Research Article

Fekete-Szegő Inequality for Analytic and Biunivalent Functions Subordinate to Gegenbauer Polynomials

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In the present paper, a subclass of analytic and biunivalent functions by means of Gegenbauer polynomials is introduced. Certain coefficients bound for functions belonging to this subclass are obtained. Furthermore, the Fekete-Szegő problem for this subclass is solved. A number of known or new results are shown to follow upon specializing the parameters involved in our main results.

1. Introduction

Orthogonal polynomials have been studied extensively as early as they were discovered by Legendre in 1784 [1]. In mathematical treatment of model problems, orthogonal polynomials arise often to find solutions of ordinary differential equations under certain conditions imposed by the model.

The importance of the orthogonal polynomials for contemporary mathematics, as well as for a wide range of their applications in physics and engineering, is beyond any doubt. It is well-known that these polynomials play an essential role in problems of the approximation theory. They occur in the theory of differential and integral equations as well as in mathematical statistics. Their applications in quantum mechanics, scattering theory, automatic control, signal analysis, and axially symmetric potential theory are also known [2, 3].

Formally speaking, polynomials P_r and P_m of order r and m are orthogonal if

$$\int_a^b \Phi(x) P_r(x) P_m(x) dx = 0 \text{ for } r \neq m, \quad (1)$$

where $\Phi(x)$ is nonnegative function in the interval (a, b) ; therefore, the integral is well-defined for all finite order polynomials $P_r(x)$.

A special case of orthogonal polynomials is Gegenbauer polynomials. They are representatively related with typically real functions T_R as discovered in [4], where the integral representation of typically real functions and generating function of Gegenbauer polynomials are using common algebraic expressions. Undoubtedly, this led to several useful inequalities appear from the Gegenbauer polynomial realm.

Typically, real functions play an important role in the geometric function theory because of the relation $T_R = \bar{c}oS_R$ and its role of estimating coefficient bounds, where S_R denotes the class of univalent functions in the unit disk with real coefficients and $\bar{c}oS_R$ denotes the closed convex hull of S_R .

This paper associates certain biunivalent functions with Gegenbauer polynomials and then explores some properties of the class in hand. Paving the way for mathematical notations and definitions, we provide the following section.

2. Definitions and Preliminaries

Let $\mathcal{A}$ denotes the class of all analytic functions f defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Thus, each $f \in \mathcal{A}$ has a Taylor-Maclaurin series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, (z \in \mathbb{U}). \quad (2)$$

Further, let $\mathcal{S}$ denotes the class of all functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$ (for details, see [5]).

A subordination between two analytic functions f and g is written as $f < g$. Conceptually, the analytic function f is subordinate to g if the image under g contains the image under f . Technically, the analytic function f is subordinate to g if there exists a Schwarz function w with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$; such that

$$f(z) = g(w(z)). \quad (3)$$

Besides, if the function g is univalent in $\mathbb{U}$, then the following equivalence holds:

$$\begin{aligned} f(z) < g(z) \text{ if and only if } f(0) &= g(0) \\ f(\mathbb{U}) &\subset g(\mathbb{U}). \end{aligned} \quad (4)$$

Further on the subordination principle, we refer to [6].

It is well-known that, if $f(z)$ is an univalent analytic function from a domain $\mathbb{D}_1$ onto a domain $\mathbb{D}_2$, then the inverse function $g(z)$ defined by

$$g(f(z)) = z, (z \in \mathbb{D}_1), \quad (5)$$

is an analytic and univalent mapping from $\mathbb{D}_2$ to $\mathbb{D}_1$. Moreover, by the familiar Koebe one-quarter theorem (for details, see [5]), we know that the image of $\mathbb{U}$ under every function $f \in \mathcal{S}$ contains a disk of radius $1/4$.

According to this, every function $f \in \mathcal{S}$ has an inverse map f^{-1} that satisfies the following conditions:

$$\begin{aligned} f^{-1}(f(z)) &= z (z \in \mathbb{U}), \\ f(f^{-1}(w)) &= w \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right). \end{aligned} \quad (6)$$

In fact, the inverse function is given by

$$\begin{aligned} g(w) = f^{-1}(w) &= w - a_2 w^2 + (2a_2^2 - a_3) w^3 \\ &\quad - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots. \end{aligned} \quad (7)$$

A function $f \in \mathcal{A}$ is said to be biunivalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denotes the class of

biunivalent functions in $\mathbb{U}$ given by (2). Examples of functions in the class Σ are

$$\frac{z}{1-z}, -\log(1-z), \frac{1}{2} \log\left(\frac{1+z}{1-z}\right), \dots \quad (8)$$

It is worth noting that the familiar Koebe function is not a member of Σ , since it maps the unit disk $\mathbb{U}$ univalently onto the entire complex plane except the part of the negative real axis from $-1/4$ to $-\infty$. Thus, clearly, the image of the domain does not contain the unit disk $\mathbb{U}$. For a brief history and some intriguing examples of functions and characterization of the class Σ , see [7–15].

