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Research Article

On the System of Coupled Nondegenerate Kirchhoff Equations with Distributed Delay: Global Existence and Exponential Decay

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This paper studies the system of coupled nondegenerate viscoelastic Kirchhoff equations with a distributed delay. By using the energy method and Faedo-Galerkin method, we prove the global existence of solutions. Furthermore, we prove the exponential stability result.

1. Introduction

Let $\mathscr{H} = \Omega \times (\tau_1, \tau_2) \times (0, \infty)$, in this work, we consider

$$\begin{cases} |u|^l u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u - \Delta u_{tt} + \int_0^t g_1(t-s)\Delta u(s)ds - \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)|\Delta u_t(x, t-\varrho)d\varrho + f_1(u, v) = 0, \\ |v|^l v_{tt} - M\left(\|\nabla v\|^2\right)\Delta v - \Delta v_{tt} + \int_0^t g_2(t-s)\Delta v(s)ds - \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\Delta v_t(x, t-\varrho)d\varrho + f_2(u, v) = 0, \end{cases} \tag{1}$$

where

$$(x, \varrho, t) \in \mathscr{H}, \tag{2}$$

under the initial and boundary conditions

$$\begin{cases} (u(x,0), v(x,0)) = (u_0(x), v_0(x)), \text{ in } \Omega, \\ (u_t(x,0), v_t(x,0)) = (u_1(x), v_1(x)), \text{ in } \Omega, \\ (u_t(x,-t), v_t(x,-t)) = (f_0(x,t), g_0(x,t)), \text{ in } \Omega \times (0, \tau_2), \\ u(x,t) = v(x,t) = 0, \text{ in } \partial\Omega \times (0, \infty), \end{cases} \tag{3}$$

where Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $l > 0$ and Δ is the Laplacian operator, and the functions $\mu_1, \mu_2 : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ are bounded, with $0 \le \tau_1 < \tau_2$, and the relaxation functions are denoted by g_1, g_2. The function M is given by

$$\begin{aligned} &M : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \\ &r \mapsto M(r) = a + br^\gamma, \end{aligned} \tag{4}$$

with $a, b > 0$, and $\gamma \ge 1$, and the functions f_1, f_2 will be defined later.

In 1976, Kirchhoff developed an equation describing the vibrations produced by a fixed series at its end, since it is considered a generalization of the d'Alembert equation, and it belongs to the wave equation models. Over time, many researchers and authors addressed these issues and problems with their continuous and rapid development, for example, see [1–4].

As for viscoelasticity, it is possible to delve into the following works for further clarification [3–10].

Also, the time or delay recorded in many natural and physical phenomena, especially problems resulting from vibrations, is an important factor for stability in general. And it has been studied extensively by many authors, including [5–7, 11–21]. Recently, in the presence of the varying delay, Mezouar and Boularrass studied system (1); for more information, see [22]. Based on these works, we in this work expand the results in [22] by adding the term of distributed delay.

We, under appropriate conditions, obtained the global existence of solutions, and we proved the exponential stability result of the system.

And we divided the paper into the following: in the second part, we set out the necessary hypotheses and the main result; in the third part, we prove the global existence of solutions, while in the fourth part, we present our result for exponential stability.

2. Preliminaries

In this section, we set the necessary hypotheses for proving the main result.

We need the following assumptions:

(A1) $g_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, i = 1, 2$ are C^1 functions satisfying

$$g(0) > 0, a - \int_0^\infty g_i(s)ds \geq k > 0, i = 1, 2. \tag{5}$$

(A2) $\exists \xi_i > 0$ satisfying

$$g_i'(t) \leq -\xi_i g_i(t), i = 1, 2, t \geq 0. \tag{6}$$

(A3) The number l satisfying $0 < l \leq \gamma$ and

$$\begin{cases} \leq \dfrac{2}{n-2} \text{ if } n > 2, \\ \gamma < \infty \quad \text{if} \quad n \leq 2. \end{cases} \tag{7}$$

(A4)

$$\begin{cases} f_1(u, v) = a_1 v + b_1 |v|^{q+1} |v|^{p-1} u, \\ f_2(u, v) = a_1 u + b_2 |u|^{q+1} |v|^{p-1} v, \end{cases} \tag{8}$$

where $a_1 > 0, b_1 = (p+1)(p+q), b_2 = (q+1)(p+q)$ such that $p$and q are onjugate $((1/p) + (1/q) = 1), p, q < \gamma - (1/2)$ and satisfy

$$2 \leq p, q \leq \begin{cases} \sqrt{\dfrac{n}{2(n-2)}} & \text{if} \quad n > 2, \\ \infty \quad \text{if} \quad n \leq 2. \end{cases} \tag{9}$$

We set the notations

$$(g \circ \Psi)(t) := \int_0^t g(t-s) \|\Psi(t) - \Psi(s)\|^2 ds. \tag{10}$$

As in [17], we introduce the new variables

$$\begin{cases} u_t(x, t - \varrho\rho) = \mathscr{Z}(x, \rho, \varrho, t), \\ v_t(x, t - \varrho\rho) = \mathscr{Y}(x, \rho, \varrho, t). \end{cases} \tag{11}$$

We have

$$\begin{cases} \varrho \mathscr{Z}_t(x, \rho, \varrho, t) + \mathscr{Z}_\rho(x, \rho, \varrho, t) = 0, \\ u_t(x, t) = \mathscr{Z}(x, 0, \varrho, t), \end{cases} \\ \begin{cases} \varrho \mathscr{Y}_t(x, \rho, \varrho, t) + \mathscr{Y}_\rho(x, \rho, \varrho, t) = 0, \\ v_t(x, t) = \mathscr{Y}(x, 0, \varrho, t). \end{cases} \tag{12}$$

Consequently, problem (1) is equivalent to

$$\begin{cases} |u|^l u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_{tt} + \int_0^t g_1(t-s)\Delta u(s)ds - \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \Delta \mathscr{Z}(x, 1, \varrho, t) d\varrho + f_1(u, v) = 0, \\ |v|^l v_{tt} - M(\|\nabla v\|^2)\Delta v - \Delta v_{tt} + \int_0^t g_2(t-s)\Delta v(s)ds - \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \Delta \mathscr{Y}(x, 1, \varrho, t) d\varrho + f_2(u, v) = 0, \\ \rho \mathscr{Z}_t(x, \rho, \varrho, t) + \mathscr{Z}_\rho(x, \rho, \varrho, t) = 0, \\ \rho \mathscr{Y}_t(x, \rho, \varrho, t) + \mathscr{Y}_\rho(x, \rho, \varrho, t) = 0, \end{cases} \tag{13}$$

where

$$(x,\rho,s,t)\in\Omega\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty), \tag{14}$$

with the initial and boundary conditions

$$\begin{cases}(u(x,0),v(x,0))=(u_0(x),v_0(x)), \text{ in } \Omega,\\ (u_t(x,0),v_t(x,0))=(u_1(x),v_1(x)), \text{ in } \Omega,\\ (u_t(x,-t),v_t(x,-t))=(f_0(x,t),g_0(x,t)), \text{ in } \Omega\times(0,\tau_2),\\ u(x,t)=v(x,t)=0, \text{ in } \partial\Omega\times(0,\infty),\\ \mathscr{Z}(x,\rho,\varrho,0)=f_0(x,\rho\varrho), \text{ in } \Omega\times(0,1)\times(0,\tau_2),\\ \mathscr{Y}(x,\rho,\varrho,0)=g_0(x,\rho\varrho).\end{cases} \tag{15}$$

We need the following lemma.

