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Research Article

Existence of Two Positive Solutions for Two Kinds of Fractional p-Laplacian Equations

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The aim of this paper is to investigate the existence of two positive solutions to subcritical and critical fractional integro-differential equations driven by a nonlocal operator $\mathscr{L}_K^p$. Specifically, we get multiple solutions to the following fractional p-Laplacian equations with the help of fibering maps and Nehari manifold. $\begin{cases} (-\Delta)_p^s u(x) = \lambda u^q + u^r, & u > 0 \text{ in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$ Our results extend the previous results in some respects.

1. Introduction

In this work, we are concerned with the existence of solutions for a nonlocal integro-differential equation

$$\begin{cases} -\mathscr{L}_K^p u(x) = \lambda u^q + u^r, & u > 0 \text{ in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$, $n > ps$ with $s \in (0,1)$, $\lambda > 0$, the exponents r and q fulfill $0 < q < 1 < r \leq p_s^* - 1$ with the critical fractional Sobolev exponent $p_s^* = (np/(n-ps))(n > ps)$, and $\mathscr{L}_K^p$ is a kind of nonlocal integro-differential operator defined by:

$$\mathscr{L}_K^p u(x) = 2 \lim_{\varepsilon \longrightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} |u(x) - u(y)|^{p-2} (u(x) - u(y)) K(x-y) dy, \tag{2}$$

$x \in \mathbb{R}^N$, and $K : \mathbb{R}^N \setminus \{0\} \longrightarrow (0, +\infty)$ is a measurable function with the following property:

$$\begin{cases} \gamma K \in L^1(\mathbb{R}^N) & \text{where } \gamma(x) = \min \{|x|^p, 1\}, \\ \text{there} & \text{exists a } k_0 > 0 \text{ such that,} \\ K(x) \geq k_0 |x|^{-(N+ps)} & \text{for any } x \in \mathbb{R}^N \setminus \{0\}, \\ K(x) = K(-x) & \text{for any} x \in \mathbb{R}^N \setminus \{0\}. \end{cases} \tag{3}$$

In recent years, the existence and multiplicity of solutions of elliptic equations in nonlinear analysis have attracted the attention of many scholars. In particular, problems with regular nolinearities like $u^q + \lambda u^p$, $p, q > 0$ and singular nonlinearities $u^{-q} + \lambda u^p$, $p, q > 0$. At the same time, elliptic problems can be divided into two categories according to their order: integer order and fractional order.

On the one hand, when $s = 1$, in [1], the authors considered a class of semilinear problems with singular nonlinearities. Many results on the existence and multiplicity of solutions for singular problems have appeared in the literature

[2–6]. For example, authors have investigated a singular problem with the kind of critical growth in [6],

$$-\Delta u = \lambda u^{-q} + u^{2^*-1}, \quad u > 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{4}$$

where $0 < q < 1$. They obtained the existence of solutions by means of the Nehari manifold method in a suitable range of λ.

On the other hand, in [7], Mukherjee and Sreenadh considered the following critical fractional Laplace operator equations with a singular nonlinearity

$$(-\Delta)^s u = \lambda a(x) u^{-q} + u^{2_s^*-1}, \quad u > 0 \text{ in } \Omega,\ u = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \tag{5}$$

They showed the existence and multiplicity of positive solutions with respect to the parameter λ for above equation by using variational methods. Furthermore, in [8], they studied a class of critical fractional problems with a lower order perturbation by means of variational and topological methods; precisely, they proved that the number of nontrivial weak solutions is at least twice the multiplicity of the eigenvalue. More details on the critical case of fractional p-Laplace equations can be referred to [9]. In subcritical case, the existence of positive solutions to the following quasi-linear problem

$$\begin{cases} (-\Delta)_p^s u = \lambda g(x,u) - f(x,u), & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{6}$$

is studied by means of truncation and comparison techniques in [10]. Zuo et al. [11] investigated a superlinear fractional elliptic equations; the existence of infinity many solutions is obtained by the fountain theorem in subcritical case. Moreover, they also get at least two solutions for a fractional p-Laplace system by the Nehari manifold method in [12]. We will adopt a new technique, considering both subcritical and critical cases in a more general operator context (see (1)).

In order to state our results, let us introduce some notations. The space

$$X = \left\{ u \mid u : \mathbb{R}^N \longrightarrow \mathbb{R} \text{ is measurable}, \; u|_\Omega \in L^p(\Omega) \text{ and } (u(x) - u(y)) \sqrt[p]{K(x-y)} \in L^p(Q) \right\}, \tag{7}$$

where $Q = \mathbb{R}^{2N} \setminus (C\Omega \times C\Omega)$ with $C\Omega = \mathbb{R}^N \setminus \Omega$. The space X is endowed with the norm,

$$\|u\|_X = \|u\|_{L^p(\Omega)} + \left(\int_Q |u(x) - u(y)|^p K(x-y)\, dx dy \right)^{1/p}, \tag{8}$$

and we define the closed linear subspace

$$X_0 = \left\{ u \in X : u = 0 \text{ almost everywhere in } \mathbb{R}^N \setminus \Omega \right\}, \tag{9}$$

with the norm

$$\|u\|_{X_0} = \left(\int_Q |u(x) - u(y)|^p K(x-y)\, dx dy \right)^{1/p}. \tag{10}$$

