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Research Article

Nonexistence of Global Solutions for Coupled System of Pseudoparabolic Equations with Variable Exponents and Weak Memories

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The most important behavior for evolution system is the blow-up phenomena because of its wide applications in modern science. The article discusses the finite time blowup that arise under an appropriate conditions. The nonsolvability of boundary value problem for damped pseudoparabolic differential equations with variable exponents is investigated. Such problem has been previously studied in the case if p and q are constants. New here is the case of variables of nonlinearity p and q which make the problem has a scientific interest.

1. Introduction and Overview

Boundary value problems for evolutionary equations of parabolic in degenerate sense are well studied (see, for example, [1–4]). In this article, we study boundary value problems for coupled system of pseudoparabolic equations with $p(x)$-Laplacian in the presence of weak viscoelasticities. Such problems have not been studied in depth. To begin with, let Ω is an open bounded domain in $\mathbb{R}^n$ for $n \geq 2$ with smooth boundary $\partial\Omega$; we then consider in $\Omega \times (0, T)$

$$\begin{cases} \partial_t w - \Delta_{q(x)} w - \Delta_x \left(\partial_t w - \sigma(t) \int_0^t \varpi_1(t-s) w \, ds \right) = f_1(w, z), \\ \partial_t z - \Delta_{q(x)} z - \Delta_x \left(\partial_t z - \sigma(t) \int_0^t \varpi_2(t-s) z \, ds \right) = f_2(w, z), \end{cases} \tag{1}$$

for $0 < T < \infty$ with initial condition

$$w(x, 0) = w_0(x) \in W_0^{1,q(\cdot)}(\Omega), \tag{2}$$

$$z(x, 0) = z_0(x) \in W_0^{1,q(\cdot)}(\Omega), \tag{3}$$

and boundary condition

$$w = z = 0 \text{ on } \partial\Omega \times (0, T). \tag{4}$$

The lack of stability of solutions of partial differential equations is a huge restriction for qualitative studies. The terms responsible for the blow-up phenomenon in our system (1) is that of more complicated nonlinearities when they dominate the damped terms, especially when it comes with the existence of a large class of Laplacian operator

$$\Delta_{q(x)} w = \operatorname{div} \left(|\nabla_x w|^{q(x)-2} \nabla_x w \right). \tag{5}$$

The functions $f_j : \mathbb{R}^2 \longrightarrow \mathbb{R}, j = 1, 2$ are given by the nonlinearities

$$f_1(\xi_1, \xi_2) = \left[|\xi_1 + \xi_2|^{p(x)-2}(\xi_1 + \xi_2) + |\xi_1|^{(p(x)-4)/2}\xi_1|\xi_2|^{p(x)/2}\right],$$
$$f_2(\xi_1, \xi_2) = \left[|\xi_1 + \xi_2|^{p(x)-2}(\xi_1 + \xi_2) + |\xi_2|^{(p(x)-4)/2}\xi_2|\xi_1|^{p(x)/2}\right], \tag{6}$$

respectively. The weak-viscoelastic term is $\sigma(t)\int_0^t \varpi_j(t-s) u(s)\, ds$.

There exists a function $\mathcal{F} \in C^1(\mathbb{R}^2, \mathbb{R})$ such that

$$\xi_1 f_1(\xi_1, \xi_2) + \xi_2 f_2(\xi_1, \xi_2) = p(x)\mathcal{F}(\xi_1, \xi_2), \ \forall (\xi_1, \xi_2) \in \mathbb{R}^2,$$
$$p(x)\mathcal{F}(\xi_1, \xi_2) = |\xi_1 + \xi_2|^{p(x)} + 2|\xi_1\xi_2|^{p(x)/2}, \tag{7}$$

where $f_1 = \partial\mathcal{F}/\partial w$ and $f_2 = \partial\mathcal{F}/\partial z$. There exist two positive constants c_0 and c_1 such that

$$c_0\left(|\xi_1|^{p(x)} + |\xi_2|^{p(x)}\right) \leq p(x)\mathcal{F}(\xi_1, \xi_2) \leq c_1\left(|\xi_1|^{p(x)} + |\xi_2|^{p(x)}\right). \tag{8}$$

For more details, see [5–8] and references therein.

With $p(x)$-Laplacian, which is nonlinear differential operator, in [9], a problem of elliptic equation is considered as

$$-\Delta_{p(x)}u - |u|^{p(x)-2}u = f(x, u), x \in \mathbb{R}^n. \tag{9}$$

The variable exponents $q(\cdot)$ and $p(\cdot)$ are two continuous functions on $\bar{\Omega}$ such that

$$2 < q_- \leq q(x) \leq q_+ < p_- \leq p(x) \leq p_+ < q_*(x), \tag{10}$$

with

$$q_*(x) = \begin{cases} \dfrac{nq(x)}{n - q(x)} & \text{if } n > q(x), \\ +\infty & \text{if } n \leq q(x), \end{cases}$$
$$p_- = \operatorname*{essinf}_{x\in\Omega} p(x),$$
$$p_+ = \operatorname*{esssup}_{x\in\Omega} p(x). \tag{11}$$

We assume that $q(x)$ satisfies the Zhikov-Fan condition, i.e., for all $x, y \in \Omega$,

$$|q(x) - q(y)| \leq \frac{K}{-\log|x - y|}, \text{with } |x - y| < \kappa, \tag{12}$$

with $K > 0,\ 0 < \kappa < 1$ and

$$\operatorname*{essinf}_{x\in\Omega}(q_*(x) - p(x)) > 0. \tag{13}$$

We state assumptions on ϖ_j and σ as follows: $\sigma, \varpi_j \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies

$$\begin{cases} 0 < \sigma'(t) < \sigma(t), \\ \varpi_j'(t) < -\varpi_j(t) < 0, \displaystyle\int_0^{+\infty} \varpi_j(s) ds < \left(\frac{1}{q_+} - \frac{1}{p_-}\right) q_- \|\sigma\|_\infty^{-1}. \end{cases} \tag{14}$$

Define positive constants α_0, α_1, E_1, and E_2 by

$$\begin{cases} \alpha_1 = \left(\dfrac{q_-}{C_1 p_+}\right)^{q_-/(p_+-q_-)}, E_1 = \dfrac{p_+ - q_-}{p_+}\alpha_1 - \rho_1, \\ \alpha_2 = \left(\dfrac{q_-}{C_2 p_+}\right)^{q_-/(p_+-q_-)}, \ E_2 = \dfrac{p_+ - q_-}{p_+}\alpha_2 - \rho_2, \end{cases} \tag{15}$$

for some constants $C_1, C_2, \rho_1, \rho_2 > 0$ which will be specified later.

Fan et al. discussed the existence and multiplicity of solutions of (9) for $u \in W^{1,p(x)}(\mathbb{R}^n)$, where $n \geq 2, p(x)$ is a function defined on $\mathbb{R}^n$.

Regarding nonlinear parabolic equation, we mention the work by [1]. The author proposed the problem.

$$\begin{cases} \partial_t u - \Delta_p u - |u|^{q-2}u = f(x, t), & x \in \Omega, \ 0 < t \leq T, \\ u = 0, & x \in \partial\Omega, \ 0 < t \leq T, \\ u(x, 0) = u_0, & x \in \Omega, \end{cases} \tag{16}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $p, q \geq 2$. Time existence of solutions of system (16) was proved.