In 1967, Lewin [16] investigated the biunivalent function class Σ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [17] conjectured that $|a_2| \leq \sqrt{2}$. On the other hand, Netanyahu [18] showed that $\max_{f \in \Sigma} |a_2| = 4/3$. The

best-known estimate for functions in Σ has been obtained in 1984 by Tan [19], that is, $|a_2| < 1.485$. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ ($n \in \mathbb{N} \setminus \{1, 2\}$) for each $f \in \Sigma$ given by (2) is presumably still an open problem.

The most important and well-investigated subclasses of the analytic and univalent function class $\mathcal{S}$ are the class $\mathcal{S}^*(\varsigma)$ of starlike functions of order ς in $\mathbb{U}$ and the class $\mathcal{K}(\varsigma)$ of convex functions of order ς in $\mathbb{U}$. By definition, we have

$$\begin{aligned} \mathcal{S}^*(\varsigma) &:= \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \varsigma, (z \in \mathbb{U}; 0 \leq \varsigma < 1) \right\}, \\ \mathcal{K}(\varsigma) &:= \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \varsigma, (z \in \mathbb{U}; 0 \leq \varsigma < 1) \right\}. \end{aligned} \quad (9)$$

For $0 \leq \varsigma < 1$, a function $f \in \Sigma$ is in the class $\mathcal{S}_\Sigma^*(\varsigma)$ of bistarlike function of order ς or $\mathcal{K}_\Sigma(\varsigma)$ of biconvex function of order ς if both f and f^{-1} are, respectively, starlike or convex functions of order ς .

Very recently, Amourah [20] considered the Gegenbauer polynomials $H_\alpha(x, z)$, which are given by the following recurrence relation:

For nonzero real constant α , a generating function of Gegenbauer polynomials is defined by

$$H_\alpha(x, z) = \frac{1}{(1 - 2xz + z^2)^\alpha}, \quad (10)$$

where $x \in [-1, 1]$ and $z \in \mathbb{U}$. For fixed x , the function H_α is analytic in $\mathbb{U}$, so it can be expanded in a Taylor series as

$$H_\alpha(x, z) = \sum_{n=0}^{\infty} C_n^\alpha(x) z^n, \quad (11)$$

where $C_n^\alpha(x)$ is Gegenbauer polynomial of degree n .

Obviously, H_α generates nothing when $\alpha = 0$. Therefore, the generating function of the Gegenbauer polynomial is set to be

$$H_0(x, z) = 1 - \log(1 - 2xz + z^2) = \sum_{n=0}^{\infty} C_n^0(x)z^n, \quad (12)$$

for $\alpha = 0$. Moreover, it is worth to mention that a normalization of α to be greater than $-1/2$ is desirable [3, 21]. Gegenbauer polynomials can also be defined by the following recurrence relations:

$$C_n^\alpha(x) = \frac{1}{n} [2x(n + \alpha - 1)C_{n-1}^\alpha(x) - (n + 2\alpha - 2)C_{n-1}^\alpha(x)], \quad (13)$$

with the initial values

$$C_0^\alpha(x) = 1, C_1^\alpha(x) = 2\alpha x \text{ and } C_2^\alpha(x) = 2\alpha(1 + \alpha)x^2 - \alpha. \quad (14)$$

First off, we present some special cases of the polynomials $C_n^\alpha(x)$:

- (1) For $\alpha = 1$, we get the Chebyshev Polynomials
- (2) For $\alpha = 1/2$, we get the Legendre Polynomials

Recently, many researchers have been exploring biunivalent functions associated with orthogonal polynomials, few to mention [22–28]. For Gegenbauer polynomial, as far as we know, there is little work associated with biunivalent functions in the literatures. Initiating an exploration on the properties of biunivalent functions associated with Gegenbauer polynomials is the main goal of this paper. To do so, we take into account, the following definitions.

