Q})$ and rank $(X) = b$, with $b \in N$. As $e_b \in \mathscr{V}$, with $b \in N$ and by the linearity of $\mathscr{V}$, one has $(s_b(X))_{b=0}^{\infty} = \sum_{a=0}^{b-1} s_a(X) e_a \in \mathscr{V}$. Therefore, $X \in \mathbb{B}_{\mathscr{V}}^{s}(\mathscr{P}, \mathscr{Q})$, this gives $\mathbb{F}(\mathscr{P}, \mathscr{Q}) \subseteq \mathbb{B}_{\mathscr{V}}^{s}(\mathscr{P}, \mathscr{Q})$

(ii) Presume $X_1, X_2 \in \mathbb{B}_{\mathscr{V}}^{s}(\mathscr{P}, \mathscr{Q})$ and $\rho_1, \rho_2 \in \mathscr{C}$, then $(s_b(X_1))_{b=0}^{\infty} \in \mathscr{V}$ and $(s_b(X_2))_{b=0}^{\infty} \in \mathscr{V}$. As $b \ge 2[b/2]$, compared to the definition of s-numbers and $s_b(X)$ is a decreasing sequence, we get $s_b(\rho_1 X_1 + \rho_2 X_2) \le s_{2[b/2]}(\rho_1 X_1 + \rho_2 X_2) \le s_{[b/2]}(\rho_1 X_1) + s_{[b/2]}(\rho_2 X_2) = |\rho_1| s_{[b/2]}(X_1) + |\rho_2| s_{[b/2]}(X_2)$, with $b \in N$. By using the linearity of $\mathscr{V}$, conditions (24) and (25), one can see $(s_b(\rho_1 X_1 + \rho_2 X_2))_{b=0}^{\infty} \in \mathscr{V}$, so $\rho_1 X_1 + \rho_2 X_2 \in \mathbb{B}_{\mathscr{V}}^{s}(\mathscr{P}, \mathscr{Q})$

(iii) Let $X \in \mathbb{B}(\mathscr{P}_0, \mathscr{P})$, $Y \in \mathbb{B}_{\mathscr{V}}^{s}(\mathscr{P}, \mathscr{Q})$, and $Z \in \mathbb{B}(\mathscr{Q}, \mathscr{Q}_0)$, one has $(s_b(Y))_{b=0}^{\infty} \in \mathscr{V}$. As $s_b(ZYX) \le \|Z\| s_b(Y) \|X\|$. By using the linearity of $\mathscr{V}$ and condition (24), we have $(s_b(ZYX))_{b=0}^{\infty} \in \mathscr{V}$, then $ZYX \in \mathbb{B}_{\mathscr{V}}^{s}(\mathscr{P}_0, \mathscr{Q}_0)$

Here and after, we will denote the space of all increasing sequences of real numbers by $\mathfrak{I}$.

Theorem 13. *$ces_r^{(t)}$ is $\mathfrak{pss}$, if $(t_l) \in \mathfrak{I} \cap \ell_\infty$ with $t_0 > 1$.*

Proof.

(1.i) Assume $f, g \in ces_r^{(t)}$. As $(t_l) \in \ell_\infty$, one has

$$\begin{aligned}&\sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l} r^z f_z + r^z g_z\right|}{[l+1]_r}\right)^{t_l}\\ &\le 2^{\hbar-1}\left(\sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l} r^z f_z\right|}{[l+1]_r}\right)^{t_l} + \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l} r^z g_z\right|}{[l+1]_r}\right)^{t_l}\right)\\ &< \infty,\end{aligned} \tag{9}$$

so $f + g \in ces_r^{(t)}$.

(1.ii) Suppose $\rho \in \mathscr{C}$, $f \in ces_r^{(t)}$, and as $(t_l) \in \ell_\infty$, we obtain

$$\sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l} r^z \rho f_z\right|}{[l+1]_r}\right)^{t_l} \le \sup_l |\rho|^{t_l} \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l} r^z f_z\right|}{[l+1]_r}\right)^{t_l} < \infty. \tag{10}$$

Hence, $\rho f \in ces_r^{(t)}$. Relative to (1-i) and (1-ii), we have $ces_r^{(t)}$ as a linear space.

Also as $(t_l) \in \mathfrak{I} \cap \ell_\infty$ with $t_0 > 1$, one has

$$\sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l} r^z (e_b)_z\right|}{[l+1]_r}\right)^{t_l} = \sum_{l=b}^{\infty}\left(\frac{r^b}{[l+1]_r}\right)^{t_l} \le \sup_l r^{bt_l} \sum_{l=b}^{\infty}\left(\frac{1}{[l+1]_r}\right)^{t_l} < \infty. \tag{11}$$

Therefore, $e_b \in ces_r^{(t)}$, with $b \in N$.

(1) If $|f_b| \le |g_b|$, for each $b \in N$ and $|g| \in ces_r^{(t)}$. One can see

$$\sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l} r^z |f_z|\right|}{[l+1]_r}\right)^{t_l} \le \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l} r^z |g_z|\right|}{[l+1]_r}\right)^{t_l} < \infty. \tag{12}$$

Hence, $|f| \in ces_r^{(t)}$.

(2) Assume $(|f_z|) \in ces_r^{(t)}$, where $(t_l) \in \mathfrak{I} \cap \ell_\infty$, we get

$$\begin{aligned}\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r^z \left|f_{[z/2]}\right|}{[l+1]_r}\right)^{t_l} &= \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{2l} r^z \left|f_{[z/2]}\right|}{[2l+1]_r}\right)^{t_{2l}}\\ &\quad + \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{2l+1} r^z \left|f_{[z/2]}\right|}{[2l+2]_r}\right)^{t_{2l+1}}\\ &\le \sum_{l=0}^{\infty}\left(\frac{1}{[2l+1]_r}\left(r^{2l}|f_l| + \sum_{z=0}^{l}\left(r^{2z} + r^{2z+1}\right)|f_z|\right)\right)^{t_l}\\ &\quad + \sum_{l=0}^{\infty}\left(\frac{1}{[2l+2]_r}\sum_{z=0}^{l}\left(r^{2z} + r^{2z+1}\right)|f_z|\right)^{t_l}\\ &\le 2^{\hbar-1}\left(\sum_{l=0}^{\infty}\left(\frac{1}{[l+1]_r}\sum_{z=0}^{l} r^z |f_z|\right)^{t_l} + \sum_{l=0}^{\infty}\left(\frac{2}{[l+1]_r}\sum_{z=0}^{l} r^z |f_z|\right)^{t_l}\right)\\ &\quad + \sum_{l=0}^{\infty}\left(\frac{2}{[l+1]_r}\sum_{z=0}^{l} r^z |f_z|\right)^{t_l}\\ &\le \left(2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar}\right)\sum_{l=0}^{\infty}\left(\frac{1}{[l+1]_r}\sum_{z=0}^{l} r^z |f_z|\right)^{t_l} < \infty,\end{aligned} \tag{13}$$

so $(|f_{[z/2]}|) \in ces_r^{(t)}$.

By using Theorem 12, we can get the next theorem.

Theorem 14. *Pick up $(t_l) \in \mathfrak{I} \cap \ell_\infty$ with $t_0 > 1$, then $\mathbb{B}^s_{ces_r^{(t)}}$ is a map ideal.*

Definition 15. A subclass of $\mathfrak{pss}$ is named a premodular $\mathfrak{pss}$, if there is a map $v : \mathscr{V} \longrightarrow [0,\infty)$ with the following settings:

(i) When $f \in \mathscr{V}$, $f = \theta \Leftrightarrow v(|f|) = 0$, with $v(f) \geq 0$, where θ is the zero element of $\mathscr{V}$

(ii) If $f \in \mathscr{V}$ and $\rho \in \mathscr{C}$, we have $E_0 \geq 1$ with $v(\rho f) \leq |\rho| E_0 v(f)$

(iii) $v(f + g) \leq G_0(v(f) + v(g))$ includes for some $G_0 \geq 1$, with $f, g \in \mathscr{V}$

(iv) For $b \in N$, $|f_b| \leq |g_b|$, we get $v((|f_b|)) \leq v((|g_b|))$

(v) The inequality, $v((|f_b|)) \leq v((|f_{[b/2]}|)) \leq D_0 v((|f_b|))$ includes, for $D_0 \geq 1$

(vi) If $\mathscr{F}$ denotes the space of all sequences with finite nonzero coordinates, then $\bar{\mathscr{F}} = \mathscr{V}_v$

(vii) We have $\varpi > 0$ so that $v(\rho, 0, 0, 0, \cdots) \geq \varpi |\rho| v(1, 0, 0, 0, \cdots)$, with $\rho \in \mathscr{C}$

Definition 16. The $\mathfrak{pss}\mathscr{V}_v$ is named a prequasi normed $\mathfrak{pss}$, if v supports the points (i)–(iii) of Definition 15. If $\mathscr{V}$ is complete equipped with v, then $\mathscr{V}_v$ is named a prequasi Banach $\mathfrak{pss}$.