Whereas, in [10], nonlinear pseudoparabolic was considered

$$\partial_t w - \Delta_{q(x)}w - \Delta_x\left(\partial_t w - \sigma(t)\int_0^t \varpi_1(t-s)w\, ds\right) = |w|^{p(x)-2}w. \tag{17}$$

Global in time nonexistence of (17) was shown under an appropriate conditions on $\varpi, p(x)$, and $q(x)$.

System (1), where the exponents $q(x) = q$ and $p(x) = p$, the existence/nonexistence results have been extensively studied (please see [11–15]).

The paper is organized as follows. In Section 2, we state the properties of the $p(x)$-growth conditions and present assumptions of the kernel functions. In Section 3, we state our main results and prove some auxiliary lemmas. In Section 4, we prove the global nonexistence of solutions given in Theorem 9. The paper is concluded by explanatory commentaries.

2. Preliminary

We try to list here some useful mathematical tools.

First, let Ω be an open bounded domain in $\mathbb{R}^n$ for $n \geq 2$ with smooth boundary $\partial\Omega$ and $p : \Omega \longrightarrow [1,\infty]$ be a measurable function. Denoting by

$$p_- = \underset{x\in\Omega}{\operatorname{essinf}}\, p(x), \quad p_+ = \underset{x\in\Omega}{\operatorname{esssup}}\, p(x). \tag{18}$$

We define the $p(\cdot)$ modular of a measurable function $w : \Omega \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ as

$$Q_{p(\cdot)}(w) = \int_{\Omega-\Omega_\infty} |w|^{p(x)} dx + \underset{x\in\Omega_\infty}{\operatorname{esssup}} |w(x)|, \tag{19}$$

where

$$\Omega_\infty = \{x \in \Omega \,;\, p(x) = \infty\}. \tag{20}$$

The special Orlicz Musielak space $L^{p(\cdot)}(\Omega)$ is a Lebesgue space with variable-exponent, and it consists of all the measurable function w defined on Ω for which

$$Q_{p(\cdot)}(\lambda w) < \infty \text{some } \lambda > 0. \tag{21}$$

Let

$$\|w\|_{p(\cdot)} = \inf \left\{\lambda > 0 \,;\, Q_{p(\cdot)}\left(\frac{w}{\lambda}\right) \leq 1\right\}, \tag{22}$$

be the Luxembourg norm on this space (see [16]).

The Sobolev space $W^{1,q(\cdot)}(\Omega)$ consists of functions $w \in L^{q(\cdot)}(\Omega)$ whose distributional gradient $\nabla_x w$ exists and satisfies $|\nabla_x w| \in L^{q(\cdot)}(\Omega)$. This space is a Banach with respect to the norm

$$\|w\|_{1,q(\cdot)} = \|w\|_{q(\cdot)} + \|\nabla_x w\|_{q(\cdot)}. \tag{23}$$

Lemma 1 (Corollary 8.2.5 in [17]).

(1) If (12) holds with $q(x)$, then

$$\|w\|_{q(\cdot)} \leq C \|\nabla_x w\|_{q(\cdot)}, \forall w \in W_0^{1,q(\cdot)}(\Omega), \tag{24}$$

where Ω is a bounded domain and C is a positive constant. The norm of the space $W_0^{1,q(\cdot)}(\Omega)$ is given by

$$\|w\|_{1,q(\cdot)} = \|\nabla_x w\|_{q(\cdot)}, \forall w \in W_0^{1,q(\cdot)}(\Omega). \tag{25}$$

(2) If

$$q \in C(\bar{\Omega}), : \Omega \longrightarrow [1,\infty), \tag{26}$$

is a measurable function and

$$\underset{x\in\Omega}{\operatorname{essinf}} (q^*(x) - p(x)) > 0, \tag{27}$$

with

$$q^* = \frac{nq(x)}{(n - q(x))_+}. \tag{28}$$

Then

$$W_0^{1,q(\cdot)}(\Omega)^{\circ\circ} L^{p(\cdot)}(\Omega), \tag{29}$$

with a continuous and compact embedding and

$$\|w\|_{p(\cdot)} \leq C_S \|\nabla_x w\|_{q(\cdot)}, \tag{30}$$

where $C_S > 0$ is an embedding constant.

Proposition 2 (Section 1 and Lemma 3.2.20 in [17]). *Let $1 < p_- \leq p_+ < +\infty$. The spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are separable, uniformly convex, and reflexive Banach spaces. The conjugate space of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$, where*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \ \forall x \in \Omega. \tag{31}$$

For $w \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have

$$\left|\int_\Omega w(x)v(x)\, dx\right| \leq \left(\frac{1}{p_-} + \frac{1}{\left(p'\right)_-}\right) \|w\|_{p(\cdot)} \|v\|_{p}{}'^{(\cdot)}. \tag{32}$$

Lemma 3 (Lemma 3.2.4 in [17]). *If $p \geq 1$ is a measurable function on Ω and $w \in L^{p(\cdot)}(\Omega)$, then $\|w\|_{p(\cdot)} \leq 1$ and $Q_{p(\cdot)}(w) \leq 1$ are equivalent. For $w \in L^{p(\cdot)}(\Omega)$, we have*

(1) $\|w\|_{p(\cdot)} \leq 1$ implies $Q_{p(\cdot)}(w) \leq \|w\|_{p(\cdot)}$

(2) $\|w\|_{p(\cdot)} > 1$ implies $Q_{p(\cdot)}(w) \geq \|w\|_{p(\cdot)}$

Lemma 4 (Section 2 in [18] and Lemma 3.2.5 in [17]). *If $p(x) \in [1,\infty)$ is a measurable function on Ω, then*

$$\min\left\{\|w\|_{p(\cdot)}^{p_-}, \|w\|_{p(\cdot)}^{p_+}\right\} \le Q_{p(\cdot)}(w) \le \max\left\{\|w\|_{p(\cdot)}^{p_-}, \|w\|_{p(\cdot)}^{p_+}\right\}, \tag{33}$$

for all $w \in L^{p(\cdot)}(\Omega)$.

Lemma 5 (Lemma 3.2.20 in [17]). *If $p_1(x) \ge p_2(x) \ge 1$ a.e. in Ω, there is a continuous inclusion $L^{p_1(\cdot)}(\Omega) \subset L^{p_2(\cdot)}(\Omega)$ and for all $w \in L^{p_1(\cdot)}(\Omega)$,*

$$\|w\|_{p_2(\cdot)} \le 2\|1\|_{r(\cdot)}\|w\|_{p_1(\cdot)}, \tag{34}$$

where

$$\frac{1}{r(x)} \equiv \frac{1}{p_2(x)} - \frac{1}{p_1(x)}. \tag{35}$$

The following notation will be used throughout this paper

$$\left(\varpi_j \circ v\right)(t) = \int_0^t \varpi_j(t-s)\|v(t)-v(s)\|_2^2\,ds, \tag{36}$$

for $v \in L^2(\Omega)$ and $t \ge 0$. We have the following technical lemma.