Let

$$K : \mathbb{R}^N \setminus \{0\} \longrightarrow (0, +\infty), \tag{11}$$

fulfill condition (3). We have that $C_0^\infty(\Omega) \subset X_0$, and $(X_0, \|\cdot\|_{X_0})$ is a reflexive Banach space (see [13]). Moreover,

$$\begin{aligned} X &\subset W^{s,p}(\Omega), \\ X_0 &\subset W^{s,p}(\mathbb{R}^N), \end{aligned} \tag{12}$$

where $W^{s,p}(\Omega)$ is the usual fractional Sobolev space endowed norm

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}}\, dx dy \right)^{1/p}, \tag{13}$$

and the embedding

$$X_0 \hookrightarrow L^{p_s^*}(\Omega), \tag{14}$$

$C_0(N, p, s)$ such that, for any $v \in X_0$, $1 < k < p_s^*$

$$\|v\|_{L^k(\Omega)} \le C_0 \|v\|_{X_0}. \tag{15}$$

Definition 1. We say that u is a weak solution of problem (1), if u fulfills

$$\begin{aligned} &\int_Q |u(x) - u(y)|^{p-2} (u(x) - u(y))(\varphi(x) - \varphi(y)) K(x-y)\, dx dy \\ &\quad = \int_\Omega \left(\lambda u_+^q(x) + u_+^p(x) \right) \varphi(x)\, \mathrm{d}x, \end{aligned} \tag{16}$$

for all $\varphi \in X_0$, where $u_+ = \max\{u, 0\}$.

The main results of this article are as follows.

Theorem 2. *Set $s \in (0,1)$, $n > ps$, K fulfilling condition (3), if $0 < q < 1$, $1 < r < p_s^*$, then there exists $\lambda_\star > 0$, such that for $\lambda \in (0, \lambda_\star)$, equation (1) has at least two positive solutions.*

Theorem 3. *Set $s \in (0,1)$, $n > ps$, and Ω be an open bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. K fulfilling the condition (3), if $0 < q < 1$, $r = p_s^* - 1$, assumes that there exists $u_0 \in X_0 \setminus \{0\}$ with $u_0 \ge 0$ almost everywhere in $\mathbb{R}^n$, such that*

$$\sup_{t \ge 0} \mathcal{J}_{K, p_s^*}(t u_0) < \frac{s}{n} S_K^{n/ps}, \tag{17}$$

where $\mathcal{I}_{K,p_s^*}$ will be introduced in Section 2. Then, there exists $\lambda_2 > 0$, such that for $\lambda \in (0, \lambda_2)$, problem (1) admits least two solutions.

2. Preliminaries

We define the energy functional

$$J_\lambda : X_0 \longrightarrow \mathbb{R}, \tag{18}$$

associated to problem (1) as

$$J_\lambda(u) = \mathcal{I}_{K,p}(u) - \mathcal{I}_\lambda(u), \tag{19}$$

with

$$\begin{aligned} \mathcal{I}_{K,p}(u) &= \frac{1}{p}\int_Q |u(x)-u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y \\ &\quad - \frac{1}{r+1}\int_\Omega u_+^{r+1}(x)\mathrm{d}x, \\ \mathcal{I}_\lambda(u) &= \frac{\lambda}{q+1}\int_\Omega u_+^{q+1}(x)\mathrm{d}x. \end{aligned} \tag{20}$$

We can see that $J_\lambda \in C^1(X_0, \mathbb{R})$ and

$$\begin{aligned} \left\langle J_\lambda'(u), \varphi \right\rangle_{X_0} &= \int_Q |u(x)-u(y)|^{p-2}(u(x)-u(y)) \\ &\quad \times (\varphi(x)-\varphi(y))K(x-y)dxdy \\ &\quad - \lambda\int_\Omega u_+^q(x)\varphi(x)\mathrm{d}x - \int_\Omega u_+^r(x)\varphi(x)\mathrm{d}x, \end{aligned} \tag{21}$$

for any $\varphi \in X_0$.

Now, we give the Nehari manifold

$$\mathcal{N}_\lambda = \left\{u \in X_0 \setminus \{0\}: \left\langle J_\lambda'(u), u \right\rangle = 0\right\}, \tag{22}$$

where $\langle , \rangle$ denotes the duality between X_0 and its dual space. Thus, $u \in \mathcal{N}_\lambda$ if and only if

$$\|u\|_{X_0}^p - \lambda\int_\Omega u_+^{q+1}(x)\mathrm{d}x - \int_\Omega u_+^{r+1}(x)\mathrm{d}x = 0. \tag{23}$$

The Nehari manifold $\mathcal{N}_\lambda$ is closely related to the following function $\varphi_u : t \mapsto J_\lambda(tu)$ for $t > 0$ defined by

$$\begin{aligned} \varphi_u(t) := J_\lambda(tu) &= \frac{t^p}{p}\|u\|_{X_0}^p - \lambda\frac{t^{q+1}}{q+1}\int_\Omega u_+^{q+1}(x)\mathrm{d}x \\ &\quad - \frac{t^{r+1}}{r+1}\int_\Omega u_+^{r+1}(x)\mathrm{d}x. \end{aligned} \tag{24}$$

Remark 4. Set $u \in X_0 \setminus \{0\}$, then $tu \in \mathcal{N}_\lambda$ if and only if $\varphi_u'(t) = 0$.