Definition 1. Let $\lambda \geq 1$, $\mu \geq 0$, $x \in (1/2, 1]$ and α is a nonzero real constant. A function $f \in \Sigma$ given by (2) is said to be in the class $\mathfrak{B}_\Sigma^\mu(\lambda, x, \alpha)$ if the following subordinations are satisfied:

$$(1 - \lambda) \left(\frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} < H_\alpha(x, z), \quad (15)$$

$$(1 - \lambda) \left(\frac{f^{-1}(w)}{w} \right)^\mu + \lambda (f^{-1}(w))' \left(\frac{f^{-1}(w)}{w} \right)^{\mu-1} < H_\alpha(x, w), \quad (16)$$

where the function $f^{-1}(w)$ is defined by (7) and H_α is the generating function of the Gegenbauer polynomial given by (10).

By suitably specializing the parameters μ , λ , and α , the class $\mathfrak{B}_\Sigma^\mu(\lambda, x, \alpha)$ leads to the following new subclasses of biunivalent functions:

Example 1. If $\mu = 1$ and a function $f \in \Sigma$ given by (2) is said to be in the class $\mathfrak{B}_\Sigma(\lambda, x, \alpha)$ if the following subordinations are satisfied:

$$(1 - \lambda) \left(\frac{f(z)}{z} \right) + \lambda f'(z) < \frac{1}{(1 - 2xz + z^2)^\alpha},$$

$$(1 - \lambda) \left(\frac{f^{-1}(w)}{w} \right) + \lambda (f^{-1}(w))' < \frac{1}{(1 - 2xz + z^2)^\alpha}, \quad (17)$$

where $x \in (1/2, 1]$ and the function $f^{-1}(w)$ is defined by (7).

Example 2. If $\lambda = 1$ and a function $f \in \Sigma$ given by (2) is said to be in the class $\mathfrak{B}_\Sigma^\mu(x, \alpha)$ if the following subordinations are satisfied:

$$f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} < \frac{1}{(1 - 2xz + z^2)^\alpha},$$

$$(f^{-1}(w))' \left(\frac{f^{-1}(w)}{w} \right)^{\mu-1} < \frac{1}{(1 - 2xz + z^2)^\alpha}, \quad (18)$$

where $x \in (1/2, 1]$ and the function $f^{-1}(w)$ is defined by (7).

Example 3. If $\lambda = 1$, $\mu = 1$, and a function $f \in \Sigma$ given by (2) is said to be in the class $\mathfrak{B}_\Sigma(x, \alpha)$ if the following subordinations are satisfied:

$$f'(z) < \frac{1}{(1 - 2xz + z^2)^\alpha},$$

$$(f^{-1}(w))' < \frac{1}{(1 - 2xz + z^2)^\alpha}, \quad (19)$$

where $x \in (1/2, 1]$ and the function $f^{-1}(w)$ is defined by (7).

Example 4. If $\lambda = 1$, $\mu = 0$ and a function $f \in \Sigma$ given by (2) is said to be in the class $\mathcal{S}_\Sigma^*(x, \alpha)$ if the following subordinations are satisfied:

$$\frac{zf'(z)}{f(z)} < \frac{1}{(1 - 2xz + z^2)^\alpha},$$

$$\frac{z(f^{-1}(w))'}{f^{-1}(w)} < \frac{1}{(1 - 2xz + z^2)^\alpha}, \quad (20)$$

where $x \in (1/2, 1]$ and the function $f^{-1}(w)$ is defined by (7).

Remark 2. The subclasses $\mathfrak{B}_\Sigma^\mu(\lambda, x, 1) = \mathfrak{B}_\Sigma^\mu(\lambda, x)$ and $\mathfrak{B}_\Sigma^1(\lambda, x, 1) = \mathfrak{B}_\Sigma(\lambda, x)$ were studied by Bulut et al. [29] and Bulut et al. [30], respectively.

In this paper, motivated by recent works of Amourah et al. [20], we use Gegenbauer polynomials to obtain the estimates on the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class $\mathfrak{B}_\Sigma^\mu(\lambda, x, \alpha)$.

Unless otherwise mentioned, we assume in the remainder of this paper that $\lambda \geq 1, \mu \geq 0, x \in (1/2, 1]$, and α is a nonzero real constant.

3. Coefficient Bounds of the Class $\mathfrak{B}_\Sigma^\mu(\lambda, x, \alpha)$

In the following theorem, we determine the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class $\mathfrak{B}_\Sigma^\mu(\lambda, x, \alpha)$.