Lemma 6. *Let $\kappa \in \mathbb{N}$. For any $\Delta_x^p v \in C^1(0,T,H_0^1(\Omega))$ with $p = 0, 1, \cdots, \kappa-1$, $j = 1, 2$, we have*

$$\begin{aligned}
&\left\langle \sigma(t)\int_0^t \varpi_j(t-s)\Delta_x^\kappa v(s), \partial_t v(t)\,ds \right\rangle_2 \\
&\quad = \frac{(-1)^{\kappa+1}}{2}\partial_t\left[\sigma(t)\left(\varpi_j \circ \nabla_x^\kappa v\right)(t)\right] \\
&\qquad + \frac{(-1)^\kappa}{2}\partial_t\left[\sigma(t)\int_0^t \varpi_j(s)\,ds\|\nabla_x^\kappa v\|_2^2\right] \\
&\qquad + \frac{(-1)^\kappa}{2}\sigma(t)\left(\partial_t\varpi_j \circ \nabla_x^\kappa v\right)(t) \\
&\qquad + \frac{(-1)^{\kappa+1}}{2}\sigma(t)\varpi_j(t)\|\nabla_x^\kappa v\|_2^2 + \frac{(-1)^\kappa}{2}\sigma'(t)\left(\varpi_j \circ \nabla_x^\kappa v\right)(t) \\
&\qquad + \frac{(-1)^{\kappa+1}}{2}\sigma'(t)\int_0^t \varpi_j(s)\,ds\|\nabla_x^\kappa v\|_2^2.
\end{aligned} \tag{37}$$

Proof. Since

$$\langle \Delta_x^\kappa v, w\rangle_2 = (-1)^\kappa\langle \nabla_x^\kappa v, \nabla_x^\kappa w\rangle_2, \tag{38}$$

holds for any $\Delta_x^p v, \Delta_x^p w \in H_0^1(\Omega)$, we have

$$\begin{aligned}
&\left\langle \sigma(t)\int_0^t \varpi_j(t-s)\Delta_x^\kappa v(s), \partial_t v(t)\right\rangle_2 ds \\
&\quad = (-1)^\kappa\sigma(t)\int_0^t \varpi_j(t-s)\langle \nabla_x^\kappa\partial_t v(t), [\nabla_x^\kappa v(s) - \nabla_x^\kappa v(t) + \nabla_x^\kappa v(t)]\rangle_2\,ds \\
&\quad = (-1)^\kappa\sigma(t)\int_0^t \varpi_j(t-s)\langle \nabla_x^\kappa\partial_t v(t), [\nabla_x^\kappa v(s) - \nabla_x^\kappa v(t)]\rangle_2\,ds \\
&\qquad + (-1)^\kappa\sigma(t)\int_0^t \varpi_j(s)\,ds\langle \nabla_x^\kappa\partial_t v(t), \nabla_x^\kappa v(t)\rangle_2 \\
&\quad = \frac{(-1)^{\kappa+1}}{2}\sigma(t)\int_0^t \varpi_j(t-s)\partial_t\|\nabla_x^\kappa v(s) - \nabla_x^\kappa v(t)\|_2^2\,ds \\
&\qquad + \frac{(-1)^\kappa}{2}\sigma(t)\int_0^t \varpi_j(s)\,ds\,\partial_t\|\nabla_x^\kappa v\|_2^2 \\
&\quad = \frac{(-1)^{\kappa+1}}{2}\partial_t\left[\sigma(t)\int_0^t \varpi_j(t-s)\|\nabla_x^\kappa v(s) - \nabla_x^\kappa v(t)\|_2^2\,ds\right] \\
&\qquad + \frac{(-1)^\kappa}{2}\partial_t\left[\sigma(t)\int_0^t \varpi_j(s)\|\nabla_x^\kappa v\|_2^2\,ds\right] \\
&\qquad + \frac{(-1)^\kappa}{2}\sigma(t)\int_0^t \partial_t\varpi_j(t-s)\|\nabla_x^\kappa v(s) - \nabla_x^\kappa v(t)\|_2^2\,ds \\
&\qquad + \frac{(-1)^{\kappa+1}}{2}\sigma(t)\varpi_j(t)\|\nabla_x^\kappa v\|_2^2 + \frac{(-1)^\kappa}{2}\sigma'(t) \\
&\qquad \times \int_0^t \varpi_j(t-s)\|\nabla_x^\kappa v(s) - \nabla_x^\kappa v(t)\|_2^2\,ds + \frac{(-1)^{\kappa+1}}{2}\sigma'(t) \\
&\qquad \times \int_0^t \varpi_j(s)\,ds\|\nabla_x^\kappa v\|_2^2.
\end{aligned} \tag{39}$$

Using the notation (36), we obtain the desired result.

3. Main Results

Before stating our main theorems, we define the weak solution for the problem (1)–(4).

Definition 7. The pair (w, z) is said to be weak solution to (1)–(4) on $[0, T_0]$ if it satisfies,

$$\begin{cases}
\langle \partial_t w, \varphi\rangle_2 + \left\langle |\nabla_x w|^{q(x)-2}\nabla_x w, \nabla_x\varphi\right\rangle_2 + \langle \partial_t z, \psi\rangle_2 + \left\langle |\nabla_x z|^{q(x)-2}\nabla_x z, \nabla_x\psi\right\rangle_2, \\
+\left\langle \nabla_x\left(\partial_t w - \sigma(t)\int_0^t \varpi_1(t-s)w\,ds\right), \nabla_x\varphi\right\rangle_2 - \langle f_1(w,z), \varphi\rangle_2, \\
+\left\langle \nabla_x\left(\partial_t z - \sigma(t)\int_0^t \varpi_2(t-s)z\,ds\right), \nabla_x\psi\right\rangle_2 - \langle f_2(w,z), \psi\rangle_2 = 0,
\end{cases} \tag{40}$$

for all test functions $\varphi, \psi \in W_0^{1,q(\cdot)}(\Omega)$, $t \in [0, T_0]$.

Here, we present without proof, the first known result concerning the local existence (in time) for the problem (1)–(4).

Theorem 8. *Assume that (10), (12), and (14) hold. Then, the problem (1)–(4) has a unique local solution* (w, z) *satisfying*

$$(w, z) \in \left[C\left([0, T_0]; W_0^{1,q(\cdot)}(\Omega)\right)\right]^2,$$
$$(\partial_t w, \partial_t z) \in \left[C\left([0, T_0], L^2(\Omega)\right) \cap L^2\left([0, T_0]; H_0^1(\Omega)\right)\right]^2, \tag{41}$$

for $T_0 > 0$ *depending on* $\|w_0\|_{1,q(\cdot)}, \|z_0\|_{1,q(\cdot)}$.

To prove the previous theorem, we can adopt the Faedo-Galerkin method which is the same procedure used in [4].

We introduce the main result concerned with the finite time blowup of solutions of the problem (1)–(4).

We define

$$\lambda_1 = \frac{1}{q_+} - \frac{1}{q_-}\|\sigma\|_\infty \int_0^\infty \varpi_1(s)\, ds,$$
$$\lambda_2 = \frac{1}{q_+} - \frac{1}{q_-}\|\sigma\|_\infty \int_0^\infty \varpi_2(s)\, ds, \tag{42}$$

and find

$$\frac{1}{p_-} < \lambda_1, \lambda_2 < \frac{1}{q_+} < \frac{1}{2}, \tag{43}$$

by (10) and (14).