Moreover,

$$\varphi_u'(t) = t^{p-1}\|u\|_{X_0}^p - \lambda t^q\int_\Omega u_+^{q+1}(x)\mathrm{d}x - t^r\int_\Omega u_+^{r+1}(x)\mathrm{d}x, \tag{25}$$

$$\begin{aligned} \varphi_u'(t) &= (p-1)t^{p-2}\|u\|_{X_0}^p - q\lambda t^{q-1}\int_\Omega u_+^{q+1}(x)\mathrm{d}x \\ &\quad - rt^{r-1}\int_\Omega u_+^{r+1}(x)\mathrm{d}x. \end{aligned} \tag{26}$$

According to (25) and Remark 4, for $u \in \mathcal{N}_\lambda$, we have

$$\begin{aligned} \varphi_u'(1) &= (p-1)\|u\|_{X_0}^p - \lambda q\int u_+^{p+1}(x)\mathrm{d}x - r\int_\Omega u_+^{r+1}(x)\mathrm{d}x \\ &= (p-r-1)\int_\Omega u_+^{r+1}(x)\mathrm{d}x + \lambda(p-q-1)\int_\Omega u_+^{q+1}(x)\mathrm{d}x \\ &= (p-q-1)\|u\|_{X_0}^p - (q+r)\int_\Omega u_+^{r+1}(x)\mathrm{d}x \\ &= (p-r-1)\|u\|_{X_0}^p - \lambda(q-r)\int_\Omega u_+^{q+1}(x)\mathrm{d}x. \end{aligned} \tag{27}$$

The $\mathcal{N}_\lambda$ is divided into three sets, which are local minimum, local maximum, and local inflection point, respectively, i.e.,

$$\begin{aligned} \mathcal{N}_\lambda^+ &= \left\{u \in \mathcal{N}_\lambda : \varphi_u'(1) > 0\right\}, \\ \mathcal{N}_\lambda^+ &= \left\{u \in \mathcal{N}_\lambda : \varphi_u'(1) > 0\right\}, \\ \mathcal{N}_\lambda^0 &= \left\{u \in \mathcal{N}_\lambda : \varphi_u'(1) = 0\right\}. \end{aligned} \tag{28}$$

To prove our result, we should start to show the following auxiliary lemmas.

Lemma 5. *If u_0 is a local minimizer of J_λ on $\mathcal{N}_\lambda$ and $u_0 \notin \mathcal{N}_\lambda^0$, then u_0 is a critical point of J_λ.*

Similar to Theorem 2.3 in [14], we can get this conclusion.

About fibering maps and the Nehari manifold, considering the function $\psi_u : \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined by

$$\psi_u(t) = t^{1-q}\|u\|_{X_0}^p - t^{r-q}\int_\Omega u_+^{r+1}dx. \tag{29}$$

Obviously, for any $t > 0$, $tu \in \mathcal{N}_\lambda$ if and only if

$$\psi_u(t) = \lambda\int_\Omega u_+^{q+1}dx. \tag{30}$$

Moreover,

$$\psi_u'(t) = (1-q)t^{-q}\|u\|_{X_0}^p - (r-q)t^{r-q-1}\int_\Omega u_+^{r+1}dx, \tag{31}$$

and moreover, we know that $tu \in \mathcal{N}_\lambda$*, then*

$$t^q \psi_u'(t) = \varphi_u'(t). \tag{32}$$

So, $tu \in \mathcal{N}_\lambda^+ (\text{or} \mathcal{N}_\lambda^-)$ *if and only if* $\psi_u'(t) > 0$ *(or <0). Assume* $u \in X_0$ *and* $u_+ \neq 0$*. In view of (29),* ψ_u *fulfills the following properties:*

(i) ψ_u *has a unique critical point at* $t = t_{\max}(u) = (((1-q)\|u\|_{X_0}^p)/((r-q)\int_\Omega u_+^{r+1} dx))^{1/(r-1)} > 0$

(ii) $\psi_u \uparrow$ *on* $(0, t_{\max}(u))$ *and* $\downarrow$ *on* $(t_{\max}(u), +\infty)$

$$\lim_{t\longrightarrow +\infty} \psi_u(t) = -\infty. \tag{33}$$

Further, it follows from $\int_\Omega u_+^{q+1} dx > 0$ *that (30) has no solutions if* λ *fulfills*

$$\lambda \int_\Omega u_+^{q+1} dx > \psi_u(t_{\max}(u)) = \left[\left(\frac{1-q}{r-q}\right)^{(1-q)/(r-1)} - \left(\frac{1-q}{r-q}\right)^{(r-q)/(r-1)}\right] \frac{\|u\|_{X_0}^{(p(r-q))/(r-1)}}{\left(\int_\Omega u_+^{r+1} dx\right)^{(1-q)/(r-1)}}. \tag{34}$$

According to (25) and (30) if λ *fulfills (34), then* $\varphi_u'(t) > 0$*. It seems* $\varphi_u'(t) < 0$ *as* λ *is sufficiently large. Therefore,* $tu \notin \mathcal{N}_\lambda$ *for any* $t > 0$*. Moreover, if* λ *fulfills*

$$0 < \lambda \int_\Omega u_+^{q+1} dx < \psi_u(t_{\max}(u)), \tag{35}$$

then there exist t_1 *and* t_2 *with* $t_1 < t_{\max}(u) < t_2$*, such that*

$$\psi_u(t_1) = \psi_u(t_2) = \lambda \int_\Omega u_+^{q+1} dx, \text{ and } \psi_u'(t_1) > 0, \psi_u'(t_2) < 0, \tag{36}$$

combining (25) and (30), which imply that $\varphi_u'(t_1) = \varphi_u'(t_2) = 0$*. It follows from (32) that* $\varphi_u'(t_1) > 0, \varphi_u'(t_2) < 0$*, which mean that the fibering map* φ_u *admits a local minimum* $t_1 u \in \mathcal{N}_\lambda^+$ *and a local maximum at* $t_2 u \in \mathcal{N}_\lambda^-$*.*

3. The Subcritical Case: $0 < q < 1 < r < p_s^* - 1$

Firstly, we prove the following lemmas.