Lemma 1. *The energy functional E, given by*

$$\begin{aligned}E(t)&=\frac{1}{l+2}\left(\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\right)+\frac{b}{2(\gamma+2)}\\&\cdot\left(\|\nabla u\|^{2(\gamma+2)}+\|\nabla v\|^{2(\gamma+2)}\right)+\frac{1}{2}\left(a-\int_0^t g_1(s)ds\right)\|\nabla u\|^2\\&+\frac{1}{2}\left(a-\int_0^t g_2(s)ds\right)\|\nabla v\|^2+\frac{1}{2}\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)\\&+\frac{1}{2}(g_1\circ\nabla u)(t)+\frac{1}{2}(g_2\circ\nabla v)(t)+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\rho(|\mu_1(\rho)|\\&\cdot\|\nabla\mathscr{Z}\|^2+|\mu_2(\rho)|\|\nabla\mathscr{Y}\|^2)d\rho d\rho+\alpha\int_\Omega uvdx\\&+(p+q)\int_\Omega|u|^{p+1}|v|^{q+1}dx,\end{aligned} \tag{16}$$

satisfies

$$\begin{aligned}E'(t)&\le-\beta\int_{\tau_1}^{\tau_2}\left(|\mu_1(\varrho)|\|\nabla\mathscr{Z}(x,1,\varrho,t)\|^2+|\mu_2(\varrho)|\right.\\&\cdot\left.\|\nabla\mathscr{Y}(x,1,\varrho,t)\|^2\right)d\varrho+\lambda\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)\\&+\frac{1}{2}\left(g_1'\circ\nabla u\right)(t)+\frac{1}{2}\left(g_2'\circ\nabla v\right)(t)-\frac{1}{2}g_1(t)\\&\cdot\|\nabla u(t)\|^2-\frac{1}{2}g_2(t)\|\nabla v(t)\|^2,\end{aligned} \tag{17}$$

where $\beta=((1-\delta_1)/2)>0$,

and $\lambda=\max\{\lambda_1=((\delta_1+1)/2)\int_{\tau_1}^{\tau_2}|\mu_1(\rho)|d\rho,\lambda_2=((\delta_1+1)/2)\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|d\rho\},\delta_1<1$.

Proof. Multiplying equation $(13)_{1,2}$ by u_t, v_t, and we use (15), one gets

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{l+2}\|u_t\|_{l+2}^{l+2}+\frac{b}{2(\gamma+1)}\|\nabla u_t\|^{2(\gamma+1)}+\frac{1}{2}\left(a-\int_0^t g_1(s)ds\right)\right.\\&\left.\cdot\|\nabla u\|^2+\frac{1}{2}\|\nabla u_t\|^2+\frac{1}{2}(g_1\circ\nabla u)(t)\right\}-\frac{1}{2}\left(g_1'\circ\nabla u\right)(t)\\&+\frac{1}{2}g_1(t)\|\nabla u(t)\|^2+\int_\Omega u_t\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\Delta\mathscr{Z}(x,1,\varrho,t)d\varrho dx\\&+\int_\Omega u_tvdx+b_1\int_\Omega u_tu|u|^{p-1}|v|^{q+1}dx+\frac{d}{dt}\left\{\frac{1}{l+2}\|v_t\|_{l+2}^{l+2}\right.\\&+\frac{b}{2(\gamma+1)}\|\nabla v_t\|^{2(\gamma+1)}+\frac{1}{2}\left(a-\int_0^t g_2(s)ds\right)\|\nabla v\|^2\\&\left.+\frac{1}{2}\|\nabla v_t\|^2+\frac{1}{2}(g_2\circ\nabla v)(t)\right\}-\frac{1}{2}\left(g_2'\circ\nabla v\right)(t)\\&+\frac{1}{2}g_2(t)\|\nabla v(t)\|^2+\int_\Omega v_t\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\Delta\mathscr{Y}(x,1,\varrho,t)d\varrho dx\\&+\int_\Omega v_tudx+b_2\int_\Omega v_tv|v|^{p-1}|u|^{q+1}dx.\end{aligned} \tag{18}$$

And multiplying equation $(13)_3$ by $-\Delta\mathscr{Z}|\mu_1(\varrho)|$, and integrating the result over $\Omega\times(0,1)\times(\tau_1,\tau_2)$, one gets

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_1(\varrho)|(\nabla\mathscr{Z})^2d\varrho d\rho dx\\&=-\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\nabla\mathscr{Z}\nabla\mathscr{Z}_\rho d\varrho d\rho dx\\&=-\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\frac{d}{d\rho}(\nabla\mathscr{Z})^2d\varrho d\rho dx\\&=\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\left((\nabla\mathscr{Z}(x,0,\varrho,t))^2-(\nabla\mathscr{Z}(x,1,\varrho,t))^2\right)d\varrho dx\\&=\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|d\rho\int_\Omega|\nabla u_t|^2dx-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\\&\cdot(\nabla\mathscr{Z}(x,1,\varrho,t))^2d\varrho dx=\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|d\varrho\right)\\&\cdot\|\nabla u_t\|^2-\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\|\nabla\mathscr{Z}(x,1,\varrho,t)\|^2d\varrho.\end{aligned} \tag{19}$$

Similarly, multiplying equation $(13)_4$ by $-\Delta\mathscr{Y}\,|\,\mu_2(\rho)\,|$, we find

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)|(\nabla\mathscr{Y})^2d\varrho d\rho dx\\&=\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\right)\|\nabla v_t\|^2\\&-\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\|\nabla\mathscr{Y}(x,1,\varrho,t)\,\|^2d\varrho,\end{aligned} \tag{20}$$

by using the inequalities of Young and Cauchy-Schwartz for $\delta_1 > 0$, we have

$$\begin{aligned}\int_\Omega \nabla u_t \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \nabla \mathscr{Z}(x,1,\varrho,t) d\varrho dx \\ \leq \frac{\delta_1}{2}\left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho\right) \|\nabla u_t\|^2 \\ + \frac{\delta_1}{2}\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \|\nabla \mathscr{Z}(x,1,\varrho,t)\|^2 d\varrho.\end{aligned} \tag{21}$$

Similarly, we get

$$\begin{aligned}\int_\Omega \nabla v_t \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \nabla \mathscr{Y}(x,1,\varrho,t) d\varrho dx \\ \leq \frac{\delta_1}{2}\left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho\right) \|\nabla v_t\|^2 + \frac{\delta_1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \\ \cdot \|\nabla \mathscr{Y}(x,1,\varrho,t)\|^2 d\varrho.\end{aligned} \tag{22}$$

By summing (18)–(20) and using (21) and (22), and choosing δ_1 such that $\delta_1 < 1$, we find (16) and (17). This completes the proof.