Theorem 9. *Assume that (10)–(14) hold. Given* $w_0, z_0 \in W_0^{1,q(\cdot)}(\Omega)$ *satisfying*

$$E(0) < (E_1 + E_2) < 0, \tag{44}$$

$$\left(\int_\Omega |\nabla_x w_0|^q\, dx, \int_\Omega |\nabla_x z_0|^q\, dx\right) > \left(\frac{\alpha_1}{\lambda_1}, \frac{\alpha_2}{\lambda_2}\right). \tag{45}$$

Then, any solution of (1) with (3) and (4) blows up in finite time $T^* < \infty$, *where*

$$T^* = \frac{L(0)^{1-(q_-/2)}}{((q_-/2) - 1)\Gamma}, \text{for } L(0), \Gamma > 0, \tag{46}$$

and E_1 *and* E_2 *are given in (66).*

In order to prove Theorem 9, we need to exploit some lemmas. First, we define the modified energy functional $E(t)$ *associated to the problem (1)–(4) by*

$$E(t) = \int_\Omega \frac{|\nabla_x w(x,t)|^{q(x)}}{q(x)}\, dx + \int_\Omega \frac{|\nabla_x z(x,t)|^{q(x)}}{q(x)}\, dx + \frac{1}{2}\sigma(t) \cdot [(\varpi_1 \circ \nabla_x w)(t) + (\varpi_2 \circ \nabla_x z)(t)] - \int_\Omega \mathcal{F}(w,z) dx - \frac{1}{2}\sigma(t) \cdot \left[\int_0^t \varpi_1(s)\, ds \|\nabla_x w\|_2^2 + \int_0^t \varpi_2(s)\, ds \|\nabla_x z\|_2^2\right]. \tag{47}$$

Lemma 10. *Let* (w, z) *be the solution of (1)–(4) with (10)–(14). Then, the energy functional satisfies*

$$\begin{aligned} 2\partial_t E(t) = {} & -2\|\partial_t w\|_2^2 - 2\|\nabla_x \partial_t w\|_2^2 - 2\|\partial_t z\|_2^2 - 2\|\nabla_x \partial_t z\|_2^2 \\ & + \sigma(t)\left[(\partial_t \varpi_1 \circ \nabla_x w)(t) - \varpi_1(t)\|\nabla_x w\|_2^2\right] \\ & + \sigma(t)\left[(\partial_t \varpi_2 \circ \nabla_x z)(t) - \varpi_2(t)\|\nabla_x z\|_2^2\right] + \sigma'(t) \\ & \cdot \left[(\varpi_1 \circ \nabla_x w)(t) - \int_0^t \varpi_1(s)\, ds \|\nabla_x w\|_2^2\right] + \sigma'(t) \\ & \cdot \left[(\varpi_2 \circ \nabla_x z)(t) - \int_0^t \varpi_2(s)\, ds \|\nabla_x z\|_2^2\right] \leq 0. \end{aligned} \tag{48}$$

Proof. Multiplying $(1)_1$ by $\partial_t w$ and $(1)_2$ by $\partial_t z$, integrating by parts over Ω, summing and using (14) and Lemma 6. Then, it is clear that $f_1(w, z)$ and $f_2(w, z)$ are bounded, so the potential $\mathcal{F}(w, z)$ exists and is given by (85) as follows

$$\mathcal{F}(w, z) = \frac{1}{p(x)}\left[|w + z|^{p(x)} + 2|wz|^{p(x)/2}\right]. \tag{49}$$

Differentiating (49) with respect to time, it follows that

$$\partial_t \mathcal{F}(w, z) = \langle f_1(w, z), \partial_t w \rangle_2 + \langle f_2(w, z), \partial_t z \rangle_2. \tag{50}$$

Hence, the proof is finished .

Lemma 11. *Let* (w, z) *be a strong solution of (1)–(4) with (10)–(14). Then, we have*

$$\begin{aligned} E(t) \geq {} & \lambda_1 \int_\Omega |\nabla_x w|^q\, dx + \lambda_2 \int_\Omega |\nabla_x z|^q\, dx - \int_\Omega \mathcal{F}(w, z)\, dx \\ & + \frac{1}{2}\sigma(t)[(\varpi_1 \circ \nabla_x w)(t) + (\varpi_2 \circ \nabla_x z)(t)] \\ & - \frac{q_+ - 2}{2q_+}\|\sigma\|_\infty |\Omega| \left(\int_0^\infty \varpi_1(s)\, ds + \int_0^\infty \varpi_2(s)\, ds\right). \end{aligned} \tag{51}$$

Proof. Using Young's inequality, we obtain

$$|\nabla_x w|^2 \leq \frac{2}{q}|\nabla_x w|^q + \frac{q-2}{q} \leq \frac{2}{q_-}|\nabla_x w|^q + \frac{q_+ - 2}{q_+},$$
$$|\nabla_x z|^2 \leq \frac{2}{q}|\nabla_x z|^q + \frac{q-2}{q} \leq \frac{2}{q_-}|\nabla_x z|^q + \frac{q_+ - 2}{q_+}, \tag{52}$$

which proves the lemma.

Lemma 12. *Let (w, z) be a strong solution of (1)–(4) with (10)–(14). Then, we have*

$$
\begin{aligned}
E(t) \geq{}& \lambda_1 \int_\Omega |\nabla_x w|^q \,dx + \lambda_2 \int_\Omega |\nabla_x z|^q \,dx - \frac{c_1}{p_-}\int_\Omega |w|^p dx \\
&- \frac{c_1}{p_-}\int_\Omega |z|^p dx + \frac{1}{2}\sigma(t)[(\varpi_1 \circ \nabla_x w)(t) + (\varpi_2 \circ \nabla_x z)(t)] \\
&- \frac{q_+ - 2}{2q_+}\|\sigma\|_\infty |\Omega| \left(\int_0^\infty \varpi_1(s)\,ds + \int_0^\infty \varpi_2(s)\,ds \right).
\end{aligned} \tag{53}
$$

Proof. By using both equivalences (8) and (10), we deduce

$$
\begin{aligned}
-\int_\Omega \mathscr{F}(w, z) &\geq -\frac{c_1}{p(x)}\left(\int_\Omega |w|^{p(x)} dx + \int_\Omega |z|^{p(x)} dx\right) \\
&\geq -\frac{c_1}{p_-}\left(\int_\Omega |w|^p dx + \int_\Omega |z|^p dx\right).
\end{aligned} \tag{54}
$$

The proof is completed by direct use of (54).

In condition (10), $\Omega_\infty = \varnothing$ holds, which yields

$$
Q_{p(x)}(w) = \int_\Omega |w|^{p(x)} \,dx. \tag{55}
$$

We derive a shaper estimate than that in Lemma 13.

Lemma 13. *Let (w, z) be a strong solution of (1)–(4) with (10)–(14). Then, we have*

$$
\begin{aligned}
E(t) \geq{}& \lambda_1 \int_\Omega |\nabla_x w|^q \,dx + \lambda_2 \int_\Omega |\nabla_x z|^q \,dx - \frac{C_S c_1}{p_-} \\
&\cdot \left[\left(\int_\Omega |\nabla_x w|^q \,dx \right)^{p_+/q_-} + \left(\int_\Omega |\nabla_x z|^q \,dx \right)^{p_+/q_-} \right] \\
&+ \frac{1}{2}\sigma(t)[(\varpi_1 \circ \nabla_x w)(t) + (\varpi_2 \circ \nabla_x z)(t)] - \frac{2c_1}{p_-}(C_S + 1) \\
&- \frac{q_+ - 2}{2q_+}\|\sigma\|_\infty |\Omega| \left(\int_0^\infty \varpi_1(s)\,ds + \int_0^\infty \varpi_2(s)\,ds \right),
\end{aligned} \tag{56}
$$

where C_S is an embedding constant defined in Lemma 1.