Theorem 17. *A prequasi normed $\mathfrak{pss}\mathscr{V}_v$, whenever it is premodular $\mathfrak{pss}$.*

Theorem 18. *$(ces_r^{(t)})_v$ is a premodular $\mathfrak{pss}$, if $(t_l) \in \mathfrak{I} \cap \ell_\infty$ with $t_0 > 1$.*

Proof.

(i) Easily, $v(f) \geq 0$ and $v(|f|) = 0 \Leftrightarrow f = \theta$

(ii) We have $E_0 = \max\{1, \sup_l |\rho|^{t_l - 1}\} \geq 1$ with $v(\rho f) \leq E_0 |\rho| v(f)$, for every $f \in \mathrm{ces}_r^{(t)}$ and $\rho \in \mathscr{C}$

(iii) One has $v(f + g) \leq 2^{\hbar - 1}(v(f) + v(g))$, for each $f, g \in \mathrm{ces}_r^{(t)}$

(iv) Definitely, from the proof part (24) of Theorem 13

(v) Indeed, from the proof part (25) of Theorem 13, $D_0 \geq 2^{2\hbar - 1} + 2^{\hbar - 1} + 2^{\hbar} \geq 1$

(vi) Obviously, $\bar{\mathscr{F}} = \mathrm{ces}_r^{(t)}$

(vii) We have $0 < \varpi \leq \sup_l |\rho|^{t_l - 1}$ with $v(\rho, 0, 0, 0, \cdots) \geq \varpi |\rho| v(1, 0, 0, 0, \cdots)$, for each $\rho \neq 0$ and $\varpi > 0$, if $\rho = 0$

By following Theorems 17 and 18, we determine the next theorem.

Theorem 19. *The space $\mathscr{V}_v$ is a prequasi normed $\mathfrak{pss}$, if $(t_l) \in \mathfrak{I} \cap \ell_\infty$ with $t_0 > 1$.*

3. Multiplication Maps on $(\mathrm{ces}_r^{(t)})_v$

In this section, we define a multiplication map on the prequasi normed $\mathfrak{pss}(\mathrm{ces}_r^{(t)})_v$ and investigate the necessity and sufficient setup on $(\mathrm{ces}_r^{(t)})_v$ so as the multiplication map is bounded, invertible, approximable, Fredholm, and closed range map.

Definition 20. If $\omega = (\omega_k) \in \mathscr{C}^N$ and $\mathscr{V}_v$ is a prequasi normed $\mathfrak{pss}$, the map $H_\omega : \mathscr{V}_v \longrightarrow \mathscr{V}_v$ is called a multiplication map on $\mathscr{V}_v$, when $H_\omega f = (\omega_b f_b) \in \mathscr{V}_v$, with $f \in \mathscr{V}_v$. The multiplication map is named created by ω, if $H_\omega \in \mathbb{B}(\mathscr{V}_v)$.

Theorem 21. *Pick up $\omega \in \mathscr{C}^N$ and $(t_l) \in \mathfrak{I} \cap \ell_\infty$ with $t_0 > 1$, then $\omega \in \ell_\infty$, if and only if $H_\omega \in \mathbb{B}((ces_r^{(t)})_v)$.*

Proof. Suppose the settings are confirmed. Let $\omega \in \ell_\infty$. Hence, there is $\nu > 0$ so as to $|\omega_b| \leq \nu$, with $b \in N$. For $f \in (\mathrm{ces}_r^{(t)})_v$, one has

$$\begin{aligned} v(H_\omega f) &= v(\omega f) \\ &= \sum_{l=0}^{\infty} \left(\frac{\left| \sum_{z=0}^{l} r^z \omega_z f_z \right|}{[l+1]_r} \right)^{t_l} \\ &\leq \sum_{l=0}^{\infty} \left(\frac{\left| \sum_{z=0}^{l} \nu r^z f_z \right|}{[l+1]_r} \right)^{t_l} \\ &\leq \sup_l \nu^{t_l} \sum_{l=0}^{\infty} \left(\frac{\left| \sum_{z=0}^{l} r^z f_z \right|}{[l+1]_r} \right)^{t_l} \\ &= \sup_l \nu^{t_l} v(f). \end{aligned} \tag{14}$$

Therefore, $H_\omega \in \mathbb{B}((\mathrm{ces}_r^{(t)})_v)$.

On the contrary, let $H_\omega \in \mathbb{B}((\mathrm{ces}_r^{(t)})_v)$ and $\omega \notin \ell_\infty$. Hence, for all $b \in N$, there are $x_b \in N$ so as $\omega_{x_b} > b$. We have

$$\begin{aligned} v(H_\omega e_{x_b}) &= v(\omega e_{x_b}) \\ &= \sum_{l=0}^{\infty} \left(\frac{\left| \sum_{z=0}^{l} r^z \omega_z (e_{x_b})_z \right|}{[l+1]_r} \right)^{t_l} \\ &= \sum_{l=x_b}^{\infty} \left(\frac{r^{x_b} |\omega_{x_b}|}{[l+1]_r} \right)^{t_l} \\ &> \sum_{l=x_b}^{\infty} \left(\frac{b r^{x_b}}{[l+1]_r} \right)^{t_l} > b^{t_0} v(e_{x_b}). \end{aligned} \tag{15}$$

Hence, $H_\omega \notin \mathbb{B}((\mathrm{ces}_r^{(t)})_v)$. So $\omega \in \ell_\infty$.

Theorem 22. *Assume $\omega \in \mathscr{C}^N$ and $(ces_r^{(t)})_v$ be a prequasi normed* $\mathfrak{pss}$. *Then, $\omega_b = g$, for every $b \in N$ and $g \in \mathscr{C}$ with $|g| = 1$, if and only if H_ω is an isometry.*

Proof. Let the sufficient condition be verified. One has

$$\begin{aligned} v(H_\omega f) &= v(\omega f) \\ &= \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} r^k \omega_k f_k\right|}{[l+1]_r}\right)^{t_l} \\ &= \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} |g| r^k f_k\right|}{[l+1]_r}\right)^{t_l} \\ &= v(f), \end{aligned} \tag{16}$$

with $f \in (\mathrm{ces}_r^{(t)})_v$. So H_ω is an isometry.

Let the necessity condition be satisfied and $|\omega_b| < 1$, for some $b = b_0$. We get

$$\begin{aligned} v(H_\omega e_{b_0}) &= v(\omega e_{b_0}) \\ &= \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} r^k \omega_k (e_{b_0})_k\right|}{[l+1]_r}\right)^{t_l} \\ &= \sum_{l=b_0}^{\infty}\left(\frac{r^{b_0}|\omega_{b_0}|}{[l+1]_r}\right)^{t_l} \\ &< \sum_{l=b_0}^{\infty}\left(\frac{r^{b_0}}{[l+1]_r}\right)^{t_l} \\ &= v(e_{b_0}). \end{aligned} \tag{17}$$

Also, when $|\omega_{b_0}| > 1$, it is easy to show that $v(H_\omega e_{b_0}) > v(e_{b_0})$, which is an inconsistency for the two cases. Therefore, $|\omega_b| = 1$, for all $b \in N$.

By $\mathfrak{F}$, we will denote the space of all sets with finite number of elements.