3. Global Existence

Theorem 2. *Suppose that (5)–(8) hold. Then, given* $(u_0, v_0) \in (H^2(\Omega) \cap H_0^1(\Omega))^2$, $(u_1, v_1) \in (H_0^1(\Omega))^2$, *and* $(f_0, g_0) \in (H^1(\Omega, (0,1), (\tau_1, \tau_2)))^2$, *there exists a weak solution* $(u, v, \mathscr{Z}, \mathscr{Y})$ *of problem (13)–(15) such that*

$$\begin{aligned}(u, v, \mathscr{Z}, \mathscr{Y}) \in L^\infty(\mathbb{R}_+, \mathscr{H}_1), u_t, v_t \\ \in L^\infty(\mathbb{R}_+, H_0^1(\Omega)), u_{tt}, v_{tt} \\ \in L^2(\mathbb{R}_+, H_0^1(\Omega)),\end{aligned} \tag{23}$$

where

$$\mathscr{H}_1 = (H^2(\Omega) \cap H_0^1(\Omega))^2 \times (H^1(\Omega, (0,1), (\tau_1, \tau_2)))^2. \tag{24}$$

Proof. Let the Galerkin basis $u_j, v_j, \mathscr{Z}_j, \mathscr{Y}_j$, for $n \geq 1$, we set

$$\begin{aligned}W_n = \operatorname{span}\{u_1, u_2, \cdots, u_n\}, \\ K_n = \operatorname{span}\{v_1, v_2, \cdots, v_n\}.\end{aligned} \tag{25}$$

The sequences $\mathscr{Z}_j(x,\tau,p), \mathscr{Y}_j(x,\tau,p)$ are defined for $1 \leq j \leq n$ by

$$\mathscr{Z}_j(x,0,p) = u_j(x), \mathscr{Y}_j(x,0,p) = v_j(x). \tag{26}$$

Then, taking $\mathscr{Z}_j(x,0,p), \mathscr{Y}_j(x,0,p)$ by over $L^2((0,1) \times (0,1) \times (\tau_1, \tau_2))$ and denoting

$$\begin{aligned}Z_n = \operatorname{span}\{\mathscr{Z}_1, \mathscr{Z}_2, \cdots, \mathscr{Z}_n\}, \\ Y_n = \operatorname{span}\{\mathscr{Y}_1, \mathscr{Y}_2, \cdots, \mathscr{Y}_n\}.\end{aligned} \tag{27}$$

Given initial data $u_0, v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1, v_1 \in H_0^1(\Omega)$, and $f_0, g_0 \in L^2((\Omega) \times (0,1) \times (\tau_1, \tau_2))$, we define the approximations

$$\begin{aligned}u_m &= \sum_{j=1}^n g_{jm}(t) u_j(x), \\ v_m &= \sum_{j=1}^n h_{jm}(t) v_j(x), \\ \mathscr{Z}_m &= \sum_{j=1}^n f_{jm}(t) \mathscr{Z}_j(x,\tau,p), \\ \mathscr{Y}_m &= \sum_{j=1}^n k_{jm}(t) \mathscr{Y}_j(x,\tau,p).\end{aligned} \tag{28}$$

It investigates the following problem:

$$\begin{aligned}&(|u_{mt}|^l u_{mtt}, u_j) + M(\|\nabla u_m(t)\|)(\nabla u_m, \nabla u_j) \\ &\quad + (\nabla u_{mtt}, \nabla u_j) + (f_1(u_m, v_m), u_j) \\ &\quad - \int_0^t g_1(t-s)(\nabla u_m(s), \nabla u_j) ds \\ &\quad + \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| (\nabla \mathscr{Z}_m(x,1,\varrho,t), \nabla u_j) d\varrho = 0, \\ &(|v_{mt}|^l v_{mtt}, v_j) + M(\|\nabla v_m(t)\|)(\nabla v_m, \nabla v_j) \\ &\quad + (\nabla v_{mtt}, \nabla v_j) + (f_2(u_m, v_m), v_j) \\ &\quad - \int_0^t g_2(t-s)(\nabla v_m(s), \nabla v_j) ds \\ &\quad + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| (\nabla \mathscr{Y}_m(x,1,\varrho,t), \nabla v_j) d\varrho = 0, \\ &(\varrho \mathscr{Z}_{mt}(x,\rho,\varrho,t), \mathscr{Z}_j) + (\mathscr{Z}_{m\rho}(x,\rho,\varrho,t), \mathscr{Z}_j) = 0, \\ &(\varrho \mathscr{Y}_{mt}(x,\rho,\varrho,t), \mathscr{Y}_j) + (\mathscr{Y}_{m\rho}(x,\rho,\varrho,t), \mathscr{Y}_j) = 0,\end{aligned} \tag{29}$$

with initial conditions

$$\begin{aligned}u_m(0) = u_0^m, u_{mt}(0) = u_1^m, \\ v_m(0) = v_1^m, v_{mt}(0) = v_1^m, \\ \mathscr{Z}_m(0) = \mathscr{Z}_0^m, \mathscr{Y}_m(0) = \mathscr{Y}_0^m,\end{aligned} \tag{30}$$

which satisfies

$$\begin{aligned} u_0^m &\longrightarrow u_0, \text{ in } H^2(\Omega)\cap H_0^1(\Omega),\\ u_1^m &\longrightarrow u_1, \text{ in } H_0^1(\Omega),\\ v_0^m &\longrightarrow v_0, \text{ in } H^2(\Omega)\cap H_0^1(\Omega),\\ v_1^m &\longrightarrow v_1, \text{ in } H_0^1(\Omega),\\ \mathscr{Z}_0^m &\longrightarrow \mathscr{Z}_0, \text{ in } L^2(\Omega\times(0,1)\times(\tau_1,\tau_2)),\\ \mathscr{Y}_0^m &\longrightarrow \mathscr{Y}_0, \text{ in } L^2(\Omega\times(0,1)\times(\tau_1,\tau_2)). \end{aligned} \tag{31}$$

Noting that $(l/(2(l+1))) + (1/(2(l+1))) + (1/2) = 1$, by using Hölder's inequality, we get

$$\begin{aligned} \left(|u_{mt}|^l u_{mtt}, u_j\right) &= \int_\Omega |u_{mt}|^l u_{mtt} u_j dx\\ &\le \left(\int_\Omega |u_{mt}|^{2(l+1)} dx\right)^{1/(2(l+1))}\\ &\quad\cdot \|u_{mtt}\|_{2(l+1)}\|u_j\|_2. \end{aligned} \tag{32}$$

As (8) holds, using the embedding of Sobolev, the terms $(|u_{mt}|^l u_{mtt}, u_j)$ and $(|v_{mt}|^l v_{mtt}, v_j)$ in (29) make sense (see [22]).