Proof. First of all, note that, for $a \geq 0$,

$$
\max\left(a^{p_-}, a^{p_+}\right) = \begin{cases} a^{p_+} & \text{if } a > 1 \\ a^{p_-} & \text{if } a \leq 1 \end{cases} \leq a^{p_+} + 1. \tag{57}
$$

Then, owing to Lemmas 1 and 4, we have

$$
\begin{cases}
\int_\Omega |w|^p \,dx = Q_p(w) \leq \max\left(\|w\|_p^{p_-}, \|w\|_p^{p_+}\right) \leq \|w\|_p^{p_+} + 1 \leq C_S \|\nabla_x w\|_q^{p_+} + 1, \\
\int_\Omega |z|^p \,dx = Q_p(z) \leq \max\left(\|z\|_p^{p_-}, \|z\|_p^{p_+}\right) \leq \|z\|_p^{p_+} + 1 \leq C_S \|\nabla_x z\|_q^{p_+} + 1.
\end{cases} \tag{58}
$$

Similarly, for $a \geq 0$, we have

$$
a = \begin{cases} \left\{\min\left(a^{q_-}, a^{q_+}\right)\right\}^{1/q_-} & \text{if } a > 1, \\ \left\{\min\left(a^{q_-}, a^{q_+}\right)\right\}^{1/q_+} & \text{if } a \leq 1. \end{cases} \tag{59}
$$

Hence, by Lemmas 3 and 4, (58) is estimated as

$$
\begin{cases}
\int_\Omega |w|^p \,dx \leq C_S \|\nabla_x w\|_q^{p_+} + 1 \leq C_S \left\{ \left(Q_p(\nabla_x w)\right)^{p_+/q_-} + 1 \right\} + 1, \\
\int_\Omega |z|^p \,dx \leq C_S \|\nabla_x z\|_q^{p_+} + 1 \leq C_S \left\{ \left(Q_p(\nabla_x z)\right)^{p_+/q_-} + 1 \right\} + 1.
\end{cases} \tag{60}
$$

The substitution of (60) in the estimate (53) ends the proof.

Now, we can see that (56) takes the following form

$$
\begin{aligned}
E(t) \geq{}& \lambda_1 \int_\Omega |\nabla_x w|^q \,dx + \lambda_2 \int_\Omega |\nabla_x z|^q \,dx + \frac{1}{2}\sigma(t) \\
&\cdot [(\varpi_1 \circ \nabla_x w)(t) + (\varpi_2 \circ \nabla_x z)(t)] - C_1 \\
&\cdot \left[\lambda_1 \int_\Omega |\nabla_x w|^q \,dx + \frac{1}{2}\sigma(t)[(\varpi_1 \circ \nabla_x w)(t)] \right]^{p_+/q_-} \\
&- \rho_1 - C_2 \left[\lambda_2 \int_\Omega |\nabla_x z|^q \,dx + \frac{1}{2}\sigma(t)[(\varpi_2 \circ \nabla_x z)(t)] \right]^{p_+/q_-} - \rho_2,
\end{aligned} \tag{61}
$$

where

$$
C_1 = \frac{C_S c_1}{p_-}\left(\frac{1}{\lambda_1}\right)^{p_+/q_-}, C_2 = \frac{C_S c_1}{p_-}\left(\frac{1}{\lambda_2}\right)^{p_+/q_-}, \tag{62}
$$

$$
\rho_1 = \frac{c_1}{p_-}(C_S + 1) + \frac{q_+ - 2}{2q_+}\|\sigma\|_\infty |\Omega| \int_0^\infty \varpi_1(s)\,ds, \tag{63}
$$

$$
\rho_2 = \frac{c_1}{p_-}(C_S + 1) + \frac{q_+ - 2}{2q_+}\|\sigma\|_\infty |\Omega| \int_0^\infty \varpi_2(s)\,ds. \tag{64}
$$

Let $f(\cdot, \cdot)$ be a function defined by

$$
\begin{aligned}
f: \quad \mathbb{R}^+ \times \mathbb{R}^+ \quad &\longrightarrow \quad \mathbb{R} \\
(\psi_1, \psi_2) \quad &\mapsto \quad \left(\psi_1 - C_1 \psi_1^{p_+/q_-} - \rho_1 + \psi_2 - C_2 \psi_2^{p_+/q_-} - \rho_2\right).
\end{aligned} \tag{65}
$$

Then, $f(\cdot, \cdot)$ is increasing in $(0, \alpha_1) \times (0, \alpha_2)$ decreasing for $(\psi_1, \psi_2) > (\alpha_1, \alpha_2)$, such that $f(\psi_1, \psi_2) \longrightarrow -\infty$ as

$(\psi_1, \psi_2) \longrightarrow (+\infty, +\infty)$, and we have

$$f(\alpha_1, \alpha_2) = \left(\underbrace{\frac{p_+ - q_-}{p_+}\alpha_1 - \rho_1}_{\equiv E_1} + \underbrace{\frac{p_+ - q_-}{p_+}\alpha_2 - \rho_2}_{\equiv E_2} \right), \quad (66)$$

for

$$(\alpha_1, \alpha_2) = \left(\left(\frac{q_-}{C_1 p_+} \right)^{q_-/(p_+ - q_-)}, \left(\frac{q_-}{C_2 p_+} \right)^{q_-/(p_+ - q_-)} \right). \quad (67)$$

Since, by the fact (43), i.e., $\{1/\lambda_1, 1/\lambda_2\} > 2$ holds. Then, by $(63)_1$, we can find

$$\min \{C_1, C_2\} > 2^{p_+/q_-} \frac{C_s c_1}{p_-}, \quad (68)$$

for all

$$\max \{\alpha_1, \alpha_2\} < \left(\frac{1}{2}\right)^{p_+/(p_+ - q_-)} \left(\frac{p_- q_-}{p_+}\right)^{q_-/(p_+ - q_-)} \left(\frac{1}{C_S c_1}\right)^{q_-/(p_+ - q_-)}. \quad (69)$$

Lemma 14. *Under the assumptions of Lemma 13. Then, the pair (E_1, E_2) defined in (66) is negative.*

Proof. Direct use of (63) and (66). Thus, we obtain

$$\begin{aligned} E_1 + \frac{q_+ - 2}{2q_+} \|\sigma\|_\infty |\Omega| \int_0^\infty \varpi_1(s)\, ds &= \frac{p_+ - q_-}{p_+}\alpha_1 - \frac{c_1}{p_-}(C_S + 1), \\ E_2 + \frac{q_+ - 2}{2q_+} \|\sigma\|_\infty |\Omega| \int_0^\infty \varpi_2(s)\, ds &= \frac{p_+ - q_-}{p_+}\alpha_2 - \frac{c_1}{p_-}(C_S + 1). \end{aligned} \quad (70)$$