Theorem 3. *Let $f \in \Sigma$ given by (2) belongs to the class $\mathfrak{B}_\Sigma^\mu(\lambda, x, \alpha)$. Then,*

$$\begin{aligned} |a_2| &\leq \frac{2|\alpha|x\sqrt{2|\alpha|x}}{\sqrt{|\alpha(\mu+\lambda)^2 - 2[\alpha(\mu+\lambda)^2(1+\alpha) - \alpha^2(\mu+2\lambda)(1+\mu)]x^2|}}, \\ |a_3| &\leq \frac{4\alpha^2x^2}{(\mu+\lambda)^2} + \frac{2|\alpha|x}{(\mu+2\lambda)}. \end{aligned} \quad (21)$$

Proof. Let $f \in \mathfrak{B}_\Sigma^\mu(\lambda, x, \alpha)$. From (15) and (16), we have

$$\begin{aligned} (1-\lambda)\left(\frac{f(z)}{z}\right)^\mu + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \\ = 1 + C_1^\alpha(x)c_1z + [C_1^\alpha(x)c_2 + C_2^\alpha(x)c_1^2]z^2 + \dots, \end{aligned} \quad (22)$$

$$\begin{aligned} (1-\lambda)\left(\frac{f^{-1}(w)}{w}\right)^\mu + \lambda (f^{-1}(w))' \left(\frac{f^{-1}(w)}{w}\right)^{\mu-1} \\ = 1 + C_1^\alpha(x)d_1w + [C_1^\alpha(x)d_2 + C_2^\alpha(x)d_1^2]w^2 + \dots, \end{aligned} \quad (23)$$

for some analytic functions

$$\begin{aligned} p(z) &= c_1z + c_2z^2 + c_3z^3 + \dots (z \in \mathbb{U}), \\ q(w) &= d_1w + d_2w^2 + d_3w^3 + \dots (w \in \mathbb{U}), \end{aligned} \quad (24)$$

such that $p(0) = q(0) = 0$ and $|p(z)| < 1, |q(w)| < 1$ for all $z, w \in \mathbb{U}$.

It is fairly well known that if $|p(z)| < 1, (z \in \mathbb{U})$ and $|q(w)| < 1, (w \in \mathbb{U})$, then

$$|c_j| \leq 1 \text{ and } |d_j| \leq 1 \text{ for all } j \in \mathbb{N}. \quad (25)$$

Thus, upon comparing the corresponding coefficients in (22) and (23), we have

$$(\mu+\lambda)a_2 = C_1^\alpha(x)c_1, \quad (26)$$

$$(\mu+2\lambda)\left[\left(\frac{\mu-1}{2}\right)a_2^2 + a_3\right] = C_1^\alpha(x)c_2 + C_2^\alpha(x)c_1^2, \quad (27)$$

$$-(\mu+\lambda)a_2 = C_1^\alpha(x)d_1, \quad (28)$$

$$(\mu+2\lambda)\left[\frac{\mu+3}{2}a_2^2 - a_3\right] = C_1^\alpha(x)d_2 + C_2^\alpha(x)d_1^2. \quad (29)$$

It follows from (26) and (28) that

$$c_1 = -d_1, \quad (30)$$

$$2(\mu+\lambda)^2a_2^2 = [C_1^\alpha(x)]^2(c_1^2 + d_1^2). \quad (31)$$

If we add (27) and (29), we get

$$(\mu+2\lambda)(1+\mu)a_2^2 = C_1^\alpha(x)(c_2 + d_2) + C_2^\alpha(x)(c_1^2 + d_1^2). \quad (32)$$

Substituting the value of $(c_1^2 + d_1^2)$ from (31) in the right-hand side of (32), we deduce that

$$\left[(\mu+2\lambda)(1+\mu) - 2(\mu+\lambda)^2 \frac{C_2^\alpha(x)}{[C_1^\alpha(x)]^2}\right]a_2^2 = C_1^\alpha(x)(c_2 + d_2). \quad (33)$$

Moreover, computations using (23), (25), and (33), we find that

$$|a_2| \leq \frac{2|\alpha|x\sqrt{2|\alpha|x}}{\sqrt{|\alpha(\mu+\lambda)^2 - 2[\alpha(\mu+\lambda)^2(1+\alpha) - \alpha^2(\mu+2\lambda)(1+\mu)]x^2|}}. \quad (34)$$

Moreover, if we subtract (29) from (27), we obtain

$$2(\mu+2\lambda)(a_3 - a_2^2) = C_1^\alpha(x)(c_2 - d_2) + C_2^\alpha(x)(c_1^2 - d_1^2). \quad (35)$$