First estimate.

As the sequences $u_0^m, v_0^m, u_1^m, v_1^m, \mathscr{Z}_0^m(.,.,0)$ and $\mathscr{Y}_0^m(.,.,0)$ converge and from (17) and Gronwall's lemma, we get $C_1 > 0$ independent of m such that

$$\begin{aligned} E_m(t) &+ \beta\int_{\tau_1}^{\tau_2} \varrho\big(|\mu_1(\varrho)|\|\nabla\mathscr{Z}_m(x,1,\varrho,t)\|^2\\ &+ |\mu_2(\varrho)|\|\nabla\mathscr{Y}_m(x,1,\varrho,t)\|^2\big)d\varrho \le C_1, \end{aligned} \tag{33}$$

where

$$\begin{aligned} E_m(t) &= \frac{1}{l+2}\left(\|u_{mt}\|_{l+2}^{l+2}+\|v_{mt}\|_{l+2}^{l+2}\right) + \frac{b}{2(\gamma+2)}\left(\|\nabla u_m\|^{2(\gamma+2)}\right.\\ &\quad\left.+ \|\nabla v_m\|^{2(\gamma+2)}\right) + \frac{1}{2}\left(a - \int_0^t g_1(s)ds\right)\|\nabla u_m\|^2\\ &\quad+ \frac{1}{2}\left(a - \int_0^t g_2(s)ds\right)\|\nabla v_m\|^2 + \frac{1}{2}\left(\|\nabla u_{mt}\|^2 + \|\nabla v_{mt}\|^2\right)\\ &\quad+ \frac{1}{2}(g_1\circ\nabla u_m)(t) + \frac{1}{2}(g_2\circ\nabla v_m)(t)\\ &\quad+ \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\varrho\big(|\mu_1(\varrho)|\|\nabla\mathscr{Z}_m\|^2 + |\mu_2(\varrho)|\|\nabla\mathscr{Y}_m\|^2\big)d\varrho d\rho\\ &\quad+ \alpha\int_\Omega u_m v_m dx + (p+q)\int_\Omega |u_m|^{p+1}|v_m|^{q+1}dx, \end{aligned} \tag{34}$$

using (33) and (8), one gets

$$\begin{aligned} &u_m, v_m \text{ are bounded in } L^\infty_{\text{loc}}\left(\mathbb{R}_+, H_0^1(\Omega)\right),\\ &u_{mt}, v_{mt} \text{ are bounded in } L^\infty_{\text{loc}}\left(\mathbb{R}_+, H_0^1(\Omega)\right),\\ &\mathscr{Z}_m(x,\rho,\varrho,t), \mathscr{Y}_m(x,\rho,\varrho,t) \text{ are bounded in}\\ &L^\infty_{\text{loc}}\left(\mathbb{R}_+, H_0^1(\Omega\times(0,1)\times(\tau_1,\tau_2))\right). \end{aligned} \tag{35}$$

The second estimate.

We multiply equation $(29)_{1,2}$ by g_{jmtt}, h_{jmtt}; by summing j from 1 to n, one gets

$$\begin{aligned} &\int_\Omega |u_{mt}|^l|u_{mtt}|^2dx + \int_\Omega M(\|\nabla u_m(t)\|)\nabla u_m\nabla u_{mtt}dx\\ &\quad+ \int_\Omega |\nabla u_{mtt}|^2dx + \int_\Omega f_1(u_m,v_m)u_{mtt}dx\\ &\quad- \int_\Omega\int_0^t g_1(t-s)\nabla u_m(s)\nabla u_{mtt}dsdx\\ &\quad+ \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\rho)|\nabla\mathscr{Z}_m(x,1,\rho,t)\nabla u_{mtt}d\rho dx = 0,\\ &\int_\Omega |v_{mt}|^l|v_{mtt}|^2dx + \int_\Omega M(\|\nabla v_m(t)\|)\nabla v_m\nabla v_{mtt}dx\\ &\quad+ \int_\Omega |\nabla v_{mtt}|^2dx + \int_\Omega f_2(u_m,v_m)v_{mtt}dx\\ &\quad- \int_\Omega\int_0^t g_2(t-s)\nabla v_m(s)\nabla v_{mtt}dsdx\\ &\quad+ \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\nabla\mathscr{Y}_m(x,1,\varrho,t)\nabla v_{mtt}d\varrho dx = 0. \end{aligned} \tag{36}$$

By differentiating $(29)_{3,4}$, we get

$$\begin{aligned} &\left(\varrho\mathscr{Z}_{mtt}(x,\rho,\varrho,t) + \mathscr{Z}_{mt\rho}(x,\rho,\varrho,t), \mathscr{Z}_j\right) = 0,\\ &\left(\varrho\mathscr{Y}_{mtt}(x,\rho,\varrho,t) + \mathscr{Y}_{mt\rho}(x,\rho,\varrho,t), \mathscr{Y}_j\right) = 0. \end{aligned} \tag{37}$$

And we multiply $(37)_1$ by $\mathscr{Z}_{jmt}$ and $(37)_2$ by $\mathscr{Y}_{jmt}$; by summing j from 1 to n, we have

$$\begin{aligned} &\frac{1}{2}\varrho\frac{d}{dt}\|\mathscr{Z}_{mt}\|^2 + \frac{1}{2}\frac{d}{d\rho}\|\mathscr{Z}_{mt}\|^2 = 0,\\ &\frac{1}{2}\varrho\frac{d}{dt}\|\mathscr{Y}_{mt}\|^2 + \frac{1}{2}\frac{d}{d\rho}\|\mathscr{Y}_{mt}\|^2 = 0. \end{aligned} \tag{38}$$

Integrating the result (38) over $(0,1)$ with respect to ρ, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1\varrho\|\mathscr{Z}_{mt}\|^2d\rho + \frac{1}{2}\|\mathscr{Z}_{mt}(x,1,\varrho,t)\|^2 - \frac{1}{2}\|u_{mtt}(x,t)\|^2 = 0,\\ &\frac{1}{2}\frac{d}{dt}\int_0^1\varrho\|\mathscr{Y}_{mt}\|^2d\rho + \frac{1}{2}\|\mathscr{Y}_{mt}(x,1,\varrho,t)\|^2 - \frac{1}{2}\|v_{mtt}(x,t)\|^2 = 0. \end{aligned} \tag{39}$$