By considering (68), (69). Then, we use Young's inequality to get

$$\begin{aligned} E_1 + \frac{q_+ - 2}{2q_+} \|\sigma\|_\infty |\Omega| \int_0^\infty \varpi_1(s)\, ds &< \frac{p_+ - q_-}{p_+} \left(\frac{1}{2}\right)^{p_+/(p_+ - q_-)} \\ &\quad \times \left(\frac{p_- q_-}{p_+}\right)^{q_-/(p_+ - q_-)} \left(\frac{1}{C_S c_1}\right)^{q_-/(p_+ - q_-)} - \frac{c_1}{p_-}(C_S + 1) \\ &\le \delta_0 \left(\frac{1}{C_S c_1}\right)^{q_-/(p_+ - q_-)} + \frac{1}{\delta_0}\frac{p_+ - q_-}{p_+}\left(\frac{1}{2}\right)^{p_+/(p_+ - q_-)} \\ &\quad \times \left(\frac{p_- q_-}{p_+}\right)^{q_-/(p_+ - q_-)} - \frac{c_1}{p_-}(C_S + 1) < 0, \end{aligned}$$

$$\begin{aligned} E_2 + \frac{q_+ - 2}{2q_+} \|\sigma\|_\infty |\Omega| \int_0^\infty \varpi_2(s)\, ds &< \frac{p_+ - q_-}{p_+} \left(\frac{1}{2}\right)^{p_+/(p_+ - q_-)} \\ &\quad \times \left(\frac{p_- q_-}{p_+}\right)^{q_-/(p_+ - q_-)} \left(\frac{1}{C_S c_1}\right)^{q_-/(p_+ - q_-)} - \frac{c_1}{p_-}(C_S + 1) \\ &\le \delta_1 \left(\frac{1}{C_S c_1}\right)^{q_-/(p_+ - q_-)} + \frac{1}{\delta_1}\frac{p_+ - q_-}{p_+}\left(\frac{1}{2}\right)^{p_+/(p_+ - q_-)} \\ &\quad \times \left(\frac{p_- q_-}{p_+}\right)^{q_-/(p_+ - q_-)} - \frac{c_1}{p_-}(C_S + 1) < 0. \end{aligned} \quad (71)$$

To provide this, we take c_1 sufficiently large and δ_0, δ_1 small enough.

Lemma 15. *Let (w, z) be a strong solution of (1)–(4) with (10)–(14) and initial condition satisfying (44) and (45). Then, there exists a constants $\beta_1 > \alpha_1$ and $\beta_2 > \alpha_2$ such that*

$$\left(\lambda_1 \int_\Omega |\nabla_x w(t)|^q\, dx + \frac{1}{2}\sigma(t)(\varpi_1 \circ \nabla_x w)(t), \lambda_2 \int_\Omega |\nabla_x z(t)|^q\, dx + \frac{1}{2}\sigma(t)(\varpi_2 \circ \nabla_x z)(t) \right) \ge (\beta_1, \beta_2). \quad (72)$$

Proof. Since

$$f\left(\lambda_1 \int_\Omega |\nabla_x w_0|^q\, dx, \lambda_2 \int_\Omega |\nabla_x z_0|^q\, dx \right) \le E(0) < (E_1 + E_2), \quad (73)$$

holds by (61). Then, there exists $(\beta_1 > \alpha_1$ and $\beta_2 > \alpha_2$ such that $f(\beta_1, \beta_2) = E(0)$. Thus, we have

$$f\left(\lambda_1 \int_\Omega |\nabla_x w_0|^q\, dx, \lambda_2 \int_\Omega |\nabla_x z_0|^q\, dx \right) \le E(0) \equiv f(\beta_1, \beta_2), \quad (74)$$

which implies that

$$\left(\lambda_1 \int_\Omega |\nabla_x w_0|^q\, dx, \lambda_2 \int_\Omega |\nabla_x z_0|^q\, dx \right) \ge (\beta_1, \beta_2). \quad (75)$$

Now, to establish (72), we suppose by contradiction that

$$\left(\lambda_1 \int_\Omega |\nabla_x w(t_0)|^q\, dx + \frac{1}{2}\sigma(t_0)(\varpi_1 \circ \nabla_x w)(t_0), \lambda_2 \int_\Omega |\nabla_x z(t_0)|^q\, dx + \frac{1}{2}\sigma(t_0)(\varpi_2 \circ \nabla_x z)(t_0) \right) < (\beta_1, \beta_2), \quad (76)$$

for some $t_0 \ge 0$ and by continuity of $f(\cdot, \cdot)$, we can choose t_0

such that

$$\left(\lambda_1\int_\Omega |\nabla_x w(t_0)|^q\,dx + \frac{1}{2}\sigma(t_0)(\varpi_1\circ\nabla_x w)(t_0), \lambda_2\int_\Omega |\nabla_x z(t_0)|^q\,dx + \frac{1}{2}\sigma(t_0)(\varpi_2\circ\nabla_x z)(t_0)\right) > (\alpha_1,\alpha_2). \tag{77}$$

Again, the use of (61) leads to

$$\begin{aligned} E(t_0) &\ge f\left(\lambda_1\int_\Omega |\nabla_x w(t_0)|^q\,dx + \frac{1}{2}\sigma(t_0)(\varpi_1\circ\nabla_x w)\right.\\ &\quad \left.\cdot(t_0), \lambda_2\int_\Omega |\nabla_x w(t_0)|^q\,dx + \frac{1}{2}\sigma(t_0)(\varpi_2\circ\nabla_x z)(t_0)\right)\\ &> f(\beta_1,\beta_2) = E(0). \end{aligned} \tag{78}$$

This is impossible, since

$$E(t) < E(0), \forall t\in[0,T_0). \tag{79}$$

Hence, (72) established.

Lemma 16. *Let (w,z) be a strong solution of (1)–(4) with (10)–(14) and (44). Then, we have*

$$\begin{aligned} \int_\Omega |w|^p\,dx + \int_\Omega |z|^p\,dx &\ge \frac{\lambda_1 p_-}{c_1}\int_\Omega |\nabla_x w|^q\,dx + \frac{\lambda_2 p_-}{c_1}\int_\Omega |\nabla_x z|^q\,dx\\ &\quad + 2(C_S+1) + \frac{p_-}{c_1}\left(C_1\beta_1^{p_+/q_-} - \beta_1 + C_2\beta_2^{p_+/q_-} - \beta_2\right). \end{aligned} \tag{80}$$

$$\int_\Omega |w|^p\,dx + \int_\Omega |z|^p\,dx \ge 2(C_S+1) + \frac{p_-}{c_1}\left(C_1\beta_1^{p_+/q_-} + C_2\beta_2^{p_+/q_-}\right), \tag{81}$$

for all $t\in[0,T_0)$.