Theorem 23. *Raise up $\omega \in \mathscr{C}^N$ and $(t_l) \in \mathfrak{I} \cap \ell_\infty$ with $t_0 > 1$. Then, $H_\omega \in \mathscr{A}((ces_r^{(t)})_v)$, if and only if $(\omega_b)_{b=0}^{\infty} \in c_0$.*

Proof. Let $H_\omega \in \mathscr{A}((\mathrm{ces}_r^{(t)})_v)$, so $H_\omega \in \mathscr{K}((\mathrm{ces}_r^{(t)})_v)$. Suppose $\lim_{b\longrightarrow\infty}\omega_b \neq 0$. Therefore, we have $\rho > 0$ so as the set $K_\rho = \{b \in N : |\omega_b| \geq \rho\} \mathfrak{a} \mathfrak{F}$, if $\{\alpha_b\}_{b\in N} \subset K_\rho$. Hence, $\{e_{\alpha_b} : \alpha_b \in K_\rho\} \in \ell_\infty$ is an infinite set in $(\mathrm{ces}_r^{(t)})_v$. Since

$$\begin{aligned} v(H_\omega e_{\alpha_a} - H_\omega e_{\alpha_b}) &= v(\omega e_{\alpha_a} - \omega e_{\alpha_b}) \\ &= \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} r^k \omega_k\left((e_{\alpha_a})_k - (e_{\alpha_b})_k\right)\right|}{[l+1]_r}\right)^{t_l} \\ &\geq \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} r^k \rho\left((e_{\alpha_a})_k - (e_{\alpha_b})_k\right)\right|}{[l+1]_r}\right)^{t_l} \\ &\geq \inf_l \rho^{t_l} v(e_{\alpha_a} - e_{\alpha_b}), \end{aligned} \tag{18}$$

with $\alpha_a, \alpha_b \in K_\rho$. Therefore, $\{e_{\alpha_b} : \alpha_b \in K_\rho\} \in \ell_\infty$, which cannot have a convergent subsequence under H_ω. Hence, $H_\omega \notin \mathscr{K}((\mathrm{ces}_r^{(t)})_v)$, which implies $H_\omega \notin \mathscr{A}((\mathrm{ces}_r^{(t)})_v)$; this gives an inconsistency. So $\lim_{b\longrightarrow\infty}\omega_b = 0$. On the other hand, let $\lim_{b\longrightarrow\infty}\omega_b = 0$. Hence, for all $\rho > 0$, one has $K_\rho = \{b \in N : |\omega_b| \geq \rho\} \subset \mathfrak{F}$. Hence, for each $\rho > 0$, we have $\dim(((\mathrm{ces}_r^{(t)})_v)_{K_\rho}) = \dim(\mathscr{C}^{K_\rho}) < \infty$. So $H_\omega \in \mathbb{F}(((\mathrm{ces}_r^{(t)})_v)_{K_\rho})$. Define $\omega_a \in \mathscr{C}^N$, for all $a \in N$, by

$$(\omega_a)_b = \begin{cases} \omega_b, & b \in K_{1/a+1}, \\ 0, & \text{otherwise}. \end{cases} \tag{19}$$

It is clear that $H_{\omega_a} \in \mathbb{F}(((\mathrm{ces}_r^{(t)})_v)_{B_{1/(a+1)}})$ as $\dim(((\mathrm{ces}_r^{(t)})_v)_{B_{1/(a+1)}}) < \infty$, for all $a \in N$. From $(t_l) \in \mathfrak{I} \cap \ell_\infty$ with $t_0 > 1$, one can see

$$\begin{aligned} v((H_\omega - H_{\omega_a})f) &= v(((\omega_b - (\omega_a)_b)f_b)_{b=0}^{\infty}) \\ &= \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{b=0}^{l} r^b(\omega_b - (\omega_a)_b)f_b\right|}{[l+1]_r}\right)^{t_l} \\ &= \sum_{l=0, l\in K_{1/(a+1)}}^{\infty}\left(\frac{\left|\sum_{b=0}^{l} r^b(\omega_b - (\omega_a)_b)f_b\right|}{[l+1]_r}\right)^{t_l} \\ &\quad + \sum_{l=0, l\notin K_{1/(a+1)}}^{\infty}\left(\frac{\left|\sum_{b=0}^{l} r^b(\omega_b - (\omega_a)_b)f_b\right|}{[l+1]_r}\right)^{t_l} \\ &= \sum_{l=0, l\notin K_{1/(a+1)}}^{\infty}\left(\frac{\left|\sum_{b=0}^{l} r^b \omega_b f_b\right|}{[l+1]_r}\right)^{t_l} \\ &\leq \frac{1}{(a+1)^{t_0}}\sum_{l=0, l\notin K_{1/(a+1)}}^{\infty}\left(\frac{\left|\sum_{b=0}^{l} r^b f_b\right|}{[l+1]_r}\right)^{t_l} \\ &< \frac{1}{(a+1)^{t_0}}\sum_{l=0}^{\infty}\left(\frac{\left|\sum_{b=0}^{l} r^b f_b\right|}{[l+1]_r}\right)^{t_l} \\ &= \frac{1}{(a+1)^{t_0}} v(f). \end{aligned} \tag{20}$$

Hence, $\|H_\omega - H_{\omega_a}\| \leq 1/(a+1)^{t_0}$. This gives that H_ω is a limit of finite rank maps. Therefore, $H_\omega \in \mathscr{A}((\mathrm{ces}_r^{(t)})_v)$.

Theorem 24. *Pick up $\omega \in \mathscr{C}^N$ and $(t_l) \in \mathfrak{I} \cap \ell_\infty$ with $t_0 > 1$. Then, $H_\omega \in \mathscr{K}((ces_r^{(t)})_v)$, if and only if $(\omega_b)_{b=0}^{\infty} \in c_0$.*

Proof. Obviously, since $\mathscr{A}((\mathrm{ces}_r^{(t)})_v) Þ \mathscr{K}((\mathrm{ces}_r^{(t)})_v)$.

Corollary 25. *If* $(t_l) \in \mathfrak{I} \cap \ell_\infty$ *with* $t_0 > 1$, *then* $\mathscr{K}((ces_r^{(t)})_v) \not\supsetneq \mathbb{B}((ces_r^{(t)})_v)$.

Proof. As $\omega = (1, 1, \cdots)$ is created the multiplication map I on $(\text{ces}_r^{(t)})_v$, $I \notin \mathscr{K}((\text{ces}_r^{(t)})_v)$ and $I \in \mathbb{B}((\text{ces}_r^{(t)})_v)$.

Theorem 26. *If* $(ces_r^{(t)})_v$ *is a prequasi Banach* $\mathfrak{pss}$ *and* $H_\omega \in \mathbb{B}((ces_r^{(t)})_v)$, *then there are* $\alpha > 0$ *and* $\eta > 0$ *so as* $\alpha < |\omega_b| < \eta$, *with* $b \in (\ker(\omega))^c$, *if and only if Range*(H_ω) *is closed.*

Proof. Assume the sufficient setup is confirmed. Hence, there is $\rho > 0$ so as $|\omega_b| \geq \rho$, with $b \in (\ker(\omega))^c$. To show that Range(H_ω) is closed, if g is a limit point of Range(H_ω), we have $H_\omega f_b \in (\text{ces}_r^{(t)})_v$, with $b \in N$ so that $\lim_{b\longrightarrow\infty} H_\omega f_b = g$. Obviously, the sequence $H_\omega f_b$ is a Cauchy sequence. As $(t_l) \in \mathfrak{I} \cap \ell_\infty$ with $t_0 > 1$, one has

$$\begin{aligned}
v(H_\omega f_a - H_\omega f_b) &= \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} r^k\left(\omega_k (f_a)_k - \omega_k (f_b)_k\right)\right|}{[l+1]_r}\right)^{t_l} \\
&= \sum_{l=0, l\in(\ker(\omega))^c}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} r^k\left(\omega_k (f_a)_k - \omega_k (f_b)_k\right)\right|}{[l+1]_r}\right)^{t_l} \\
&\quad + \sum_{l=0, l\notin(\ker(\omega))^c}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} r^k\left(\omega_k (f_a)_k - \omega_k (f_b)_k\right)\right|}{[l+1]_r}\right)^{t_l} \\
&\geq \sum_{l=0, l\in(\ker(\omega))^c}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} r^k\left(\omega_k (f_a)_k - \omega_k (f_b)_k\right)\right|}{[l+1]_r}\right)^{t_l} \\
&= \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} r^k\left(\omega_k (u_a)_k - \omega_k (u_b)_k\right)\right|}{[l+1]_r}\right)^{t_l} \\
&> \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} r^k \rho\left((u_a)_k - (u_b)_k\right)\right|}{[l+1]_r}\right)^{t_l} \\
&\geq \inf_l \rho^{t_l} v(u_a - u_b),
\end{aligned} \tag{21}$$

where

$$(u_a)_k = \begin{cases} (f_a)_k, & k \in (\ker(\omega))^c, \\ 0, & k \notin (\ker(\omega))^c. \end{cases} \tag{22}$$