Lemma 6. *There exists* $\lambda_* > 0$*, such that for any* $\lambda \in (0, \lambda_*)$*, we have* $\mathcal{N}_\lambda^0 = \emptyset$*.*

Proof. Using the inverse method, if $\mathcal{N}_\lambda^0 \neq \emptyset$ for any $\lambda > 0$. Then,

$$\begin{aligned} \left\langle J_\lambda'(u), u \right\rangle_{X_0} &= 0, \\ \varphi_u'(1) &= 0. \end{aligned} \tag{37}$$

for $u \in \mathcal{N}_\lambda^0$.

Namely,

$$\begin{aligned} \|u\|_{X_0}^p &= \lambda \int_\Omega u_+^{q+1} ds + \int_\Omega u_+^{r+1} dx, \text{ and} \|u\|_{X_0}^2 \\ &= \lambda q \int_\Omega u_+^{q+1} ds + r \int_\Omega u_+^{r+1} dx. \end{aligned} \tag{38}$$

Thus,

$$\begin{aligned} (1-q)\|u\|_{X_0}^p &= (r-q) \int_\Omega u_+^{r+1} dx, \text{ and } (r-1)\|u\|_{X_0}^p \\ &= \lambda (r-q) \int_\Omega u_+^{q+1} dx. \end{aligned} \tag{39}$$

Using the Hölder inequality and Remark 4, there exist two positive constants C_1, C_2 such that

$$\|u\|_{X_0}^p \le C_1 \|u\|_{X_0}^{r+1} \text{and} \|u\|_{X_0}^p \le \lambda C_2 \|u\|_{X_0}^{q+1}. \tag{40}$$

It yields that $C_1^{1/(p-r-1)} \le \|u\|_{X_0} \le (\lambda C_2)^{1/(p-q-1)}$. If λ is small enough, then it is impossible. Thus, assuming no, the original set is empty.

Lemma 7. J_λ *is coercive and bounded from below on* $\mathcal{N}_\lambda$ *for* $\lambda \in (0, \lambda_*)$*.*

Proof. Let $u \in \mathcal{N}_\lambda$, (19) and (23) we get

$$J_\lambda(u) = \left(\frac{1}{p} - \frac{1}{r+1}\right) \|u\|_{X_0}^p - \lambda \left(\frac{1}{q+1} - \frac{1}{r+1}\right) \int_\Omega u_+^{q+1} dx. \tag{41}$$

Using Remark 4 and Hölder inequality, we get

$$\int_\Omega u_+^{q+1} dx \le C_{n,q,s,\theta,|\Omega|} \|u\|_{X_0}^{q+1}. \tag{42}$$

Prove complete due to $0 < q < 1 < r$.

By Lemmas 6 and 7, for any $\lambda \in (0, \lambda_*)$, we get $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$, and so, J_λ is coercive and bounded from below on $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$. Therefore, we define

$$\alpha_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u), \ \alpha_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u). \tag{43}$$

We have the following result.

Proposition 8. *If $0 < \lambda < \lambda_*$, then the functional J_λ has a minimizer u_1 in $\mathcal{N}_\lambda^+$ and satisfies*

(1) $J_\lambda(u_1) = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u) < 0$.

(2) u_1 is a solution of problem (1).

Proof. Since the bounded from below of J_λ on $\mathcal{N}_\lambda^+$, there exists a minimizing sequence $\{u_k\} \subset \mathcal{N}_\lambda^+$, such that

$$\lim_{k\longrightarrow\infty} J_\lambda(u_k) = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u). \tag{44}$$

We know that the sequence $\{u_k\}$ is bounded in X_0 by Lemma 7. $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space (see Lemma 7 in [15]); thus, there exists $u_1 \in X_0$ such that, up to a subsequence,

$$\begin{aligned} &\int_Q |u_k(x) - u_k(y)|^{p-2}(u_k(x) - u_k(y))(\phi(x) - \phi(y))K(x-y)\mathrm{d}x\mathrm{d}y \\ &\longrightarrow \int_Q |u_1(x) - u_2(y)|^{p-2}(u_1(x) - u_1(y)) \\ &\quad\cdot (\phi(x) - \phi(y))K(x-y)\mathrm{d}x\mathrm{d}y \,\text{for}\forall\phi \in X_0, \end{aligned} \tag{45}$$

when $k \longrightarrow \infty$. Further, by Lemma 8 in [15], we have

$$u_k \longrightarrow u_1 \text{ in } L^r(\mathbb{R}^n), \quad u_k \longrightarrow u_1 \text{ a.e } \mathbb{R}^n, \tag{46}$$

as $k \longrightarrow \infty$, and by ([16], TheoremIV – 9), there exists $\ell \in L^r(\mathbb{R}^n)$ such that

$$|u_k(x)| \le \ell(x) \text{ a.e in } \mathbb{R}^n, \tag{47}$$

for any $1 \le r < p_s^* = np/(n-ps)(n > ps)$. It follows from the dominated convergence theorem that

$$\begin{aligned} &\int_\Omega (u_k)_+^{q+1}\mathrm{d}x \longrightarrow \int_\Omega (u_1)_+^{q+1}\mathrm{d}x, \text{ and } \int_\Omega (u_k)_+^{r+1}\mathrm{d}x \\ &\qquad \longrightarrow \int_\Omega (u_1)_+^{r+1}\mathrm{d}x, \end{aligned} \tag{48}$$

as$k \longrightarrow \infty$. So, there exists t_1 such that $t_1u_1 \in \mathcal{N}_\lambda^+$ and $J_\lambda(t_1u_1) < 0$. Therefore, we get $\inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u) < 0$.