Proof. To prove (80), we exploit Lemmas 10 and 13 to obtain

$$\begin{aligned} \frac{c_1}{p_-}\left(\int_\Omega |w|^p\,dx + \int_\Omega |z|^p\,dx\right) &= I(t) \ge \lambda_1\int_\Omega |\nabla_x w|^q\,dx + \lambda_2\int_\Omega |\nabla_x z|^q\,dx - E(0) + \frac{1}{2}\sigma(t)\\ &\quad \times [(\varpi_1\circ\nabla_x w)(t) + \varpi_2\circ\nabla_x z)(t)]\\ &\quad - \frac{q_+-2}{2q_+}\|\sigma\|_\infty|\Omega| \min\left\{\int_0^\infty \varpi_1(s)\,ds, \int_0^\infty \varpi_2(s)\,ds\right\}. \end{aligned} \tag{82}$$

Then, by the definitions of ρ_1,ρ_2,f, and E_0. So, inequality (82) can be estimated as follows

$$\begin{aligned} I(t) &\ge \lambda_1\int_\Omega |\nabla_x w|^q\,dx + \lambda_2\int_\Omega |\nabla_x z|^q\,dx - f(\beta_1,\beta_2)\\ &\quad - \frac{q_+-2}{2q_+}\|\sigma\|_\infty|\Omega|\left(\int_0^\infty \varpi_1(s)\,ds + \int_0^\infty \varpi_2(s)\,ds\right)\\ &\ge \lambda_1\int_\Omega |\nabla_x w|^q\,dx - \frac{q_+-2}{2q_+}\|\sigma\|_\infty|\Omega|\int_0^\infty \varpi_1(s)\,ds\\ &\quad - \left(\beta_1 - C_1\beta_1^{\frac{p_+}{q_-}} - \rho_1\right) + \lambda_2\int_\Omega |\nabla_x z|^q\,dx\\ &\quad - \frac{q_+-2}{2q_+}\|\sigma\|_\infty|\Omega|\int_0^\infty \varpi_2(s)\,ds - \left(\beta_2 - C_2\beta_2^{p_+/q_-} - \rho_2\right)\\ &\ge \lambda_1\int_\Omega |\nabla_x w|^q\,dx + \lambda_2\int_\Omega |\nabla_x z|^q\,dx + \frac{2c_1}{p_-}(C_S+1)\\ &\quad + C_1\beta_1^{p_+/q_-} - \beta_2 + C_2\beta_2^{p_+/q_-} - \beta_2. \end{aligned} \tag{83}$$

From the previous estimate, we deduce

$$\begin{aligned} I(t) &\ge \beta_1 - \frac{q_+-2}{2q_+}\|\sigma\|_\infty|\Omega|\int_0^\infty \varpi_1(s)\,ds - \left(\beta_1 - C_1\beta_1^{p_+/q_-} - \rho_1\right)\\ &\quad + \beta_2 - \frac{q_+-2}{2q_+}\|\sigma\|_\infty|\Omega|\int_0^\infty \varpi_2(s)\,ds - \left(\beta_2 - C_2\beta_2^{p_+/q_-} - \rho_2\right)\\ &\ge \frac{2c_1}{p_-}(C_S+1) + C_1\beta_1^{p_+/q_-} + C_2\beta_2^{p_+/q_-}. \end{aligned} \tag{84}$$

Thanks to (72) and (84) and the definition of different constants, then the desired results are proved.

4. Proof of Theorem 9

We are now ready to prove Theorem 9.

Proof. Let the auxiliary function defined as

$$L(t) = \frac{1}{2}\left(\|w(t)\|_2^2 + \|z(t)\|_2^2 + \|\nabla_x w(t)\|_2^2 + \|\nabla_x z(t)\|_2^2\right). \tag{85}$$

By differentiating the functional L, we get

$$\begin{aligned} \partial_t L(t) &= \int_\Omega w\partial_t w\,dx + \int_\Omega z\partial_t z\,dx + \int_\Omega \nabla_x w\nabla_x\partial_t w\,dx + \int_\Omega \nabla_x z\nabla_x\partial_t z\,dx\\ &= \int_\Omega w\left[\operatorname{div}\left(|\nabla_x w|^{q(x)-2}\nabla_x w\right) + \Delta\partial_t w - \sigma(t)\int_0^t \varpi_1(t-s)\Delta_x w(s)\,ds + f_1(w,z)\right]dx\\ &\quad + \int_\Omega \nabla_x w\nabla_x\partial_t w\,dx + \int_\Omega z\left[\operatorname{div}\left(|\nabla_x z|^{q(x)-2}\nabla_x z\right) + \Delta\partial_t z - \sigma(t)\right.\\ &\quad \left.\cdot\int_0^t \varpi_2(t-s)\Delta_x z(s)\,ds + f_2(w,z)\right]dx + \int_\Omega \nabla_x z\nabla_x\partial_t z\,dx\\ &= -\int_\Omega |\nabla_x w(t)|^q\,dx - \int_\Omega |\nabla_x z(t)|^q\,dx + \underbrace{\int_\Omega |w(t)+z(t)|^p + 2|wz|^{p/2}\,dx}_{=\int_\Omega p(x)\mathcal{F}(w,z)dx}\\ &\quad + \int_\Omega \sigma(t)\int_0^t \varpi_1(t-s)\nabla_x w(s)\cdot\nabla_x w(t)\,ds\,dx + \int_\Omega \sigma(t)\\ &\quad \cdot\int_0^t \varpi_2(t-s)\nabla_x z(s)\cdot\nabla_x z(t)\,ds\,dx. \end{aligned} \tag{86}$$

By using (8), we get

$$\begin{aligned}\partial_t L(t) \geq & -\int_\Omega |\nabla_x w(t)|^q\,dx - \int_\Omega |\nabla_x z(t)|^q\,dx + c_0 \\ & \cdot \left(\int_\Omega |w(t)|^p dx + \int_\Omega |z(t)|^p dx\right) \\ & + \int_\Omega \sigma(t)\int_0^t \varpi_1(t-s)\nabla_x w(s)\cdot\nabla_x w(t)\,ds\,dx \\ & + \int_\Omega \sigma(t)\int_0^t \varpi_2(t-s)\nabla_x z(s)\cdot\nabla_x z(t)\,ds\,dx.\end{aligned} \tag{87}$$

Thanks to Cauchy Schwarz's inequality, we get

$$\int_\Omega \nabla_x w(s)\cdot\nabla_x w(t)\,dx \geq \frac{2C_0^2-1}{2C_0^2}\|\nabla_x w(t)\|_2^2 - \frac{C_0^2}{2}\|\nabla_x w(t)-\nabla_x w(s)\|_2^2, \tag{88}$$

$$\int_\Omega \nabla_x z(s)\cdot\nabla_x w(t)\,dx \geq \frac{2C_0^2-1}{2C_0^2}\|\nabla_x z(t)\|_2^2 - \frac{C_0^2}{2}\|\nabla_x z(t)-\nabla_x z(s)\|_2^2, \tag{89}$$

holds for some positive constant $C_0 > 0$ to be determined later.