This implies that $\{u_a\}$ is a Cauchy sequence in $(\text{ces}_r^{(t)})_v$. As $(\text{ces}_r^{(t)})_v$ is complete, there is $f \in (\text{ces}_r^{(t)})_v$ so that $\lim_{b\longrightarrow\infty} u_b = f$. Since $H_\omega \in \mathbb{B}((\text{ces}_r^{(t)})_v)$, we have $\lim_{b\longrightarrow\infty} H_\omega u_b = H_\omega f$. But $\lim_{b\longrightarrow\infty} H_\omega u_b = \lim_{b\longrightarrow\infty} H_\omega f_b = g$. So $H_\omega f = g$. Hence, $g \in \text{Range}(H_\omega)$. Therefore, Range$(H_\omega)$ is closed. Next, suppose the necessity setup is satisfied. So there is $\rho > 0$ so as $v(H_\omega f) \geq \rho v(f)$, with $f \in ((\text{ces}_r^{(t)})_v)_{(\ker(\omega))^c}$. If $K = \{b \in (\ker(\omega))^c : |\omega_b| < \rho\} \neq \varnothing$, then for $a_0 \in K$, one has

$$\begin{aligned}
v\left(H_\omega e_{a_0}\right) &= v\left(\left(\omega_b \left(e_{a_0}\right)_b\right)_{b=0}^{\infty}\right) \\
&= \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{b=0}^{l} r^b \omega_b \left(e_{a_0}\right)_b\right|}{[l+1]_r}\right)^{t_l} \\
&< \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{b=0}^{l} r^b \left(e_{a_0}\right)_b \rho\right|}{[l+1]_r}\right)^{t_l} \\
&\leq \sup_l \rho^{t_l} v\left(e_{a_0}\right).
\end{aligned} \tag{23}$$

This gives an inconsistency. Therefore, $K = \phi$, we have $|\omega_b| \geq \rho$, with $b \in (\ker(\omega))^c$. This proves the theorem.

Theorem 27. *Pick up* $\omega \in \mathscr{C}^N$ *and* $(ces_r^{(t)})_v$ *be a prequasi Banach* $\mathfrak{pss}$. *Then, there is* $\alpha > 0$ *and* $\eta > 0$ *so that* $\alpha < |\omega_b| < \eta$, *with* $b \in N$, *if and only if* $H_\omega \in \mathbb{B}((ces_r^{(t)})_v)$ *is invertible.*

Proof. Let the setup be true. Assume $\kappa \in \mathscr{C}^{\mathrm{N}}$ with $\kappa_b = 1/\omega_b$. By using Theorem 21, the maps H_ω and H_κ are bounded linear. We have $H_\omega \cdot H_\kappa = H_\kappa \cdot H_\omega = I$. Therefore, $H_\kappa = H_\omega^{-1}$. Next, let H_ω be invertible. So Range$(H_\omega) = ((\text{ces}_r^{(t)})_v)_N$. Hence, Range$(H_\omega)$ is closed. Therefore, by Theorem 26, there is $\alpha > 0$ so that $|\omega_b| \geq \alpha$, for each $b \in (\ker(\omega))^c$. We have $\ker(\omega) = \varnothing$, if $\omega_{b_0} = 0$, with $b_0 \in N$; this gives $e_{b_0} \in \ker(H_\omega)$ which is an inconsistency, as $\ker(H_\omega)$ is trivial. Therefore, $|\omega_b| \geq \alpha$, with $b \in N$. As $H_\omega \in \ell_\infty$, from Theorem 21, there is $\eta > 0$ so that $|\omega_b| \leq \eta$, with $b \in N$. Hence, one has $\alpha \leq |\omega_b| \leq \eta$, with $b \in N$.

Theorem 28. *Raise up* $(ces_r^{(t)})_v$ *to be a prequasi Banach* $\mathfrak{pss}$ *and* $H_\omega \in \mathbb{B}((ces_r^{(t)})_v)$. *Then,* H_ω *is a Fredholm map, if and only if (i)* $\ker(\omega)UN$ *is finite and (ii)* $|\omega_b| \geq \rho$, *with* $b \in (\ker(\omega))^c$.

Proof. Assume the sufficient condition is satisfied. Let $\ker(\omega)$UN be infinite; hence, $e_b \in \ker(H_\omega)$, with $b \in \ker(\omega)$. Since e_b's are linearly independent, this gives $\dim(\ker(H_\omega)) = \infty$; this implies an inconsistency. Hence, $\ker(\omega)$UN must be finite. The condition (ii) comes from Theorem 26. Next, let the setup (i) and (ii) be confirmed. From Theorem 26, the setup (ii) implies that Range(H_ω) is closed. The setting (i) gives that $\dim(\ker(H_\omega)) < \infty$ and $\dim((\text{Range}(H_\omega))^c) < \infty$. This implies that H_ω is Fredholm.

4. Prequasi Ideal

In this section, firstly, we introduce the sufficient setting (not necessary) on $(\text{ces}_r^{(t)})_v$ such that $\mathbb{F}$ is dense in $\mathbb{B}^s_{(\text{ces}_r^{(t)})_v}$. This investigates a negative answer of Rhoades [32] open problem about the linearity of s-type $(\text{ces}_r^{(t)})_v$ spaces. Secondly, for which conditions on $(\text{ces}_r^{(t)})_v$ are $\mathbb{B}^s_{(\text{ces}_r^{(t)})_v}$ complete and closed? Thirdly, we give the sufficient setup on $(\text{ces}_r^{(t)})_v$ such that $\mathbb{B}^{\alpha}_{(\mathrm{ces}_r^{(t)})_v}$ is strictly contained for different powers. We explain the settings in order that $\mathbb{B}^{\alpha}_{(\mathrm{ces}_r^{(t)})_v}$ is minimum. Fourthly, we explain the conditions so that the Banach prequasi ideal $\mathbb{B}^{s}_{(\mathrm{ces}_r^{(t)})_v}$ is simple. Fifthly, we give the sufficient conditions on $(\mathrm{ces}_r^{(t)})_v$ such that the space of all bounded linear maps which sequence of eigenvalues in $(\mathrm{ces}_r^{(t)})_v$ equals $\mathbb{B}^{s}_{(\mathrm{ces}_r^{(t)})_v}$.