In order to prove that $u_k \longrightarrow u_1$ strongly in X_0. Still use the arc method if not, then $\|u_1\|_{X_0} < \liminf_{k\longrightarrow\infty}\|u_k\|_{X_0}$. Hence, for $\{u_k\} \in \mathcal{N}_\lambda^+$, we get

$$\begin{aligned} \lim_{k\longrightarrow\infty} \varphi'_{u_k}(t_1) &= \lim_{k\longrightarrow\infty}\left(t_1\|u_k\|_{X_0}^p - \lambda t_1^q \int_\Omega (u_k)_+^{q+1}\mathrm{d}x \right. \\ &\quad \left. - t_1^r \int_\Omega (u_k)_+^{r+1}\mathrm{d}x\right) > t_1\|u_1\|_{X_0}^p \\ &\quad - \lambda t_1^q \int_\Omega (u_1)_+^{q+1}\mathrm{d}x - t_1^r \int_\Omega (u_1 x)_+^{r+1}\mathrm{d}x \\ &= \varphi'_{u_1}(t_1) = 0. \end{aligned} \tag{49}$$

That is, $\varphi'_{u_k}(t_1) > 0$ for k large enough. Since $u_k = 1, u_k \in \mathcal{N}_\lambda^+$, we infer that $\varphi'_{u_k}(t) < 0$ for $t \in (0,1)$ and $\varphi'_{u_k}(1) = 0$ for all k. So, must be $t_1 > 1$. In addition because $\varphi_{u_1}(t)$ is decreasing on $(0, t_1)$, and so,

$$J_\lambda(t_1u_1) \le J_\lambda(u_1) < \lim_{k\longrightarrow\infty} J_\lambda(u_k) = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u). \tag{50}$$

Obviously, the above equation is a contradiction. Therefore, $u_k \longrightarrow u_1$ strongly in X_0. It means that

$$J_\lambda(u_k) \longrightarrow J_\lambda(u_1) = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u) \text{ as } k \longrightarrow \infty, \tag{51}$$

i.e., u_1 is a minimizer if J_λ on $\mathcal{N}_\lambda^+$. By Lemma 5, u_1 is a solution to problem (1).

Proposition 9. *If $0 < \lambda < \lambda_1$, then J_λ admits a minimizer u_2 in $\mathcal{N}_\lambda^-$ and satisfies*

(1) $J_\lambda(u_2) = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u) > 0$.

(2) u_2 is a solution to problem (1).

Proof. Since the bounded from below of J_λ on $\mathcal{N}_\lambda^-$, there exists a minimizing sequence $\{\tilde{u}_k\} \subset \mathcal{N}_\lambda^-$, such that

$$\lim_{k\longrightarrow\infty} J_\lambda(\tilde{u}_k) = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u). \tag{52}$$

Similar to Proposition 8, there exists $u_2 \in X_0$ such that

$$\begin{aligned} &\int_Q |\tilde{u}_k(x) - \tilde{u}_k(y)|^{p-2}(\tilde{u}_k(x) - \tilde{u}_k(y))(\phi(x) - \phi(y))K(x-y)\mathrm{d}x\mathrm{d}y \\ &\longrightarrow \int_Q |u_2(x) - u_2(y)|^{p-2}(u_2(x) - u_2(y)) \\ &\quad\cdot (\phi(x) - \phi(y))K(x-y)\mathrm{d}x\mathrm{d}y \,\text{for}\forall\phi \in X_0, \end{aligned} \tag{53}$$

as $k \longrightarrow \infty$, and

$$\begin{aligned} &\int_\Omega (\tilde{u}_k)_+^{q+1}\mathrm{d}x \longrightarrow \int_\Omega (u_2)_+^{q+1}\mathrm{d}x, \text{ and} \\ &\int_\Omega (\tilde{u}_k)_+^{r+1}\mathrm{d}x \longrightarrow \int_\Omega (u_2)_+^{r+1}\mathrm{d}x, \end{aligned} \tag{54}$$

as $k \longrightarrow \infty$. Moreover, from the nature of the fibering maps $\varphi_u(t)$, we infer that there exist t_1, t_2 with $t_1 < t_{\max}(u) < t_2$ such that $t_1 u \in \mathcal{N}_\lambda^+$, $t_2 u \in \mathcal{N}_\lambda^-$, and $J_\lambda(t_1 u) \le J_\lambda(tu) \le J_\lambda(t_2 u)$.

Next, we show that $\tilde{u}_k \longrightarrow u_2$ strongly in X_0. If not, then $\|u_2\|_{X_0} < \liminf_{k\longrightarrow\infty}\|\tilde{u}_k\|_{X_0}$. Thus, for $\{\tilde{u}_k\} \in \mathcal{N}_\lambda^-$, we have $J_\lambda(\tilde{u}_k) \ge J_\lambda(t\tilde{u}_k)$ for all $t \ge t_{\max}(u)$, and

$$\begin{aligned} J_\lambda(t_2 u_2) &= \frac{t_2^p}{p}\|u_2\|_{X_0}^p - \lambda\frac{t_2^{q+1}}{q+1}\int_\Omega (u_2)_+^{q+1}\mathrm{d}x - \frac{t_2^{r+1}}{r+1}\int_\Omega (u_2)_+^{r+1}\mathrm{d}x \\ &< \lim_{k\longrightarrow\infty}\left(\frac{t_2^p}{p}\|\tilde{u}_k\|_{X_0}^p - \lambda\frac{t_2^{q+1}}{q+1}\int_\Omega (\tilde{u}_k)_+^{q+1}\mathrm{d}x \right.\\ &\quad \left. - \frac{t_2^{r+1}}{r+1}\int_\Omega (\tilde{u}_k)_+^{r+1}\mathrm{d}x\right) = \lim_{k\longrightarrow\infty} J_\lambda(t_2\tilde{u}_k) \\ &\le J_\lambda(\tilde{u}_k) = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u), \end{aligned} \tag{55}$$

in a similar way, we still can get a contradiction. Thus, $\tilde{u}_k \longrightarrow u_2$ strongly in X_0. It means that

$$J_\lambda(\tilde{u}_k) \longrightarrow J_\lambda(u_2) = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u) \text{ as } k \longrightarrow \infty. \tag{56}$$

Namely, u_2 is a minimizer if J_λ on $\mathcal{N}_\lambda^-$.u_2 is a solution to problem (1) according to Lemma 5.