Summing (36) and (39) and using $M(r) \geq a$, we get

$$\begin{aligned}
&\int_\Omega |u_{mt}|^l |u_{mtt}|^2 dx + \|\nabla u_{mtt}\|^2 \\
&\quad + \frac{1}{2}\frac{d}{dt}\int_0^1 \varrho \|\mathscr{Z}_{mt}\|^2 d\rho + \frac{1}{2}\|\mathscr{Z}_{mt}(x,1,\rho,t)\|^2 \\
&\leq \frac{1}{2}\|u_{mtt}\|^2 - \int_\Omega a\nabla u_m \nabla u_{mtt} dx - \int_\Omega f_1(u_m, v_m) u_{mtt} dx \\
&\quad + \int_\Omega \int_0^t g_1(t-s)\nabla u_m(s)\nabla u_{mtt} ds dx \\
&\quad - \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \nabla \mathscr{Z}_m(x,1,\varrho,t)\nabla u_{mtt} d\varrho dx, \\
&\int_\Omega |v_{mt}|^l |v_{mtt}|^2 dx + \|\nabla v_{mtt}\|^2 \\
&\quad + \frac{1}{2}\frac{d}{dt}\int_0^1 \varrho \|\mathscr{Y}_{mt}\|^2 d\rho + \frac{1}{2}\|\mathscr{Y}_{mt}(x,1,\varrho,t)\|^2 \\
&\leq \frac{1}{2}\|v_{mtt}(x,t)\|^2 - \int_\Omega a\nabla v_m \nabla v_{mtt} dx \\
&\quad - \int_\Omega f_2(u_m, v_m) v_{mtt} dx + \int_\Omega \int_0^t g_2(t-s)\nabla v_m(s)\nabla v_{mtt} ds dx \\
&\quad - \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \nabla \mathscr{Y}_m(x,1,\varrho,t)\nabla v_{mtt} d\rho dx.
\end{aligned} \tag{40}$$

At this point, we estimate the RHS of (40).

Integrating by parts, and using Young's and Poincare' inequalities, one gets

$$\begin{aligned}
\left| -\int_\Omega f_1(u_m, v_m) u_{mtt} dx \right| &\leq \frac{C_*^2 \alpha}{2}\left(\|\nabla u_{mt}\|^2 + \|\nabla v_{mt}\|^2\right) \\
&\quad + \frac{b_1 \eta C_*^{4q(q+1)}}{2}|\Omega|^{\frac{q-1}{2q}}\|\nabla v_m\|^{4q(q+1)} \\
&\quad + \frac{b_1 C_*^{2p^2}}{8\eta}\|\nabla u_m\|^{2p^2} + \frac{b_1 C_*^2}{2}\|\nabla u_{mtt}\|^2.
\end{aligned} \tag{41}$$

Similarly, we get

$$\begin{aligned}
\left| -\int_\Omega f_2(u_m, v_m) v_{mtt} dx \right| &\leq \frac{C_*^2 \alpha}{2}\left(\|\nabla u_{mt}\|^2 + \|\nabla v_{mt}\|^2\right) \\
&\quad + \frac{b_1 \eta C_*^{4p(p+1)}}{2}|\Omega|^{(p-1)/2p}\|\nabla u_m\|^{4p(p+1)} \\
&\quad + \frac{b_1 C_*^{2q^2}}{8\eta}\|\nabla v_m\|^{2q^2} + \frac{b_1 C_*^2}{2}\|\nabla v_{mtt}\|^2.
\end{aligned} \tag{42}$$

And, by using the inequality of Young, we get

$$\begin{aligned}
\left|\int_\Omega a\nabla u_m \nabla u_{mtt} dx\right| &\leq \eta \|\nabla u_{mtt}\|^2 + \frac{a^2}{4\eta}\|\nabla u_m\|^2, \\
\left|\int_\Omega a\nabla v_m \nabla v_{mtt} dx\right| &\leq \eta \|\nabla v_{mtt}\|^2 + \frac{a^2}{4\eta}\|\nabla v_m\|^2,
\end{aligned} \tag{43}$$

we have

$$\begin{aligned}
&\left|\int_\Omega \int_0^t g_1(t-s)\nabla u_m(s)\nabla u_{mtt} ds dx\right| \\
&\quad \leq \eta \|\nabla u_{mtt}\|^2 + \frac{(a-k)g_1(0)}{4\eta}\int_0^t \|\nabla u_m(s)\|^2 ds, \\
&\left|\int_\Omega \int_0^t g_2(t-s)\nabla v_m(s)\nabla v_{mtt} ds dx\right| \\
&\quad \leq \eta \|\nabla v_{mtt}\|^2 + \frac{(a-k)g_2(0)}{4\eta}\int_0^t \|\nabla v_m(s)\|^2 ds.
\end{aligned} \tag{44}$$

Similarly, we get

$$\begin{aligned}
&\left|\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \nabla \mathscr{Z}_m(x,1,\varrho,t)\nabla u_{mtt} d\varrho dx\right| \\
&\quad \leq \eta\lambda_1 \|\nabla u_{mtt}\|^2 + \frac{1}{4\eta}\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \|\nabla \mathscr{Z}_m(x,1,\varrho,t)\|^2 d\varrho \\
&\quad \cdot \left|\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \nabla \mathscr{Y}_m(x,1,\varrho,t)\nabla v_{mtt} d\varrho dx\right| \\
&\quad \leq \eta\lambda_2 \|\nabla v_{mtt}\|^2 + \frac{1}{4\eta}\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \|\nabla \mathscr{Y}_m(x,1,\varrho,t)\|^2 d\varrho,
\end{aligned} \tag{45}$$

substituting (41)–(45) into (40), and using (17), one gets

$$\begin{aligned}
&\int_\Omega |u_{mt}|^l |u_{mtt}|^2 dx + \left(1 - \left\{\eta(\lambda_1 + 2) + \frac{(1+b_1)C_*^2}{2}\right\}\right) \\
&\quad \cdot \|\nabla u_{mtt}\|^2 + \frac{1}{2}\frac{d}{dt}\int_0^1 \varrho \|\mathscr{Z}_{mt}\|^2 d\varrho + \frac{1}{2}\|\mathscr{Z}_{mt}(x,1,\varrho,t)\|^2 \\
&\leq C_2 + \frac{1}{4\eta}(a-k)g_1(0)C_1 T, \\
&\int_\Omega |v_{mt}|^l |v_{mtt}|^2 dx + \left(1 - \left\{\eta(\lambda_2 + 2) + \frac{(1+b_2)C_*^2}{2}\right\}\right) \\
&\quad \cdot \|\nabla v_{mtt}\|^2 + \frac{1}{2}\frac{d}{dt}\int_0^1 \varrho \|\mathscr{Y}_{mt}\|^2 d\varrho + \frac{1}{2}\|\mathscr{Y}_{mt}(x,1,\varrho,t)\|^2 \\
&\leq C_2 + \frac{1}{4\eta}(a-k)g_2(0)C_1 T,
\end{aligned} \tag{46}$$

where $C_2 > 0$ depends on $\eta, \alpha, a, C_*, b_1, b_2, p, q, C_1$.