Then, in view of (14) and (31), Eq. (35) becomes

$$a_3 = \frac{[C_1^\alpha(x)]^2}{2(\mu+\lambda)^2}(c_1^2 + d_1^2) + \frac{C_1^\alpha(x)}{2(\mu+2\lambda)}(c_2 - d_2). \quad (36)$$

Thus, applying (14), we conclude that

$$|a_3| \leq \frac{4\alpha^2x^2}{(\mu+\lambda)^2} + \frac{2|\alpha|x}{(\mu+2\lambda)}. \quad (37)$$

4. Fekete-Szegő Problem for the Function Class $\mathfrak{B}_\Sigma^\mu(\lambda, x, \alpha)$

Fekete-Szegő inequality is one of the famous problems related to coefficients of univalent analytic functions. It was first given by [31], who stated that, if $f \in \Sigma$, then

$$|a_3 - \eta a_2^2| \leq 1 + 2e^{-2\eta/(1-\mu)}. \quad (38)$$

This bound is sharp when η is real.

In this section, we aim to provide Fekete-Szegő inequalities for functions in the class $\mathfrak{B}_{\Sigma}^{\mu}(\lambda, x, \alpha)$. These inequalities are given in the following theorem.

Theorem 4. Let $f \in \Sigma$ given by (2) belongs to the class $\mathfrak{B}_{\Sigma}^{\mu}(\lambda, x, \alpha)$. Then,

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2|\alpha|x}{(\mu+2\lambda)}, \\ \frac{8|\alpha x|^3|1-\eta|}{|\alpha(\mu+\lambda)^2 - 2[\alpha(\mu+\lambda)^2(1+\alpha) - \alpha^2(\mu+2\lambda)(1+\mu)]x^2|}, \end{cases} \quad \begin{cases} |\eta-1| \leq \left| \frac{\alpha(\mu+\lambda)^2 - 2[\alpha(\mu+\lambda)^2(1+\alpha) - \alpha^2(\mu+2\lambda)(1+\mu)]x^2}{4\alpha(\mu+2\lambda)x^2} \right| \\ |\eta-1| \geq \left| \frac{\alpha(\mu+\lambda)^2 - 2[\alpha(\mu+\lambda)^2(1+\alpha) - \alpha^2(\mu+2\lambda)(1+\mu)]x^2}{4\alpha(\mu+2\lambda)x^2} \right|. \end{cases} \quad (39)$$

Proof. From (33) and (35)

$$\begin{aligned} a_3 - \eta a_2^2 &= (1-\eta) \frac{[C_1^{\alpha}(x)]^3(c_2 + d_2)}{[(\mu+2\lambda)[1+\mu][C_1^{\alpha}(x)]^2 - 2(\mu+\lambda)^2 C_2^{\alpha}(x)]} \\ &\quad + \frac{C_1^{\alpha}(x)}{2(\mu+2\lambda)}(c_2 - d_2) \\ &= C_1^{\alpha}(x) \left[\left[h(\eta) + \frac{1}{2(\mu+2\lambda)} \right] c_2 + \left[h(\eta) - \frac{1}{2(\mu+2\lambda)} \right] d_2 \right], \end{aligned} \quad (40)$$

where

$$h(\eta) = \frac{[C_1^{\alpha}(x)]^2(1-\eta)}{[(\mu+2\lambda)(1+\mu)[C_1^{\alpha}(x)]^2 - 2(\mu+\lambda)^2 C_2^{\alpha}(x)]}, \quad (41)$$

Then, in view of (14), we conclude that

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|C_1^{\alpha}(x)|}{(\mu+2\lambda)} & 0 \leq |h(\eta)| \leq \frac{1}{2(\mu+2\lambda)}, \\ 2|C_1^{\alpha}(x)||h(\eta)| & |h(\eta)| \geq \frac{1}{2(\mu+2\lambda)}, \end{cases} \quad (42)$$

which completes the proof of Theorem 4.

5. Corollaries and Consequences

In this section, we apply our main results in order to deduce each of the following new corollaries and consequences.