4.1. Finite Rank Prequasi Ideal

Theorem 29. *$\mathbb{B}^{s}_{(ces_r^{(t)})_v}(\mathscr{P},\mathscr{Q}) = \mathbb{F}(\bar{\mathscr{P}},\mathscr{Q})$, whenever $(t_l) \in \mathfrak{I} \cap \ell_\infty$ with $t_0 > 1$. But the converse is not necessarily true.*

Proof. To show that $\mathbb{F}(\bar{\mathscr{P}},\mathscr{Q}) \subseteq \mathbb{B}^{s}_{(\mathrm{ces}_r^{(t)})_v}(\mathscr{P},\mathscr{Q})$, as $e_l \in (\mathrm{ces}_r^{(t)})_v$, with $l \in N$ and $(\mathrm{ces}_r^{(t)})_v$ is a linear space, let $Z \in \mathbb{F}(\mathscr{P},\mathscr{Q})$, one has $(s_l(Z))_{l=0}^{\infty} \in \mathscr{F}$. To show that $\mathbb{B}^{s}_{(\mathrm{ces}_r^{(t)})_v}(\mathscr{P},\mathscr{Q}) \subseteq \mathbb{F}(\bar{\mathscr{P}},\mathscr{Q})$, as $t_0 > 1$ and $(t_l) \in \mathfrak{I} \cap \ell_\infty$, one can see $\sum_{l=0}^{\infty}(1/[l+1]_r)^{t_l} < \infty$. For $Z \in \mathbb{B}^{s}_{(\mathrm{ces}_r^{(t)})_v}(\mathscr{P},\mathscr{Q})$, we have $(s_l(Z))_{l=0}^{\infty} \in (\mathrm{ces}_r^{(t)})_v$. As $v(s_l(Z))_{l=0}^{\infty} < \infty$, suppose $\rho \in (0,1)$, then there is $l_0 \in N - \{0\}$ with $v((s_l(Z))_{l=l_0}^{\infty}) < \rho/2^{\hbar+3}\eta d$, for some $d \geq 1$, where $\eta = \max\{1, \sum_{l=l_0}^{\infty}(1/[l+1]_r)^{t_l}\}$. As $s_l(Z)$ is decreasing, one has

$$\sum_{l=l_0+1}^{2l_0}\left(\frac{\sum_{j=0}^{l} r^j s_{2l_0}(Z)}{[l+1]_r}\right)^{t_l} \leq \sum_{l=l_0+1}^{2l_0}\left(\frac{\sum_{j=0}^{l} r^j s_j(Z)}{[l+1]_r}\right)^{t_l} \leq \sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l} r^j s_j(Z)}{[l+1]_r}\right)^{t_l} < \frac{\rho}{2^{\hbar+3}\eta d}. \tag{24}$$

Therefore, there is $Y \in \mathbb{F}_{2l_0}(\mathscr{P},\mathscr{Q})$ so that $\mathrm{rank}(Y) \leq 2l_0$ and

$$\sum_{l=2l_0+1}^{3l_0}\left(\frac{\sum_{j=0}^{l} r^j \|Z-Y\|}{[l+1]_r}\right)^{t_l} \leq \sum_{l=l_0+1}^{2l_0}\left(\frac{\sum_{j=0}^{l} r^j \|Z-Y\|}{[l+1]_r}\right)^{t_l} < \frac{\rho}{2^{\hbar+3}\eta d}, \tag{25}$$

since $(t_l) \in \mathfrak{I} \cap \ell_\infty$, we have

$$\sup_{l=l_0}^{\infty}\left(\sum_{j=0}^{l_0} r^j \|Z-Y\|\right)^{t_l} < \frac{\rho}{2^{2\hbar+2}\eta}. \tag{26}$$

Therefore, one has

$$\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l} r^j \|Z-Y\|}{[l+1]_r}\right)^{t_l} < \frac{\rho}{2^{\hbar+3}\eta d}. \tag{27}$$

As $(t_l) \in \mathfrak{I} \cap \ell_\infty$, $t_0 > 1$ and by using inequalities (5) and (24)–(27), one has

$$\begin{aligned}
d(Z,Y) &= v(s_l(Z-Y))_{l=0}^{\infty} \\
&= \sum_{l=0}^{3l_0-1}\left(\frac{\sum_{j=0}^{l} r^j s_j(Z-Y)}{[l+1]_r}\right)^{t_l} + \sum_{l=3l_0}^{\infty}\left(\frac{\sum_{j=0}^{l} r^j s_j(Z-Y)}{[l+1]_r}\right)^{t_l} \\
&\leq \sum_{l=0}^{3l_0}\left(\frac{\sum_{j=0}^{l} r^j \|Z-Y\|}{[l+1]_r}\right)^{t_l} + \sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l+2l_0} r^j s_j(Z-Y)}{[l+2l_0+1]_r}\right)^{t_{l+2l_0}} \\
&\leq \sum_{l=0}^{3l_0}\left(\frac{\sum_{j=0}^{l} r^j \|Z-Y\|}{[l+1]_r}\right)^{t_l} + \sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l+2l_0} r^j s_j(Z-Y)}{[l+1]_r}\right)^{t_l} \\
&\leq 3\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{i} r^j \|Z-Y\|}{[l+1]_r}\right)^{t_l} \\
&\quad + \sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{2l_0-1} r^j s_j(Z-Y) + \sum_{j=2l_0}^{l+2l_0} r^j s_j(Z-Y)}{[l+1]_r}\right)^{t_l} \\
&\leq 3\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l} r^j \|Z-Y\|}{[l+1]_r}\right)^{t_l} \\
&\quad + 2^{\hbar-1}\left[\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{2l_0-1} r^j s_j(Z-Y)}{[l+1]_r}\right)^{t_l} + \sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=2l_0}^{l+2l_0} r^j s_j(Z-Y)}{[l+1]_r}\right)^{t_l}\right] \\
&\leq 3\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l} r^j \|Z-Y\|}{[l+1]_r}\right)^{t_l} \\
&\quad + 2^{\hbar-1}\left[\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{2l_0-1} r l^j \|Z-Y\|}{[l+1]_r}\right)^{t_l} + \sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l} r^{j+2l_0} s_{j+2l_0}(Z-Y)}{[l+1]_r}\right)^{t_l}\right] \\
&\leq 3\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l} r^j \|Z-Y\|}{[l+1]_r}\right)^{t_l} + 2^{2\hbar-1}\sup_{l=l_0}^{\infty}\left(\sum_{j=0}^{l_0} r^j \|Z-Y\|\right)^{t_l}\sum_{l=l_0}^{\infty}\left([l+1]_r\right)^{-t_l} \\
&\quad + 2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l} r^j s_j(Z)}{[l+1]_r}\right)^{t_l} < \rho.
\end{aligned} \tag{28}$$

Conversely, we give a counterexample as $I_4 \in \mathbb{B}^{s}_{(\mathrm{ces}_0^{(t)})_v}(\mathscr{P},\mathscr{Q})$ but $0 < r < 1$ is not verified. This confirms the proof.

4.2. Banach and Closed Prequasi Ideal

Theorem 30. *The function Ψ is a prequasi norm on $\mathbb{B}^{s}_{(\mathscr{V})_v}$, where $\Psi(Z) = v(s_b(Z))_{b=0}^{\infty}$, for all $Z \in \mathbb{B}^{s}_{(\mathscr{V})_v}(\mathscr{P},\mathscr{Q})$, if $(\mathscr{V})_v$ is a premodular $\mathfrak{pss}$.*

Proof. Let $(\mathscr{V})_v$ be a premodular $\mathfrak{pss}$; hence, Ψ satisfies the following conditions:

(1) For all $X \in \mathbb{B}^{s}_{(\mathscr{V})_v}(\mathscr{P},\mathscr{Q})$, $\Psi(X) = v(s_b(X))_{b=0}^{\infty} \geq 0$ and $\Psi(X) = v(s_b(X))_{b=0}^{\infty} = 0$, if and only if $s_b(X) = 0$, for all $b \in N$, if and only if $X = 0$

(2) There is $E_0 \geq 1$ such that $\Psi(\rho X) = v(s_b(\rho X))_{b=0}^{\infty} \leq E_0 |\rho| \Psi(X)$, for all $X \in \mathbb{B}^{s}_{(\mathscr{V})_v}(\mathscr{P},\mathscr{Q})$ and $\rho \in \mathscr{C}$

(3) There is $D \geq 1$ such that for all $X_1, X_2 \in \mathbb{B}^{s}_{(\mathscr{V})_v}(\mathscr{P},\mathscr{Q})$, we have

$$\begin{aligned}
\Psi(X_1+X_2) &= v(s_b(X_1+X_2))_{b=0}^{\infty} \\
&\leq G_0\left(v\left(s_{[b/2]}(X_1)\right)_{b=0}^{\infty} + v\left(s_{[b/2]}(X_2)\right)_{b=0}^{\infty}\right) \\
&\leq G_0 D_0\left(v(s_b(X_1))_{b=0}^{\infty} + v(s_b(X_2))_{b=0}^{\infty}\right) \\
&\leq D[\Psi(X_1) + \Psi(X_2)].
\end{aligned} \tag{29}$$

(4) There is $\rho \geq 1$ such that if $X \in \mathbb{B}(\mathscr{P}_0, \mathscr{P})$, $Y \in \mathbb{B}^s_{(\mathscr{V})_v}(\mathscr{P}, \mathscr{Q})$, and $Z \in \mathbb{B}(\mathscr{Q}, \mathscr{Q}_0)$, then $\Psi(ZYX) = v(s_b(ZYX))_{b=0}^{\infty} \leq v(\|X\|\|Z\|s_b(Y))_{b=0}^{\infty} \leq \rho\|X\|\Psi(Y)\|Z\|$

Theorem 31. *Pick up* $(t_l) \in \mathfrak{I} \cap \ell_\infty$ *with* $t_0 > 1$, *then* $(\mathbb{B}^s_{(ces_r^{(t)})_v}, \Psi)$ *is a prequasi Banach ideal, where* $\Psi(X) = v((s_l(X))_{l=0}^{\infty})$.