Integrating (41) over $(0,t)$, we get

$$\begin{aligned}
&\int_0^t\int_\Omega |u_{mt}(\sigma)|^l |u_{mtt}(\sigma)|^2 dx d\sigma \\
&\quad + \left(1 - \left\{\eta(\lambda_1+2) + \frac{(1+b_1)C_*^2}{2}\right\}\right) \\
&\quad \cdot \int_0^t \|\nabla u_{mtt}(\sigma)\|^2 d\sigma + \frac{1}{2}\int_0^1 \varrho \|\mathscr{Z}_{mt}\|^2 d\varrho \\
&\quad + \frac{1}{2}\int_0^t \|\mathscr{Z}_{mt}(x,1,\varrho,\sigma)\|^2 d\sigma \\
&\le \left(C_2 + \frac{1}{4\eta}(a-k)g_1(0)C_1 T\right)T, \\
&\int_0^t\int_\Omega \left|v_{mt}(\sigma)^l\right| v_{mtt}(\sigma)\Big|^2 dx d\sigma \\
&\quad + \left(1 - \left\{\eta(\lambda_2+2) + \frac{(1+b_2)C_*^2}{2}\right\}\right) \\
&\quad \cdot \int_0^t \|\nabla v_{mtt}(\sigma)\|^2 d\sigma + \frac{1}{2}\int_0^1 \varrho \|\mathscr{Y}_{mt}\|^2 d\varrho \\
&\quad + \frac{1}{2}\int_0^t \|\mathscr{Y}_{mt}(x,1,\varrho,\sigma)\|^2 d\sigma \\
&\le \left(C_2 + \frac{1}{4\eta}(a-k)g_2(0)C_1 T\right)T.
\end{aligned} \tag{47}$$

At this stage, choosing $\eta > 0$ such that

$$\left(1 - \left\{\eta(\lambda_i+2) + \frac{(1+b_i)C_*^2}{2}\right\}\right) > 0, \text{ for } i = 1, 2, \tag{48}$$

we find

$$\begin{aligned}
&\int_0^t \left(\|\nabla u_{mtt}(\sigma)\|^2 + \|\nabla v_{mtt}(\sigma)\|^2\right) d\sigma \\
&\quad + \frac{1}{2}\int_0^1 \rho\left(\|\mathscr{Z}_{mt}\|^2 + \|\mathscr{Y}_{mt}\|^2\right) d\rho \le C_3.
\end{aligned} \tag{49}$$

We have from (17) and (49) that there exist subsequences (u_k) of (u_m) and (v_k) of (v_m) such that

$$\begin{aligned}
&(u_k, v_k) \rightharpoonup (u,v) \text{ weakly star in } L^\infty\left(0,T,H_0^1(\Omega)\right), \\
&(u_{kt}, v_{kt}) \rightharpoonup (u_t, v_t) \text{ weakly star in } L^\infty\left(0,T,H_0^1(\Omega)\right), \\
&(u_{ktt}, v_{ktt}) \rightharpoonup (u_{tt}, v_{tt}) \text{ weakly star in } L^2\left(0,T,H_0^1(\Omega)\right), \\
&(\mathscr{Z}_k, \mathscr{Y}_k) \rightharpoonup (\mathscr{Z}, \mathscr{Y}) \text{ weakly star in } \\
&\qquad L^\infty\left(0,T,L^2(\Omega\times(0,1)\times(\tau_1,\tau_2)\right), \\
&(\mathscr{Z}_{kt}, \mathscr{Y}_{kt}) \rightharpoonup (\mathscr{Z}_t, \mathscr{Y}_t) \textit{ weakly star} \text{ in } \\
&\qquad L^\infty\left(0,T,L^2(\Omega\times(0,1)\times(\tau_1,\tau_2)\right).
\end{aligned} \tag{50}$$

We work now with the nonlinear term. From (17), we find

$$\begin{aligned}
\left\| |u_{kt}|^l u_{kt} \right\|_{L^2(0,T,L^2(\Omega))} &= \int_0^T \|u_{kt}\|_{2(l+1)}^{2(l+1)} dt \\
&\le C_*^{2(l+1)} \int_0^T \|u_{kt}\|_2^{2(l+1)} dt \le C_4,
\end{aligned} \tag{51}$$

where C_4 depends only on C_*, C_1, T, l.

And from the theorem of Aubin-Lions (see Lions [23]), we deduce that there exists a subsequence of (u_k), given by (u_k), such that

$$u_{kt} \longrightarrow u_t \text{ stongly in } L^2\left(0,T,L^2(\Omega)\right), \tag{52}$$

we get

$$u_{kt} \longrightarrow u_t \text{ almost everywhere in } \Omega \times \mathbb{R}_+. \tag{53}$$

Hence,

$$|u_{kt}|^l u_{kt} \longrightarrow |u_t|^l u_t \text{ almost everywhere in } \Omega \times \mathbb{R}_+. \tag{54}$$

Thus, using (46) and (48) and the Lions lemma, we derive

$$|u_{kt}|^l u_{kt} \rightharpoonup |u_t|^l u_t \text{ weakly in } L^2\left(0,T,L^2(\Omega)\right). \tag{55}$$

Similarly,

$$|v_{kt}|^l v_{kt} \rightharpoonup |v_t|^l v_t \text{ weakly in } L^2\left(0,T,L^2(\Omega)\right), \tag{56}$$

$$\begin{aligned}
(\mathscr{Z}_k, \mathscr{Y}_k) &\longrightarrow (\mathscr{Z}, \mathscr{Y}) \text{ stongly in} \\
&\quad L^2\left(0,T,L^2(\Omega\times(0,1)\times(\tau_1,\tau_2)\right),
\end{aligned} \tag{57}$$

which implies

$$\begin{aligned}
(\mathscr{Z}_k, \mathscr{Y}_k) &\longrightarrow (z,y) \text{ almost everywhere in } \Omega\times(0,1) \\
&\quad \times (\tau_1,\tau_2) \times \mathbb{R}_+.
\end{aligned} \tag{58}$$

The sequences (u_k) and (v_k) satisfy

$$\begin{aligned}
&f_1(u_k, v_k) \longrightarrow f_1(u,v) \text{ stongly in } L^2\left(0,T,L^2(\Omega)\right), \\
&f_2(u_k, v_k) \longrightarrow f_2(u,v) \text{ stongly in } L^2\left(0,T,L^2(\Omega)\right).
\end{aligned} \tag{59}$$

We have

$$\|f_1(u_k,v_k) - f_1(u,v)\|^2 = \int_\Omega \left| |v_m|^{q+1} |u_m|^p u_m - |v|^{q+1} |u|^p u \right|^2 dx. \tag{60}$$

Noting that $(l/2p) + (1/2q) + (1/2) = 1$, by applying the generalized Hölder's and Young's inequalities, and (8), we get

$$\|f_1(u_k,v_k) - f_1(u,v)\|^2 \le C\left[\|\nabla(u_m-u)\|^2 + \|\nabla(v_m-v)\|^2\right]. \tag{61}$$

As (u_k) and (v_k) are Cauchy sequences in $L^\infty(0, T, H_0^1(\Omega))$ (prove it as in [1]), then we get $(59)_1$. Similarly, we get the convergence $(59)_2$.