Proof of Theorem 10. We obtain that problem (1) has two solutions $u_1 \in \mathcal{N}_\lambda^+$ and $u_2 \in \mathcal{N}_\lambda^-$ in X_0 due to the Propositions 8 and 9 and Lemma 5; moreover, we know that two solutions are distinct since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \varnothing$.

4. The Critical Case: $0 < q < 1, r = p_s^* - 1$

For the critical case, since the embedding $X_0 \hookrightarrow L^{p_s^*}(\Omega)$ is not compact, then the energy functional does not satisfy the Palais-Smale condition globally, but it is true for the energy functional in a suitable range related to the best fractional critical Sobolev constant in the embedding $X_0 \hookrightarrow L^{p_s^*}(\mathbb{R}^n)$. For this, we define fractional Sobolev best constant S_K as

$$S_K = \inf_{v\in X_0\setminus\{0\}} \frac{\int_{\mathbb{R}^{2n}}|v(x)-v(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y}{\left(\int_\Omega |v(x)|^{p_s^*}\right)^{p/p_s^*}} \text{ for } v \in X_0 \setminus \{0\}. \tag{57}$$

Before we give the Proof of Theorem 13, we start by some auxiliary results. Firstly, using the same proofs of Lemma 6, we deduce that there exists $\lambda_* > 0$ such that $\mathcal{N}_\lambda^0 = \varnothing$ for each $\lambda \in (0, \lambda_*)$. Also, it is clear that J_λ is coercive and bounded from below on $\mathcal{N}_\lambda$ for $\lambda \in (0, \lambda_*)$ by Lemma 7. So, for any $\lambda \in (0, \lambda_*)$, we also obtain that $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$, and J_λ is coercive and bounded from below on $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$. We define

$$\tilde{\alpha}_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u), \tilde{\alpha}_\lambda^- = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u). \tag{58}$$

Proposition 11. *Let $\{u_k\} \subset X_0$ be a $(PS)_c$ sequence for J_λ with*

$$c < \frac{s}{n} S_K^{n/ps} - M\lambda^{p_s^*/p_s^\star - q}, \tag{59}$$

then there exists a subsequence of $\{u_k\}$, which converges strongly in X_0, where S_K is defined in (57) and $M > 0$ is defined by

$$\begin{aligned} M = &\frac{(2n-(n-2s)(q+1))(1-q)}{4(q+1)} \\ &\cdot \left(\frac{(1-q)(n-2s)}{4s}\right)^{(q+1)/(p_s^\star-(q+1))}|\Omega|. \end{aligned} \tag{60}$$

Proof. It follows from $\{u_k\}$ is bounded in X_0 that there exists $u_\infty \in X_0$ such that $u_k \longrightarrow u_\infty$ weakly in X_0, that is

$$\begin{aligned} &\int_Q |u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))(\phi(x)-\phi(y))K(x-y)\mathrm{d}x\mathrm{d}y \\ &\longrightarrow \int_Q |u_\infty(x)-u_\infty(y)|^{p-2}(u_\infty(x)-u_\infty(y)) \\ &\quad\cdot(\phi(x)-\phi(y))K(x-y)\mathrm{d}x\mathrm{d}y \text{ for} \forall\phi \in X_0, \end{aligned} \tag{61}$$

as $k \longrightarrow \infty$. Moreover, using the same arguments as lemma 9 ([17]), we get that

$$\begin{aligned} &u_k \longrightarrow u_\infty \text{weakly in } L^{p_s^*}(\mathbb{R}^n); \\ &u_k \longrightarrow u_\infty \text{ in } L^r(\mathbb{R}^n); \\ &u_k \longrightarrow u_\infty \text{ a.e.in } \mathbb{R}^n, \end{aligned} \tag{62}$$

as $k \longrightarrow \infty$, and by ([16], TheoremIV – 9], there exists $\ell \in L^r(\mathbb{R}^n)$ such that

$$|u_k(x)| \le \ell(x) \text{ a.e.in } \mathbb{R}^n, \tag{63}$$

for any $1 \le r < p_s^* = np/(n-ps)(n > ps)$. Then, using dominated convergence theorem, we have that

$$\int_\Omega (u_k)_+^{q+1}\mathrm{d}x \longrightarrow \int_\Omega (u_\infty)_+^{q+1}\mathrm{d}x. \tag{64}$$