Substituting (89) in (86) to get

$$\begin{aligned}\partial_t L(t) \geq & -\int_\Omega |\nabla_x w(t)|^q\,dx - \int_\Omega |\nabla_x z(t)|^q\,dx + c_0 \\ & \cdot \left(\int_\Omega |w(t)|^p\,dx + \int_\Omega |z(t)|^p\,dx\right) + \frac{2C_0^2-1}{2C_0^2}\sigma(t) \\ & \cdot \left(\int_0^t \varpi_1(s)\,ds\|\nabla_x w(t)\|_2^2 + \int_0^t \varpi_2(s)\,ds\|\nabla_x z(t)\|_2^2\right) \\ & - \frac{C_0^2}{2}\sigma(t)[(\varpi_1\circ\nabla_x w)(t) + (\varpi_2\circ\nabla_x z)(t)].\end{aligned} \tag{90}$$

Using Lemma 12 to estimate the terms $(\varpi_1\circ\nabla_x w)$ and $(\varpi_2\circ\nabla_x z)$, we get

$$\begin{aligned}\partial_t L(t) \geq & \left(C_0^2\lambda_1 - 1\right)Q_q(\nabla_x w) + \left(C_0^2\lambda_2 - 1\right)Q_q(\nabla_x z) \\ & + \left(c_0 - \frac{c_1 C_0^2}{p_-}\right)\left(Q_p(w) + Q_p(z)\right) + \frac{2C_0^2-1}{2C_0^2}\sigma(t) \\ & \cdot \left(\int_0^t \varpi_1(s)\,ds\|\nabla_x w(t)\|_2^2 + \int_0^t \varpi_2(s)\,ds\|\nabla_x z(t)\|_2^2\right) \\ & + C_0^2\left(-E_1 - \frac{q_+-2}{2q_+}\|\sigma\|_\infty|\Omega|\int_0^\infty \varpi_1(s)\,ds\right) \\ & + C_0^2\left(-E_2 - \frac{q_+-2}{2q_+}\|\sigma\|_\infty|\Omega|\int_0^\infty \varpi_2(s)\,ds\right).\end{aligned} \tag{91}$$

Owing to (43), we can choose C_0 as

$$2 < \frac{1}{\lambda_1}, \frac{1}{\lambda_2} < C_0^2 < p_-, \tag{92}$$

where c_0 is sufficiently large such that $c_0 > C_0^2 c_1/p_-$, by Lemma 14, we have

$$\begin{aligned}\partial_t L(t) > & \left(C_0^2\lambda_1 - 1\right)Q_q(\nabla_x w) + \left(C_0^2\lambda_2 - 1\right)Q_q(\nabla_x z) \\ & + \left(c_0 - \frac{c_1 C_0^2}{p_-}\right)\left(Q_p(w) + Q_p(z)\right) \\ > & \left(C_0^2\lambda_{\min} - 1\right)\left(Q_q(\nabla_x w) + Q_q(\nabla_x z)\right) \\ & + \left(c_0 - \frac{c_1 C_0^2}{p_-}\right)\left(Q_p(w) + Q_p(z)\right) > 0,\end{aligned} \tag{93}$$

where $\lambda_{\min} = \min\{\lambda_1, \lambda_2\}$.

We derive the differential inequality from (93). First, owing to Poincaré's inequality, we have

$$\begin{aligned}\|w\|_2 &\leq C_P\|\nabla_x w\|_2, \\ \|z\|_2 &\leq C_P\|\nabla_x z\|_2,\end{aligned} \tag{94}$$

holds. For some constant $C_P > 0$, we have

$$\begin{aligned}L(t) &\leq \frac{1+C_P^2}{2}\left(\|\nabla_x w(t)\|_2^2 + \|\nabla_x z(t)\|_2^2\right) \\ &\leq 2\left(1+C_P^2\right)\|1\|_{2q/(q-2)}^2\left(\|\nabla_x w(t)\|_q^2 + \|\nabla_x z(t)\|_q^2\right).\end{aligned} \tag{95}$$

By using the following Minkowski's inequality

$$(a+b)^p \leq 2^{p-1}\left(a^p + b^p\right), \tag{96}$$

we can obtain

$$\begin{aligned}\int_\Omega |\nabla_x w|^q dx + \int_\Omega |\nabla_x z|^q dx &\geq 2^{1-(q_-/2)}\left(\|\nabla_x w(t)\|_q^2 + \|\nabla_x z(t)\|_q^2\right)^{q_-/2} \\ &\geq 2^{1-(q_-/2)}\left(\frac{L(t)}{2\left(1+C_P^2\right)\|1\|_{2q/(q-2)}^2}\right)^{(q_-/2)}.\end{aligned} \tag{97}$$

Then, by Lemma 5, we have $L(0) \geq 0$ and

$$\partial_t L(t) > \left(c_0 - \frac{c_1 C_0^2}{p_-}\right)\left(2(C_S+1) + \frac{p_-}{c_1}\left(C_1\beta_1^{p_+/q_-} + C_2\beta_2^{p_+/q_-}\right)\right) > 0, \tag{98}$$

by (82) and (93), we can assume that $L(t)$ is sufficiently large for $t \in [0, T_0)$. Hence, thanks to (95), $\|\nabla_x w(t)\|_q > 1$ and $\|\nabla_x z(t)\|_q > 1$ hold for $t \in [0, T_0)$. Then, using Lemma 4 to get

$$\|\nabla_x w(t)\|_q^{q_-} = \min\left(\|\nabla_x w(t)\|_q^{q_-}, \|\nabla_x w(t)\|_q^{q_+}\right) \le Q_p(\nabla_x w),$$
$$\|\nabla_x z(t)\|_q^{q_-} = \min\left(\|\nabla_x z(t)\|_q^{q_-}, \|\nabla_x z(t)\|_q^{q_+}\right) \le Q_p(\nabla_x z), \tag{99}$$

which gives

$$\partial_t L(t) > \Gamma (L(t))^{q_-/2} \text{ for all } t > 0, \tag{100}$$

where

$$\Gamma = \left(C_0^2 \lambda_{\min} - 1\right)\left(\frac{2^{(2-q_-)/q_-}}{2\left(1+C_P^2\right)}\|1\|_{2q/q-2}^{-2}\right)^{q_-/2} > 0. \tag{101}$$

A direct integration of (100) over $[0, t]$ then

$$(L(t))^{1-(q_-/2)} < \left(1 - \frac{q_-}{2}\right)\Gamma t + (L(0))^{1-(q_-/2)}, \tag{102}$$

yields, which implies that

$$L(t) > \frac{1}{\left((L(0))^{1-(q_-/2)} - ((q_-/2)-1)\Gamma t\right)^{2/(q_- -2)}}, \tag{103}$$

along with $1-(q_-/2) < 0$. Finally, we have

$$L(t) \longrightarrow +\infty \text{ when } t \longrightarrow T^{*-}, \tag{104}$$

where

$$T^* = \frac{L(0)^{1-(q_-/2)}}{((q_-/2)-1)\Gamma}. \tag{105}$$

Then, the proof is now completed.

5. Conclusion

In the present paper, we examined the global nonexistence of solutions for a class of coupled pseudoparabolic equations with weak memories. In fact, the influence of the memory terms are unable to guarantee the stability of our problem. More precisely, we derived some conditions on the functions $p(x), q(x), \varpi_1(t)$, and $\varpi_2(t)$ that could occurs blowing up solutions under conditions (10) and (45) with negative initial data. The novelty of our work is to outlined the effects of weak-memory terms, i.e., it lies primarily in the use of a new relation between the relaxation functions and the exponent of the nonlinear sources to get some necessary conditions, which make the problem very interesting from the application point of view, in particular related to the energy systems (including heat systems).

The damping terms in this paper are composed with weak-viscoelastic terms

$$\sigma(t)\int_0^t \varpi_j(t-s)u(s)\,ds, \tag{106}$$

and strong damping, which let our problem dissipative named pseudoparabolic. The functions $f_j : \mathbb{R}^2 \longrightarrow \mathbb{R}, j = 1, 2$ are given in more complicated nonlinearities. We used a large class of Laplacian operator, with variable exponents.

After restriction on initial data, we found that when the sources dominate, we get a blow-up finite time, even though the existence of the damping terms which well known to ensure the stability of solutions.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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