Multiplying (29) by $\Psi(t) \in \mathscr{D}(0, T)$ and integrating the result over $(0, T)$, we get

$$\begin{aligned}
&-\frac{1}{l+1}\int_0^T \left(|u_{mt}|^l u_{mtt}, u_j\right)\Psi'(t)dt \\
&+\int_0^T M(\|\nabla u_m(t)\|)(\nabla u_m,\nabla u_j)\Psi(t)dt \\
&+\int_0^T (\nabla u_{mtt},\nabla u_j)\Psi(t)dt + \int_0^T (f_1(u_m, v_m), u_j)\Psi(t)dt \\
&-\int_0^T\int_0^t g_1(t-s)(\nabla u_m(s),\nabla u_j)\Psi(t)dsdt \\
&+\int_0^T\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)|(\nabla \mathscr{Z}_m(x,1,\varrho,t),\nabla u_j)\Psi(t)d\varrho dt = 0, \\
&-\frac{1}{l+1}\int_0^T \left(|v_{mt}|^l v_{mtt}, v_j\right)\Psi'(t)dt \\
&+\int_0^T M(\|\nabla v_m(t)\|)(\nabla v_m,\nabla v_j)\Psi(t)dt \\
&+\int_0^T (\nabla v_{mtt},\nabla v_j)\Psi(t)dt + \int_0^T (f_2(u_m, v_m), v_j)\Psi(t)dt \\
&-\int_0^T\int_0^t g_2(t-s)(\nabla v_m(s),\nabla v_j)\Psi(t)dsdt \\
&+\int_0^T\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|(\nabla \mathscr{Y}_m(x,1,\varrho,t),\nabla v_j)\Psi(t)d\varrho dt = 0, \\
&\int_0^T (\varrho \mathscr{Z}_{mt}(x,\rho,\varrho,t) + \mathscr{Z}_{m\rho}(x,\rho,\varrho,t), \mathscr{Z}_j)\Psi(t)dt = 0, \\
&\int_0^T (\rho \mathscr{Y}_{mt}(x,\rho,\varrho,t) + \mathscr{Y}_{m\rho}(x,\rho,\varrho,t), \mathscr{Y}_j)\Psi(t)dt = 0, \\
&\forall j = 1, \cdots, m.
\end{aligned} \tag{62}$$

We obtain (62) by the convergence of (50), (54), (56), and (59). This completes the proof.

4. Exponential Decay

In this section, the stability result of the system (13)–(15) is proved.

We need the following lemmas.

Lemma 3. *The functional*

$$\begin{aligned}
F_1(t) &:= \frac{1}{l+1}\int_\Omega \left(|u_t|^l u_t u + |v_t|^l v_t v\right)dx \\
&+ \int_\Omega (\nabla u_t \nabla u + \nabla v_t \nabla v)dx,
\end{aligned} \tag{63}$$

satisfies

$$\begin{aligned}
F_1(t) &\le \frac{1}{l+2}\left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}\right) + \frac{1}{2}\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) \\
&+ \left(\frac{(l+1)^{-1}}{l+2}C_*^{l+2} + \frac{c}{2}\right)\left(\|\nabla u\|^{l+2} + \|\nabla v\|^{l+2}\right),
\end{aligned} \tag{64}$$

$$\begin{aligned}
F_1'(t) &\le \frac{1}{l+1}\left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}\right) + \left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) \\
&+ \left\{\varepsilon_1(a-k+\lambda) - k + \left(\frac{b_1+b_2}{2} + \alpha\right)C_*^2\right\} \\
&\cdot \left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + \frac{1}{4\varepsilon_1}\int_{\tau_1}^{\tau_2}\left(|\mu_1(\varrho)|\|\nabla \mathscr{Z}(x,1,\varrho,t)\|^2\right. \\
&\left. + |\mu_2(\varrho)|\|\nabla \mathscr{Y}(x,1,\varrho,t)\|^2\right)d\varrho + \frac{1}{4\varepsilon_1}(g_1\circ\nabla u + g_2\circ\nabla v).
\end{aligned} \tag{65}$$

Proof.

(1) By applying the inequalities of Young and Poincare', we find

$$\begin{aligned}
|F_1(t)| &\le \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2}\|u\|^{l+2} + \frac{1}{l+2}\|v_t\|_{l+2}^{l+2} \\
&+ \frac{(l+1)^{-1}}{l+2}\|v\|^{l+2} + \frac{1}{2}\left(\|\nabla u_t\|^2 + \|\nabla u\|^2\right) \\
&+ \frac{1}{2}\left(\|\nabla v_t\|^2 + \|\nabla v\|^2\right) \le \frac{1}{l+2}\left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}\right) \\
&+ \frac{1}{2}\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) + \left(\frac{(l+1)^{-1}}{l+2}C_*^{l+2} + \frac{c}{2}\right) \\
&\cdot \left(\|\nabla u\|^{l+2} + \|\nabla v\|^{l+2}\right)
\end{aligned} \tag{66}$$

(2) Direct computation using integration by parts, we get

$$\begin{aligned}
F_1'(t) &= \int_\Omega \left(|u_t|^l u_{tt}\right) u dx + \frac{1}{l+1}\|u_t\|_{l+2}^{l+2} + \int_\Omega \left(|v_t|^l v_{tt}\right) v dx \\
&+ \frac{1}{l+1}\|v_t\|_{l+2}^{l+2} - \int_\Omega \Delta u_{tt} u dx + \|\nabla u_t\|^2 \\
&- \int_\Omega \Delta v_{tt} v dx + \|\nabla v_t\|^2 = \frac{1}{l+1}\left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}\right) \\
&- M\left(\|\nabla u\|^2\right)\|\nabla u\|^2 - M\left(\|\nabla v\|^2\right)\|\nabla v\|^2 \\
&+ \int_\Omega \nabla u \int_0^t g_1(t-s)\nabla u(s)dsdx \\
&+ \int_\Omega \nabla v \int_0^t g_2(t-s)\nabla v(s)dsdx
\end{aligned}$$

$$-\int_\Omega \nabla u \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \nabla \mathscr{Z}(x,1,\varrho,t)\, d\varrho\, dx + \|\nabla u_t\|^2 - \int_\Omega \nabla v \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \nabla \mathscr{Y}(x,1,\varrho,t)\, d\varrho\, dx + \|\nabla v_t\|^2 - (b_1+b_2)\int_\Omega |v|^{q+1}|u|^{p+1} dx - 2\alpha \int_\Omega uv\, dx, \tag{67}$$

estimate (65) easily follows by using $M(r) \ge a$, Young's inequality for $\varepsilon_1 > 0$, and (8).