Also, by the same method as in ([18], Lemma 1.32), we get

$$\begin{aligned}&\int_Q |u_k(x)-u_k(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y\\&\longrightarrow \int_Q |u_k(x)-u_\infty(x)-u_k(y)+u_\infty(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y\\&\quad+\int_Q |u_\infty(x)-u_\infty(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y+o(1),\\&\int_\Omega (u_k(x))_+^{p_s^*}\mathrm{d}x=\int_\Omega ((u_k-u_\infty)(x))_+^{p_s^*}\mathrm{d}x\\&\quad+\int_\Omega (u_\infty(x))_+^{p_s^*}\mathrm{d}x+o(1),\end{aligned}\tag{65}$$

as $k \longrightarrow \infty$. Then,

$$\begin{aligned}\left\langle J_\lambda'(u_k),u_k\right\rangle_{X_0}&=\int_Q |u_k(x)-u_k(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y\\&\quad-\lambda\int (u_k(x))_+^{q+1}\mathrm{d}x-\int_\Omega (u_k(x))_+^{p_s^*}\mathrm{d}x\\&=\int_\Omega |u_k(x)-u_\infty(x)-u_k(y)\\&\quad+u_\infty(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y\\&\quad+\int_Q |u_\infty(x)-u_\infty(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y\\&\quad-\lambda\int_\Omega (u_k(x))_+^{q+1}\mathrm{d}x-\left(\int_\Omega ((u_k-u_\infty)\right.\\&\quad\left.\cdot (x))_+^{p_s^*}\mathrm{d}x+\int_\Omega (u_\infty)_+^{r}\mathrm{d}x+o(1)\right)+o(1)\\&=\int_Q |(u_k-u_\infty)(x)-(u_k-u_\infty)(y)|^p\\&\quad\cdot K(x-y)\mathrm{d}x\mathrm{d}y-\int_\Omega ((u_k-u_\infty)(x))_+^{p_s^*}\mathrm{d}x\\&\quad+\left\langle J_\lambda'(u_\infty),u_\infty\right\rangle X_0+o(1).\end{aligned}\tag{66}$$

By $\langle J_\lambda'(u_\infty),u_\infty\rangle_{X_0}=0$ and $\langle J_\lambda'(u_k),u_k\rangle_{X_0}\longrightarrow 0$ as $k \longrightarrow \infty$, we know that

$$\begin{aligned}\|u_k-u_\infty\|_{X_0}^p&=\int_Q |(u_k-u_\infty)(x)-(u_k-u_\infty)(y)|^p\\&\quad\cdot K(x-y)\mathrm{d}x\mathrm{d}y\longrightarrow b,\end{aligned}\tag{67}$$

$$\int_\Omega ((u_k-u_\infty)(x))_+^{p_s^*}\mathrm{d}x\longrightarrow b,\quad \text{as } k\longrightarrow\infty.$$

If $b=0$, is clearly true. If $b>0$, in view of the definition of S_K in 17, we get

$$\|u_k-u_\infty\|_{X_0}^p\geq S_K\left(\int_\Omega ((u_k-u_\infty)(x))_+^{p_s^*}\mathrm{d}x\right)^{p/p_s^2}.\tag{68}$$

Thus, we have $b\geq S_K b^{p/p_s^*}$. That is, $b\geq S_K^{n/ps}$. On the other hand, we have

$$\begin{aligned}c=\lim_{k\longrightarrow\infty}J_\lambda(u_k)&=\lim_{k\longrightarrow\infty}\left(\frac{1}{p}\|u_k\|_{X_0}^p-\lambda\frac{1}{q+1}\int_\Omega (u_k(x))_+^{q+1}\mathrm{d}x\right.\\&\quad\left.-\frac{1}{r+1}\int_\Omega (u_k(x))_+^{p_\star}\mathrm{d}x\right)\geq J_\lambda(u_\infty)+\frac{s}{n}S_K^{n/ps}.\end{aligned}\tag{69}$$

By the assumption that $c<(s/n)S_K^{n/ps}$, we have $J_\lambda(u_\infty)<0$. In particular, $u_\infty\neq 0$ and

$$0<\frac{1}{p}\|u_\infty\|_{X_0}^p<\frac{1}{p_s^*}\int_\Omega (u_\infty(x))_+^{p_s^*}\mathrm{d}x+\lambda\frac{1}{q+1}\int_\Omega (u_\infty(x))_+^{q+1}\mathrm{d}x.\tag{70}$$

Then,

$$\begin{aligned}c&=\lim_{k\longrightarrow\infty}J_\lambda(u_k)=\lim_{k\longrightarrow\infty}\left(J_\lambda(u_k)-\frac{1}{p}\left\langle J_\lambda'(u_k),u_k\right\rangle_{X_0}\right)\\&=\lim_{k\longrightarrow\infty}\left(\frac{s}{n}\int_\Omega ((u_k-u_\infty)(x))_+^{p_s^*}\mathrm{d}x+\frac{s}{n}\int_\Omega (u_\infty(x))_+^{p_s^*}\mathrm{d}x\right.\\&\quad\left.+\lambda\left(\frac{1}{p}-\frac{1}{q+1}\right)\int_\Omega (u_k(x))_+^{q+1}\mathrm{d}x\right)\\&=\frac{s}{n}b+\frac{s}{n}\int_\Omega (u_\infty(x))_+^{p_s^*}\mathrm{d}x+\lambda\left(\frac{1}{p}-\frac{1}{q+1}\right)\int_\Omega (u_\infty(x))_+^{q+1}\mathrm{d}x\\&\geq\frac{s}{n}S_K^{n/ps}+\frac{s}{n}\int_\Omega (u_\infty(x))_+^{p^*}\mathrm{d}x+\lambda\left(\frac{1}{p}-\frac{1}{q+1}\right)\int_\Omega (u_\infty(x))_+^{q+1}\mathrm{d}x.\end{aligned}\tag{71}$$

Moreover, by Hölder inequality, we have

$$\int_\Omega (u_\infty(x))_+^{q+1}\mathrm{d}x\leq|\Omega|^{(p_s^*-(q+1))/p_s^*}\left(\int_\Omega (u_\infty(x))_+^{p_s^*}\mathrm{d}x\right)^{(q+1)/p_s^*}.\tag{72}$$