Proof. As $(\mathrm{ces}_r^{(t)})_v$ is a premodular $\mathfrak{pss}$, hence from Theorem 30, Ψ is a prequasi norm on $\mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v}$. Suppose $(X_b)_{b\in N}$ is a Cauchy sequence in $\mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v}(\mathscr{P}, \mathscr{Q})$. As $\mathbb{B}(\mathscr{P}, \mathscr{Q}) \supseteq \mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v}(\mathscr{P}, \mathscr{Q})$, one has

$$\begin{aligned}\Psi(X_a - X_b) &= \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r^z s_z(X_a - X_b)}{[l+1]_r}\right)^{t_l} \\ &\geq \sum_{l=0}^{\infty}\left(\frac{\|X_a - X_b\|}{[l+1]_r}\right)^{t_l} \\ &\geq \|X_a - X_b\|^{t_0},\end{aligned} \tag{30}$$

so $(X_b)_{b\in N}$ is a Cauchy sequence in $\mathbb{B}(\mathscr{P}, \mathscr{Q})$. Since $\mathbb{B}(\mathscr{P}, \mathscr{Q})$ is a Banach space, there is $X \in \mathbb{B}(\mathscr{P}, \mathscr{Q})$ with $\lim_{b\longrightarrow\infty}\|X_b - X\| = 0$. Since $(s_l(X_b))_{l=0}^{\infty} \in (\mathrm{ces}_r^{(t)})_v$, for all $b \in N$, from Definition 15 parts (ii), (iii), and (v), one can see

$$\begin{aligned}\Psi(X) &= \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r^z s_z(X)}{[l+1]_r}\right)^{t_l} \\ &\leq 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r^z s_{[z/2]}(X - X_b)}{[l+1]_r}\right)^{t_l} \\ &\quad + 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r^z s_{[z/2]}(X_b)}{[l+1]_r}\right)^{t_l} \\ &\leq 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r^z \|X - X_b\|}{[l+1]_r}\right)^{t_l} \\ &\quad + 2^{\hbar-1}D_0\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r^z s_z(X_b)}{[l+1]_r}\right)^{t_l} < \infty.\end{aligned} \tag{31}$$

Hence, $(s_l(X))_{l=0}^{\infty} \in (\mathrm{ces}_r^{(t)})_v$, so $X \in \mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v}(\mathscr{P}, \mathscr{Q})$.

Theorem 32. *Suppose* $\mathscr{P}$, $\mathscr{Q}$ *be normed spaces and* $(t_l) \in \mathfrak{I} \cap \ell_\infty$ *with* $t_0 > 1$, *then* $(\mathbb{B}^s_{(ces_r^{(t)})_v}, \Psi)$ *is a prequasi closed ideal, where* $\Psi(X) = v((s_l(X))_{l=0}^{\infty})$.

Proof. As $(\mathrm{ces}_r^{(t)})_v$ is a premodular $\mathfrak{pss}$, from Theorem 30, Ψ is a prequasi norm on $\mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v}$. Assume $X_b \in \mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v}(\mathscr{P}, \mathscr{Q})$, for each $b \in N$ and $\lim_{b\longrightarrow\infty}\Psi(X_b - X) = 0$. As $\mathbb{B}(\mathscr{P}, \mathscr{Q}) \supseteq \mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v}(\mathscr{P}, \mathscr{Q})$, we have

$$\begin{aligned}\Psi(X - X_b) &= \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r^z s_z(X - X_b)}{[l+1]_r}\right)^{t_l} \\ &\geq \sum_{l=0}^{\infty}\left(\frac{\|X - X_b\|}{[l+1]_r}\right)^{t_l} \\ &\geq \|X - X_b\|^{t_0}.\end{aligned} \tag{32}$$

Hence, $(X_b)_{b\in N}$ is a convergent sequence in $\mathbb{B}(\mathscr{P}, \mathscr{Q})$. Since $(s_l(X_b))_{l=0}^{\infty} \in (\mathrm{ces}_r^{(t)})_v$, for every $b \in N$, by using Definition 15 parts (ii), (iii), and (v), one can see

$$\begin{aligned}\Psi(X) &= \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r^z s_z(X)}{[l+1]_r}\right)^{t_l} \\ &\leq 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r^z s_{[z/2]}(X - X_b)}{[l+1]_r}\right)^{t_l} \\ &\quad + 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r^z s_{[z/2]}(X_b)}{[l+1]_r}\right)^{t_l} \\ &\leq 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r^z \|X - X_b\|}{[l+1]_r}\right)^{t_l} \\ &\quad + 2^{\hbar-1}D_0\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r^z s_z(X_b)}{[l+1]_r}\right)^{t_l} < \infty.\end{aligned} \tag{33}$$

We get $(s_l(X))_{l=0}^{\infty} \in (\mathrm{ces}_r^{(t)})_v$, so $X \in \mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v}(\mathscr{P}, \mathscr{Q})$.

4.3. Minimum Prequasi Ideal

Theorem 33. *For any infinite dimensional Banach spaces* $\mathscr{P}$, $\mathscr{Q}$ *and* $(t_l^{(1)}) \in \ell_\infty$, $(t_l^{(2)}) \in \ell_\infty$ *with* $1 < t_l^{(1)} < t_l^{(2)}$, *for all* $l \in N$, *then*

$$\mathbb{B}^s_{\left(ces_r^{\left(t_n^{(1)}\right)}\right)_v}(\mathscr{P}, \mathscr{Q}) \text{Þ} \mathbb{B}^s_{\left(ces_r^{\left(t_n^{(2)}\right)}\right)_v}(\mathscr{P}, \mathscr{Q}) \text{Ů} \mathbb{B}(\mathscr{P}, \mathscr{Q}). \tag{34}$$

Proof. Suppose $Z \in \mathbb{B}^s_{(\mathrm{ces}_r^{(t_n^{(1)})})_v}(\mathscr{P}, \mathscr{Q})$, then $(s_l(Z)) \in (\mathrm{ces}_r^{(t_n^{(1)})})_v$. One has

$$\sum_{l=0}^{\infty}\left(\frac{1}{[l+1]_r}\sum_{z=0}^{l} r^z s_z(Z)\right)^{t_l^{(2)}} < \sum_{l=0}^{\infty}\left(\frac{1}{[l+1]_r}\sum_{z=0}^{l} r^z s_z(Z)\right)^{t_l^{(1)}} < \infty. \tag{35}$$

Then, $Z \in \mathbb{B}^s_{(\mathrm{ces}_r^{(t_n^{(2)})})_v}(\mathscr{P}, \mathscr{Q})$. After, if we choose $(s_l(Z))_{l=0}^{\infty}$ so as $\sum_{z=0}^{l} r^z s_z(Z) = [l+1]_r / \sqrt[t_l^{(1)}]{l+1}$, we have $Z \in \mathbb{B}(\mathscr{P}, \mathscr{Q})$ such that

$$\sum_{l=0}^{\infty}\left(\frac{1}{[l+1]_r}\sum_{z=0}^{l} r^z s_z(Z)\right)^{t_l^{(1)}} = \sum_{l=0}^{\infty}\frac{1}{l+1}=\infty,$$
$$\sum_{l=0}^{\infty}\left(\frac{1}{[l+1]_r}\sum_{z=0}^{l} r^z s_z(Z)\right)^{t_l^{(2)}} = \sum_{l=0}^{\infty}\left(\frac{1}{l+1}\right)^{t_l^{(2)}/t_l^{(1)}}<\infty. \tag{36}$$

So $X \notin \mathbb{B}^s_{(\mathrm{ces}_r^{(t_n^{(1)})})_v}(\mathscr{P},\mathscr{Q})$ and $X \in \mathbb{B}^s_{(\mathrm{ces}_r^{(t_n^{(2)})})_v}(\mathscr{P},\mathscr{Q})$. Clearly, $\mathbb{B}^s_{(\mathrm{ces}_r^{(t_n^{(2)})})_v}(\mathscr{P},\mathscr{Q}) \subset \mathbb{B}(\mathscr{P},\mathscr{Q})$. Next, if we take $(s_l(X))_{l=0}^{\infty}$ such that $\sum_{z=0}^{l} r^z s_z(Z) = [l+1]_r/\sqrt[t_l^{(2)}]{l+1}$. We have $X \in \mathbb{B}(\mathscr{P},\mathscr{Q})$ so that $X \notin \mathbb{B}^s_{(\mathrm{ces}_r^{(t_n^{(2)})})_v}(\mathscr{P},\mathscr{Q})$. This confirms the proof.