Corollary 5. Let $f \in \Sigma$ given by (2) belongs to the class $\mathfrak{B}_{\Sigma}^{\mu}(\lambda, x, 1/2)$. Then,

$$\begin{aligned} |a_2| &\leq \frac{x\sqrt{x}}{\sqrt{|1/2(\mu+\lambda)^2 - [3/2(\mu+\lambda)^2 - 1/2(\mu+2\lambda)(1+\mu)]x^2|}}, \\ |a_3| &\leq \frac{x^2}{(\mu+\lambda)^2} + \frac{x}{(\mu+2\lambda)}, \\ |a_3 - \eta a_2^2| &\leq \begin{cases} \frac{x}{(\mu+2\lambda)}, \\ \frac{x^3|1-\eta|}{|1/2(\mu+\lambda)^2 - [3/2(\mu+\lambda)^2 - 1/2(\mu+2\lambda)(1+\mu)]x^2|}, \end{cases} \quad \begin{cases} |\eta-1| \leq \left| \frac{1/2(\mu+\lambda)^2 - [3/2(\mu+\lambda)^2 - 1/2(\mu+2\lambda)(1+\mu)]x^2}{2(\mu+2\lambda)x^2} \right| \\ |\eta-1| \geq \left| \frac{1/2(\mu+\lambda)^2 - [3/2(\mu+\lambda)^2 - 1/2(\mu+2\lambda)(1+\mu)]x^2}{2(\mu+2\lambda)x^2} \right|. \end{cases} \end{aligned} \quad (43)$$

Corollary 6. Let $f \in \Sigma$ given by (2) belongs to the class $\mathfrak{B}_{\Sigma}^1(\lambda, x, \alpha) = \mathfrak{B}_{\Sigma}(\lambda, x, \alpha)$. Then,

$$\begin{aligned}
 |a_2| &\leq \frac{2|\alpha|x\sqrt{2|\alpha|x}}{\sqrt{|\alpha(1+\lambda)^2 - 2[\alpha(1+\lambda)^2(1+\alpha) - 2\alpha^2(1+2\lambda)]x^2|}}, \\
 |a_3| &\leq \frac{4\alpha^2x^2}{(1+\lambda)^2} + \frac{2|\alpha|x}{(1+2\lambda)}, \\
 |a_3 - \eta a_2^2| &\leq \begin{cases} \frac{2|\alpha|x}{(1+2\lambda)}, & |\eta - 1| \leq \left| \frac{\alpha(1+\lambda)^2 - 2[\alpha(1+\lambda)^2(1+\alpha) - 2\alpha^2(1+2\lambda)]x^2}{4\alpha(1+2\lambda)x^2} \right| \\ \frac{8|\alpha x|^3|1-\eta|}{|\alpha(1+\lambda)^2 - 2[\alpha(1+\lambda)^2(1+\alpha) - 2\alpha^2(1+2\lambda)]x^2|}, & |\eta - 1| \geq \left| \frac{\alpha(1+\lambda)^2 - 2[\alpha(1+\lambda)^2(1+\alpha) - 2\alpha^2(1+2\lambda)]x^2}{4\alpha(1+2\lambda)x^2} \right|. \end{cases}
 \end{aligned} \tag{44}$$

Corollary 7. Let $f \in \Sigma$ given by (2) belongs to the class $\mathfrak{B}_{\Sigma}^{\mu}(1, x, \alpha) = \mathfrak{B}_{\Sigma}^{\mu}(x, \alpha)$. Then,

$$\begin{aligned}
 |a_2| &\leq \frac{2|\alpha|x\sqrt{2|\alpha|x}}{\sqrt{|\alpha(\mu+1)^2 - 2[\alpha(\mu+1)^2(1+\alpha) - \alpha^2(\mu+2)(1+\mu)]x^2|}}, \\
 |a_3| &\leq \frac{4\alpha^2x^2}{(\mu+1)^2} + \frac{2|\alpha|x}{(\mu+2)}, \\
 |a_3 - \eta a_2^2| &\leq \begin{cases} \frac{2|\alpha|x}{(\mu+2)}, & |\eta - 1| \leq \left| \frac{\alpha(\mu+1)^2 - 2[\alpha(\mu+1)^2(1+\alpha) - \alpha^2(\mu+2)(1+\mu)]x^2}{4\alpha(\mu+2)x^2} \right| \\ \frac{8|\alpha x|^3|1-\eta|}{|\alpha(\mu+1)^2 - 2[\alpha(\mu+1)^2(1+\alpha) - \alpha^2(\mu+2)(1+\mu)]x^2|}, & |\eta - 1| \geq \left| \frac{\alpha(\mu+1)^2 - 2[\alpha(\mu+1)^2(1+\alpha) - \alpha^2(\mu+2)(1+\mu)]x^2}{4\alpha(\mu+2)x^2} \right|. \end{cases}
 \end{aligned} \tag{45}$$