Thus,

$$\begin{aligned}c&\geq\frac{s}{n}S_K^{n/ps}+\frac{s}{n}\left(\int_\Omega (u_\infty(x))_+^{p_s^*}\mathrm{d}x\right)\\&\quad+\lambda\left(\frac{1}{p}-\frac{1}{q+1}\right)|\Omega|^{(p_s^*-(q+1))/p_s^*}\\&\quad\cdot\left(\int_\Omega (u_\infty(x))_+^{p_s^*}\mathrm{d}x\right)^{(q+1)/p_s}\coloneqq\frac{s}{n}S_K^{n/ps}+h(\eta),\end{aligned}\tag{73}$$

where

$$\begin{aligned}h(\eta)&=\frac{s}{n}\eta^{p_s^*}+\lambda\left(\frac{1}{p}-\frac{1}{q+1}\right)|\Omega|^{(p_s^*-(q+1))/p_s^*}\eta^{q+1}\text{ with }\eta\\&=\left(\int_\Omega (u_\infty(x))_+^{p_s^*}\mathrm{d}x\right)^{1/p_s^*}.\end{aligned}\tag{74}$$

So, $h(\eta)$ attains its minimum at $\eta_0 = (\lambda(p-q-1)(n-ps)/2ps)^{1/(p_s^\star-(q+1))}|\Omega|^{1/p_s^\star}$ and

$$h(\eta_0) = -\frac{(2n-(n-ps)(q+1))(p-q-1)}{2p(q+1)} \cdot \left(\frac{(p-1-q)(n-2s)}{2ps}\right)^{(q+1)/(p_s^\star-(q+1))} |\Omega|\lambda^{p_s^*/(p_s^*-(q+1))} = -M\lambda^{p_s^*/p_s^*-(q+1)}. \tag{75}$$

Therefore,

$$c \geq \frac{s}{n}S_K^{n/ps} - M\lambda^{p_s^*/(p_s^*-(q+1))}, \tag{76}$$

which is a contradiction. Therefore, $b=0$ and we obtain that $u_k \longrightarrow u_\infty$ strongly in X_0.

Proposition 12. *There exists $\lambda_2 > 0$ and $u_0 \in X_0$ such that*

$$\sup_{t>0} J_\lambda(tu_0) < \frac{s}{n}S_K^{n/ps} - M\lambda^{p_s^*/(p_s^*-(q+1))}, \tag{77}$$

for $\lambda \in (0, \lambda_2)$. In particular

$$\tilde{\alpha}_\lambda^- < \frac{s}{n}S_K^{n/ps} - M\lambda^{p_s^*/(p_s^*-(q+1))}. \tag{78}$$

Proof. We suppose there exists $\lambda_{**} > 0$ such that $(s/n)S_K^{n/ps} - M\lambda^{p_s^*/(p_s^\star-(q+1))} > 0$ for all $\lambda \in (0,\lambda_{**})$. By condition (17) we have that there is $u_0 \in X_0 \setminus \{0\}$ such that

$$\begin{aligned} J_\lambda(tu_0) &\leq \sup_{t\geq 0}\mathcal{I}_{K,p_s^*}(tu_0) - \lambda\frac{t^{q+1}}{q+1}\int_\Omega (u_0)_+^{q+1}\mathrm{d}x \\ &< \frac{s}{n}S_K^{n/ps} - \lambda\frac{t_0^{q+1}}{q+1}\int_\Omega (u_0)_+^{q+1}\mathrm{d}x. \end{aligned} \tag{79}$$

Let $\lambda_{***} \coloneqq (t_0^{q+1}\int_\Omega (u_0)_+^{q+1}\mathrm{d}x/(M(q+1)))^{(p_s^*-(q+1))/(q+1)}$. Therefore, for $\lambda \in (0,\lambda_{***})$, we obtain that

$$-\frac{t_0^{q+1}}{q+1}\lambda\int_\Omega (u_0)_+^{q+1}\mathrm{d}x < -M\lambda^{p_s^*/(p_s^*-(q+1))}. \tag{80}$$

Then, we have (77) holds.

Finally, let $\lambda_2 = \min\{\lambda_*,\lambda_{**},\lambda_{***}\}$, we obtain that

$$\tilde{\alpha}_\lambda^- < \frac{s}{n}S_K^{n/ps} - M\lambda^{p_s^*/(p_s^*-(q+1))}, \tag{81}$$

for $\lambda \in (0, \lambda_2)$ by the nature of fibering maps $\varphi_u(t) = J_\lambda(tu)$.

Proof of Theorem 13. There exist two sequences $\{u_k^+\}$ and $\{u_k^-\}$ in X_0 such that

$$J_\lambda(u_k^+) \longrightarrow \tilde{\alpha}_\lambda^+, J'_\lambda(u_k^+) \longrightarrow 0 \text{ and } J_\lambda(u_k^-) \longrightarrow \tilde{\alpha}_\lambda^-, J'_\lambda(u_k^-) \longrightarrow 0, \tag{82}$$

as $k \longrightarrow \infty$ because of Propositions 11 and 12. From related properties of fibering maps $\varphi_u(t)$, we have $\tilde{\alpha}_\lambda^+ < 0$. Similar to the Proof of Theorem 10, problem (1) admits two solutions $\tilde{u}_1 \in \mathcal{N}_\lambda^+$ and $\tilde{u}_2 \in \mathcal{N}_1^-$ in X_0. Moreover, these two solutions are distinct since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \varnothing$.

Conflicts of Interest

The authors declare that they have no competing interests.

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