Theorem 34. *Pick up any infinite dimensional Banach spaces $\mathscr{P}$, $\mathscr{Q}$ and $(t_l) \in \mathfrak{I} \cap \ell_\infty$ with $t_0 > 1$; hence, $\mathbb{B}^{\alpha}_{(ces_r^{(t)})_v}$ is minimum.*

Proof. Assume the setup is confirmed. So $(\mathbb{B}^{\alpha}_{\mathrm{ces}_r^{(t)}}, \Psi)$, where $\Psi(Z)=\sum_{l=0}^{\infty}((1/[l+1]_r)\sum_{z=0}^{l} r^z \alpha_z(Z))^{t_l}$, is a prequasi Banach ideal. Let $\mathbb{B}^{\alpha}_{\mathrm{ces}_r^{(t)}}(\mathscr{P},\mathscr{Q}) = \mathbb{B}(\mathscr{P},\mathscr{Q})$; hence, there is $\eta > 0$ so as $\Psi(Z) \le \eta\|Z\|$, for each $Z \in \mathbb{B}(\mathscr{P},\mathscr{Q})$. Then, by Dvoretzky's theorem [33] with $b \in N$, one has quotient spaces $\mathscr{P}/Y_b$ and subspaces M_b of $\mathscr{Q}$ which can be mapped onto ℓ_2^b by isomorphisms V_b and X_b with $\|V_b\|\|V_b^{-1}\| \le 2$ and $\|X_b\|\|X_b^{-1}\| \le 2$. If I_b is the identity map on ℓ_2^b, T_b is the quotient map from $\mathscr{P}$ onto $\mathscr{P}/Y_b$ and J_b is the natural embedding map from M_b into $\mathscr{Q}$. Assume m_z be the Bernstein numbers [34]; hence,

$$\begin{aligned}
1 = m_z(I_b) &= m_z\left(X_b X_b^{-1} I_b V_b V_b^{-1}\right)\\
&\le \|X_b\| m_z\left(X_b^{-1} I_b V_b\right)\left\|V_b^{-1}\right\|\\
&= \|X_b\| m_z\left(J_b X_b^{-1} I_b V_b\right)\left\|V_b^{-1}\right\|\\
&\le \|X_b\| d_z\left(J_b X_b^{-1} I_b V_b\right)\left\|V_b^{-1}\right\|\\
&= \|X_b\| d_z\left(J_b X_b^{-1} I_b V_b T_b\right)\left\|V_b^{-1}\right\|\\
&\le \|X_b\| \alpha_z\left(J_b X_b^{-1} I_b V_b T_b\right)\left\|V_b^{-1}\right\|,
\end{aligned} \tag{37}$$

for $0 \le l \le b$. We have

$$\begin{aligned}
\sum_{z=0}^{l} r^z &\le \sum_{z=0}^{l} \|X_b\| r^z \alpha_z\left(J_b X_b^{-1} I_b V_b T_b\right)\left\|V_b^{-1}\right\| \Rightarrow 1\\
&\le \left(\|X_b\|\left\|V_b^{-1}\right\|\right)^{t_l}\left(\frac{\sum_{z=0}^{l} r^z \alpha_z\left(J_b X_b^{-1} I_b V_b T_b\right)}{[l+1]_r}\right)^{t_l}.
\end{aligned} \tag{38}$$

Hence, for some $\rho \ge 1$, one has

$$\begin{aligned}
b+1 &\le \rho\|X_b\|\left\|V_b^{-1}\right\|\sum_{l=0}^{b}\left(\frac{\sum_{z=0}^{l} r^z \alpha_z\left(J_b X_b^{-1} I_b V_b T_b\right)}{[l+1]_r}\right)^{t_l} \Rightarrow b+1\\
&\le \rho\|X_b\|\left\|V_b^{-1}\right\|\Psi\left(J_b X_b^{-1} I_b V_b T_b\right) \Rightarrow b+1\\
&\le \rho\eta\|X_b\|\left\|V_b^{-1}\right\|\left\|J_b X_b^{-1} I_b V_b T_b\right\| \Rightarrow b+1\\
&\le \rho\eta\|X_b\|\left\|V_b^{-1}\right\|\left\|J_b X_b^{-1}\right\|\|I_b\|\|V_b T_b\|\\
&= \rho\eta\|X_b\|\left\|V_b^{-1}\right\|\left\|X_b^{-1}\right\|\|I_b\|\|V_b\| \le 4\rho\eta.
\end{aligned} \tag{39}$$

We have an inconsistency, as b is an arbitrary. Then, $\mathscr{P}$ and $\mathscr{Q}$ both cannot be infinite dimensional when $\mathbb{B}^{\alpha}_{\mathrm{ces}_r^{(t)}}(\mathscr{P},\mathscr{Q}) = \mathbb{B}(\mathscr{P},\mathscr{Q})$. This completes the proof.

Theorem 35. *Upon any infinite dimensional Banach spaces $\mathscr{P}$, $\mathscr{Q}$ and $(t_l) \in \mathfrak{I} \cap \ell_\infty$ with $t_0 > 1$, then $\mathbb{B}^{d}_{ces_r^{(t)}}$ is minimum.*

4.4. Simple Banach Prequasi Ideal

Theorem 36. *Presume $\mathscr{P}$, $\mathscr{Q}$ be infinite dimensional Banach spaces. Let $(t_l^{(1)}) \in \ell_\infty$ and $(t_l^{(2)}) \in \ell_\infty$ with $1 < t_l^{(1)} < t_l^{(2)}$, with $l \in N$, then*

$$\begin{aligned}
&\mathbb{B}\left(\mathbb{B}^s_{\left(ces_r^{\left(t_n^{(2)}\right)}\right)_v}(\mathscr{P},\mathscr{Q}), \mathbb{B}^s_{\left(ces_r^{\left(t_n^{(1)}\right)}\right)_v}(\mathscr{P},\mathscr{Q})\right)\\
&= \mathscr{A}\left(\mathbb{B}^s_{\left(ces_r^{\left(t_n^{(2)}\right)}\right)_v}(\mathscr{P},\mathscr{Q}), \mathbb{B}^s_{\left(ces_r^{\left(t_n^{(1)}\right)}\right)_v}(\mathscr{P},\mathscr{Q})\right).
\end{aligned} \tag{40}$$