Corollary 8. Let $f \in \Sigma$ given by (2) belongs to the class $\mathfrak{B}_{\Sigma}^0(1, x, \alpha) = \mathcal{S}_{\Sigma}^*(x, \alpha)$. Then,

$$\begin{aligned}
 |a_2| &\leq \frac{2|\alpha|x\sqrt{2x}}{\sqrt{|1-2(1-\alpha)x^2|}}, \\
 |a_3| &\leq 4\alpha^2x^2 + |\alpha|x, \\
 |a_3 - \eta a_2^2| &\leq \begin{cases} |\alpha|x, & |\eta - 1| \leq \left| \frac{1-2(1-\alpha)x^2}{8x^2} \right| \\ \frac{8\alpha^2|x^3|1-\eta|}{|1-2(1-\alpha)x^2|}, & |\eta - 1| \geq \left| \frac{1-2(1-\alpha)x^2}{8x^2} \right|. \end{cases}
 \end{aligned} \tag{46}$$

Corollary 9. Let $f \in \Sigma$ given by (2) belongs to the class $\mathfrak{B}_{\Sigma}^1(1, x, \alpha) = \mathfrak{B}_{\Sigma}(x, \alpha)$. Then,

$$\begin{aligned}
 |a_2| &\leq \frac{|\alpha|x\sqrt{2x}}{\sqrt{|1-(2-\alpha)x^2|}}, \\
 |a_3| &\leq \alpha^2x^2 + \frac{2|\alpha|x}{3}, \\
 |a_3 - \eta a_2^2| &\leq \begin{cases} \frac{2|\alpha|x}{3}, & |\eta - 1| \leq \left| \frac{1-(2-\alpha)x^2}{3x^2} \right| \\ \frac{2\alpha^2x^3|1-\eta|}{|1-(2-\alpha)x^2|}, & |\eta - 1| \geq \left| \frac{1-(2-\alpha)x^2}{3x^2} \right|. \end{cases}
 \end{aligned} \tag{47}$$

6. Concluding Remark

By taking $\alpha = 1$ and specializing $\mu = 1$ or $\lambda = 1$, one can deduce the above results for various subclasses of Σ studied by Bulut et al. [29] and by Altinkaya and Yalcin [32].

Data Availability

Data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors jointly worked on the results and they read and approved the final manuscript.