Proof. For $X \in \mathbb{B}(\mathbb{B}^s_{(\mathrm{ces}_r^{(t_n^{(2)})})_v}(\mathscr{P},\mathscr{Q}), \mathbb{B}^s_{(\mathrm{ces}_r^{(t_n^{(1)})})_v}(\mathscr{P},\mathscr{Q}))$ and $X \notin \mathscr{A}(\mathbb{B}^s_{(\mathrm{ces}_r^{(t_n^{(2)})})_v}(\mathscr{P},\mathscr{Q}), \mathbb{B}^s_{(\mathrm{ces}_r^{(t_n^{(1)})})_v}(\mathscr{P},\mathscr{Q}))$. From Lemma 1, one has $Y \in \mathbb{B}(\mathbb{B}^s_{(\mathrm{ces}_r^{(t_n^{(2)})})_v}(\mathscr{P},\mathscr{Q}))$ and $Z \in \mathbb{B}(\mathbb{B}^s_{(\mathrm{ces}_r^{(t_n^{(1)})})_v}(\mathscr{P},\mathscr{Q}))$ with $ZXYI_b = I_b$. Therefore, for each $b \in N$, we have

$$\begin{aligned}
\|I_b\|_{\mathbb{B}^s_{\left(ces_r^{\left(t_l^{(1)}\right)}\right)_v}(\mathscr{P},\mathscr{Q})} &= \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r^z s_z(I_b)}{[l+1]_r}\right)^{t_l^{(1)}}\\
&\le \|ZXY\|\|I_b\|_{\mathbb{B}^s_{\left(ces_r^{\left(t_l^{(2)}\right)}\right)_v}(\mathscr{P},\mathscr{Q})}\\
&\le \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{i} r^z s_z(I_j)}{[l+1]_r}\right)^{t_i^{(2)}}.
\end{aligned} \tag{41}$$

This defies Theorem 33. Then, $X \in \mathscr{A}(\mathbb{B}^s_{(\mathrm{ces}_r^{(t_n^{(2)})})_v}(\mathscr{P},\mathscr{Q}), \mathbb{B}^s_{(\mathrm{ces}_r^{(t_n^{(1)})})_v}(\mathscr{P},\mathscr{Q}))$, which affects the proof.

Corollary 37. *Upon any infinite dimensional Banach spaces $\mathscr{P}$ and $\mathscr{Q}$, if $(t_l^{(1)}) \in \ell_\infty$ and $(t_l^{(2)}) \in \ell_\infty$ with $1 < t_l^{(1)} < t_l^{(2)}$, for every $l \in N$, then*

$$\mathbb{B}\left(\mathbb{B}^s_{\left(ces_r^{\left(t_n^{(2)}\right)}\right)_v}(\mathscr{P},\mathscr{Q}), \mathbb{B}^s_{\left(ces_r^{\left(t_n^{(1)}\right)}\right)_v}(\mathscr{P},\mathscr{Q})\right) = \mathscr{K}\left(\mathbb{B}^s_{\left(ces_r^{\left(t_n^{(2)}\right)}\right)_v}(\mathscr{P},\mathscr{Q}), \mathbb{B}^s_{\left(ces_r^{\left(t_n^{(1)}\right)}\right)_v}(\mathscr{P},\mathscr{Q})\right). \quad (42)$$

Proof. Easily, as $\mathscr{A} \subset \mathscr{K}$.

Theorem 38. *Raise up $(t_l) \in \mathfrak{I} \cap \ell_\infty$ with $t_0 > 1$, and then, $\mathbb{B}^s_{(ces_r^{(t)})_v}$ be simple.*

Proof. Let the closed ideal $\mathscr{K}(\mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v}(\mathscr{P},\mathscr{Q}))$ include a map $X \notin \mathscr{A}(\mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v}(\mathscr{P},\mathscr{Q}))$. From Lemma 1, one has Y, $Z \in \mathbb{B}(\mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v}(\mathscr{P},\mathscr{Q}))$ with $ZXYI_b = I_b$. This gives that $I_{\mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v}(\mathscr{P},\mathscr{Q})} \in \mathscr{K}(\mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v}(\mathscr{P},\mathscr{Q}))$. Accordingly, $\mathbb{B}(\mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v}(\mathscr{P},\mathscr{Q})) = \mathscr{K}(\mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v}(\mathscr{P},\mathscr{Q}))$. Hence, $\mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v}$ is a simple Banach space.

4.5. Eigenvalues of s-Type Mappings

Notation 39.

$$\begin{aligned}(\mathbb{B}^s_{\mathscr{V}})^\rho &:= \left\{(\mathbb{B}^s_{\mathscr{V}})^\rho(\mathscr{P},\mathscr{Q})\,;\, \mathscr{P} \text{ and } \mathscr{Q} \text{ are Banach Spaces}\right\}, \text{where } (\mathbb{B}^s_{\mathscr{V}})^\rho(\mathscr{P},\mathscr{Q}) \\ &:= \left\{X \in \mathbb{B}(\mathscr{P},\mathscr{Q}) \colon (\rho_l(X))_{n=0}^\infty \in \mathscr{V} \text{ and } \|X - \rho_l(X)I\| \text{ is not invertible, for all } l \in N\right\}.\end{aligned} \quad (43)$$

Theorem 40. *Pick up any infinite dimensional Banach spaces $\mathscr{P}$ and $\mathscr{Q}$. Suppose $(t_l) \in \mathfrak{I} \cap \ell_\infty$ with $t_0 > 1$, then*

$$\left(\mathbb{B}^s_{(ces_r^{(t)})_v}\right)^\rho(\mathscr{P},\mathscr{Q}) = \mathbb{B}^s_{(ces_r^{(t)})_v}(\mathscr{P},\mathscr{Q}). \quad (44)$$

Proof. Let $X \in (\mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v})^\rho(\mathscr{P},\mathscr{Q})$; hence, $(\rho_l(X))_{l=0}^\infty \in (\mathrm{ces}_r^{(t)})_v$ and $\|X - \rho_l(X)I\| = 0$, for all $l \in N$. We have $X = \rho_l(X)I$, with $l \in N$, so $s_l(X) = s_l(\rho_l(X)I) = |\rho_l(X)|$, with $l \in N$. Therefore, $(s_l(X))_{l=0}^\infty \in (\mathrm{ces}_r^{(t)})_v$, so $X \in \mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v}(\mathscr{P},\mathscr{Q})$.

Secondly, let $X \in \mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v}(\mathscr{P},\mathscr{Q})$. Therefore, $(s_l(X))_{l=0}^\infty \in (\mathrm{ces}_r^{(t)})_v$. Hence, we have

$$\sum_{l=0}^\infty \left(\frac{\sum_{z=0}^l r^z s_z(X)}{\sum_{z=0}^l r^z}\right)^{t_l} \geq \sum_{l=0}^\infty [s_l(X)]^{t_l}. \quad (45)$$

So $\lim_{l\longrightarrow\infty} s_l(X) = 0$. Assume $\|X - s_l(X)I\|^{-1}$ exists, for every $l \in N$. Therefore, $\|X - s_l(X)I\|^{-1}$ exists and bounded, for every $l \in N$. So $\lim_{l\longrightarrow\infty}\|X - s_l(X)I\|^{-1} = \|X\|^{-1}$ exists and bounded. From the prequasi operator ideal of $(\mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v}, \Psi)$, we obtain

$$I = XX^{-1} \in \mathbb{B}^s_{\left(\mathrm{ces}_r^{(t)}\right)_v}(\mathscr{P},\mathscr{Q}) \Rightarrow (s_l(I))_{l=0}^\infty \in \Xi(r,t) \Rightarrow \lim_{l\longrightarrow\infty} s_l(I) = 0. \quad (46)$$

But $\lim_{l\longrightarrow\infty} s_l(I) = 1$. Therefore, $\|X - s_l(X)I\| = 0$, for every $l \in N$. This gives $X \in (\mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v})^\rho(\mathscr{P},\mathscr{Q})$. This provides the proof.

5. Conclusion

In this article, we explain some topological and geometric structure, of the multiplication maps defined on $(\mathrm{ces}_r^{(t)})_v$, of the class $\mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v}$, and of the class $(\mathbb{B}^s_{(\mathrm{ces}_r^{(t)})_v})^\rho$. This article has a number of advantages for researchers such as studying the fixed points of any contraction maps on this prequasi normed sequence space which is a generalization of the quasi normed sequence spaces, a new general space of solutions for many difference equations, the spectrum of any bounded linear operators between any two Banach spaces with s-numbers in this sequence space and noting that the closed map ideals are certain to play an effective function in the principle of Banach lattices. We open the way for many authors to generalize the results by a sequence (r_l) and generate $(\mathrm{ces}_{(r_l)}^{(t_l)})_v$ of nonabsolute type.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

This work was funded by the University of Jeddah, Saudi Arabia, under grant no. UJ-20-078-DR. The authors, therefore, acknowledge with thanks the university's technical and financial support.

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