References

- [1] A. Legendre, *Recherches sur laattraction des sphéroides homogènes*, vol. 10, Mémoires présentes par divers savants a l'Académie des Sciences de l'Institut de France, Paris, 1785.
- [2] H. Bateman, *Higher Transcendental Functions*, McGraw-Hill, 1953.
- [3] B. Doman, *The Classical Orthogonal Polynomials*, World Scientific, 2015.
- [4] K. Kiepiela, I. Naraniecka, and J. Szynal, "The Gegenbauer polynomials and typically real functions," *Journal of Computational and Applied Mathematics*, vol. 153, no. 1-2, pp. 273–282, 2003.
- [5] P. L. Duren, "Univalent Functions," in *Grundlehren der Mathematischen Wissenschaften, Band 259*, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.
- [6] S. Miller and P. Mocanu, *Differential Subordination: Theory and Applications*, CRC Press, New York, 2000.
- [7] S. Bulut, "Coefficient estimates for a class of analytic and bi-univalent functions," *Novi Sad Journal of Mathematics*, vol. 43, pp. 59–65, 2013.
- [8] B. A. Frasin, "Coefficient bounds for certain classes of bi-univalent functions," *Hacettepe Journal of Mathematics and Statistics*, vol. 43, no. 3, pp. 383–389, 2014.
- [9] B. A. Frasin and M. K. Aouf, "New subclasses of bi-univalent functions," *Applied Mathematics Letters*, vol. 24, no. 9, pp. 1569–1573, 2011.
- [10] I. Aldawish, T. Al-Hawary, and B. A. Frasin, "Subclasses of bi-univalent functions defined by Frasin differential operator," *Mathematics*, vol. 8, no. 5, p. 783, 2020.
- [11] G. Murugusundaramoorthy, N. Magesh, and V. Prameela, "Coefficient bounds for certain subclasses of bi-univalent function," *Abstract and Applied Analysis*, vol. 2013, Article ID 573017, 3 pages, 2013.
- [12] Z. Peng, G. Murugusundaramoorthy, and T. Janani, "Coefficient estimate of biunivalent functions of complex order associated with the Hohlov operator," *Journal of Complex Analysis*, vol. 2014, Article ID 693908, 6 pages, 2014.
- [13] H. M. Srivastava, A. K. Mishra, and P. Gochhayat, "Certain subclasses of analytic and bi-univalent functions," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1188–1192, 2010.
- [14] F. Yousef, B. A. Frasin, and T. Al-Hawary, "Fekete-Szegő inequality for analytic and bi-univalent functions subordinate to Chebyshev polynomials," *Univerzitet u Nišu*, vol. 32, no. 9, pp. 3229–3236, 2018.
- [15] F. Yousef, T. Al-Hawary, and G. Murugusundaramoorthy, "Fekete-Szegő functional problems for some subclasses of bi-univalent functions defined by Frasin differential operator," *Afrika Matematika*, vol. 30, no. 3-4, pp. 495–503, 2019.
- [16] M. Lewin, "On a coefficient problem for bi-univalent functions," *Proceedings of the American Mathematical Society*, vol. 18, no. 1, pp. 63–68, 1967.
- [17] D. A. Brannan and J. G. Clunie, *Aspects of Contemporary Complex Analysis (Proceedings of the NATO Advanced Study Institute Held at the University of Durham, Durham; July 120, 1979)*, Academic Press, New York and London, 1980.
- [18] E. Netanyahu, "The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$," *Archive for Rational Mechanics and Analysis*, vol. 32, no. 2, pp. 100–112, 1969.
- [19] D. L. Tan, "Coefficient estimates for bi-univalent functions," *Chinese Annals of Mathematics. Series*, vol. 5, pp. 559–568, 1984.
- [20] A. Amourah, A. Alamoush, and M. Al-Kaseasbeh, "Gegenbauer polynomials and bi-univalent functions," *Palestine Journal of Mathematics*, 2021.
- [21] M. Reimer, *Multivariate polynomial approximation*, Birkhäuser, 2012.
- [22] A. Amourah, "Initial bounds for analytic and bi-univalent functions by means of (p,q) -Chebyshev polynomials defined by differential operator," *General Letters in Mathematics*, vol. 7, no. 2, pp. 45–51, 2019.
- [23] A. Amourah, "Fekete-Szegő inequality for analytic and bi-univalent functions subordinate to (p,q) -Lucas polynomials," 2020, <https://arxiv.org/abs/2004.00409>.
- [24] A. Amourah, "Faber polynomial coefficient estimates for a class of analytic bi-univalent functions," in *AIP Conference Proceedings*, vol. 2096, Second International Conference of Mathematics (SICME 2019) Erbil, Iraq, 2019, no. 1.
- [25] A. Amourah, T. Al-Hawary, and B. A. Frasin, "Application of Chebyshev polynomials to certain class of bi-Bazilevic functions of order $\alpha+i\beta$," *Afrika Matematika*, vol. 32, pp. 1–8, 2021.
- [26] A. Amourah, B. A. Frasin, G. Murugusundaramoorthy, and T. Al-Hawary, "Bi-Bazilevič functions of order $\vartheta + i\delta$ associated with (p, q) -Lucas polynomials," *AIMS Mathematics*, vol. 6, no. 5, pp. 4296–4305, 2021.
- [27] A. Amourah, "A comprehensive subclass of analytic and bi-univalent functions associated with subordination," *Palestine Journal of Mathematics*, vol. 9, no. 1, pp. 187–193, 2020.
- [28] F. Yousef, S. Alroud, and M. Illafe, "New subclasses of analytic and bi-univalent functions endowed with coefficient estimate problems," *Analysis and Mathematical Physics*, vol. 11, no. 2, p. 58, 2021.
- [29] S. Bulut, N. Magesh, and C. Abirami, "A comprehensive class of analytic bi-univalent functions by means of Chebyshev polynomials," *Journal of Fractional Calculus and Applications*, vol. 8, no. 2, pp. 32–39, 2017.
- [30] S. Bulut, N. Magesh, and V. K. Balaji, "Initial bounds for analytic and bi-univalent functions by means of chebyshev polynomials," *Analysis*, vol. 11, no. 1, pp. 83–89, 2017.
- [31] M. Fekete and G. Szegő, "Eine Bemerkung über ungerade schlichte Funktionen," *Journal of the London Mathematical Society*, vol. 1, no. 2, pp. 85–89, 1933.
- [32] S. Altinkaya and S. Yalcin, "Estimates on coefficients of a general subclass of bi-univalent functions associated with symmetric q -derivative operator by means of the Chebyshev polynomials," *Asia Pacific Journal of Mathematics*, vol. 4, no. 2, pp. 90–99, 2017.