Lemma 4. *The functional*

$$F_2(t) := \int_\Omega \left(\Delta u_t - \frac{1}{l+1}|u_t|^l u_t\right) \int_0^t g_1(t-s)(u(t)-u(s))\, ds\, dx + \int_\Omega \left(\Delta v_t - \frac{1}{l+1}|v_t|^l v_t\right) \int_0^t g_2(t-s)(v(t)-v(s))\, ds\, dx, \tag{68}$$

satisfies

$$F_2(t) \le \frac{1}{l+2}\left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}\right) + \frac{1}{2}\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) + \left(\frac{(l+1)^{-1}}{l+2}(a-k)^{l+2} C_*^{l+2} 2^{2l+1}\right) \cdot \left(\|\nabla u\|^{2(l+1)} + \|\nabla v\|^{2(l+1)}\right) + \frac{1}{2}(a-k) \cdot \left\{1 + \frac{(l+1)^{-1}}{l+2}(a-k)^l C_*^{l+2}\right\}(g_1 \circ \nabla u + g_2 \circ \nabla v), \tag{69}$$

and for any $\varepsilon_2 > 0$,

$$\begin{aligned} F_2'(t) \le{}& \frac{1}{l+1}\left[\left(1 - \int_0^t g_1(s) ds\right)\|u_t\|_{l+2}^{l+2} + \left(1 - \int_0^t g_2(s) ds\right) \cdot \|v_t\|_{l+2}^{l+2}\right] + \left(2\varepsilon_2 (a-k)^2 + \frac{\alpha C_*^2}{2}\right)\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\ &+ \varepsilon_2 \left\{(a-k) + \frac{(l+1)^{-1}}{l+2}(g_1(0))^{l+2} C_*^{l+2} 2^{2(l+1)} + b_2 \frac{C_*^{4(p+1)}}{2} + b_1 \frac{C_*^{2p}}{2}\right\} M\left(\|\nabla u\|^2\right)\|\nabla u\|^2 \\ &+ \varepsilon_2 \left\{(a-k) + \frac{(l+1)^{-1}}{l+2}(g_2(0))^{l+2} C_*^{l+2} 2^{2(l+1)} + b_1 \frac{C_*^{4(q+1)}}{2} + b_2 \frac{C_*^{2q}}{2}\right\} M\left(\|\nabla v\|^2\right)\|\nabla v\|^2 \\ &+ \left(\varepsilon_2 - \int_0^t g_1(s) ds\right)\|\nabla u_t\|^2 + \varepsilon_2 \int_{\tau_1}^{\tau_2} |\mu_1(\rho)| \cdot \|\nabla \mathscr{Z}(x,1,\rho,t)\|^2 d\rho + \left(\varepsilon_2 - \int_0^t g_2(s) ds\right)\|\nabla v_t\|^2 \\ &+ \varepsilon_2 \int_{\tau_1}^{\tau_2} |\mu_2(\rho)| \|\nabla \mathscr{Y}(x,1,\rho,t)\|^2 d\rho + \left\{\frac{M\left(\|\nabla u\|^2\right)}{4\varepsilon_2} + \left(2\varepsilon_2 + \frac{\lambda_1}{4\varepsilon_2} + \frac{\alpha C_*^2}{2}\right)(a-k)\right\} \\ &\cdot (g_1 \circ \nabla u) + \left\{\frac{M\left(\|\nabla v\|^2\right)}{4\varepsilon_2} + \left(2\varepsilon_2 + \frac{\lambda_2}{4\varepsilon_2} + \frac{\alpha C_*^2}{2}\right) \cdot (a-k)\right\}(g_2 \circ \nabla v) - \frac{g_1(0)}{4\varepsilon_2}\left(1 + \frac{(l+1)^{-1}}{l+2} \cdot (g_1(0))^l C_*^{l+2}\right)\left(g_1' \circ \nabla u\right) - \frac{g_2(0)}{4\varepsilon_2} \\ &\cdot \left(1 + \frac{(l+1)^{-1}}{l+2}(g_2(0))^l C_*^{l+2}\right)\left(g_2' \circ \nabla v\right). \end{aligned} \tag{70}$$

Proof.

(1) By using Young's inequality and the conjugate exponents $p' = (l+2)/(l+1)$, $q' = l+2$, and Hölder's inequality, we obtain

$$\left|-\int_\Omega \frac{1}{l+1}|u_t|^l u_t \int_0^t g_1(t-s)(u(t)-u(s))\, ds\, dx\right| \le \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2}\left[(a-k)^{l+1} C_*^{l+2} \cdot \left(2^{2l+1}(a-k)\|\nabla u\|^{2(l+1)} + \frac{1}{2}(g_1 \circ \nabla u)\right)\right], \tag{71}$$

$$\left|-\int_\Omega \nabla u_t \int_0^t g_1(t-s)(\nabla u(t) - \nabla u(s))\, ds\, dx\right| \le \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}(a-k)(g_1 \circ \nabla u_t). \tag{72}$$

Similarly, we get

$$\left|-\int_\Omega \frac{1}{l+1}|v_t|^l v_t \int_0^t g_2(t-s)(v(t)-v(s))\, ds\, dx\right| \le \frac{1}{l+2}\|v_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2}\left[(a-k)^{l+1} C_*^{l+2} \cdot \left(2^{2l+1}(a-k)\|\nabla v\|^{2(l+1)} + \frac{1}{2}(g_2 \circ \nabla v)\right)\right], \tag{73}$$

$$\left|-\int_\Omega \nabla v_t \int_0^t g_2(t-s)(\nabla v(t)-\nabla v(s))dsdx\right| \le \frac{1}{2}\|\nabla v_t\|^2 + \frac{1}{2}(a-k)(g_2\circ\nabla v_t) \tag{74}$$

By combining (71)–(74), we find (69).

(2) By derivation of F_2, and integrating by parts and (15), we find

$$\begin{aligned}
F_2'(t) &= \int_\Omega M(\|\nabla u\|^2)\nabla u \int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))dsdx \\
&- \int_\Omega\int_0^t g_1(t-s)\nabla u(s)ds\int_0^t g_1(t-s) \\
&\cdot (\nabla u(t)-\nabla u(s))dsdx + \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\rho)|\nabla \mathscr{Z}(x,1,\rho,t) \\
&\cdot \left(\int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))ds\right)d\rho dx \\
&+ \int_\Omega f_1(u,v)\int_0^t g_1(t-s)(u(t)-u(s))dsdx \\
&- \int_\Omega \nabla u_t \int_0^t g_1'(t-s)(\nabla u(t)-\nabla u(s))dsdx \\
&- \int_\Omega \frac{1}{l+1}|u_t|^l u_t \int_0^t g_1'(t-s)(u(t)-u(s))dsdx \\
&- \left(\|\nabla u_t\|^2 + \frac{1}{l+1}\|u_t\|_{l+2}^{l+2}\right)\left(\int_0^t g_1(s)ds\right) \\
&+ \int_\Omega M(\|\nabla v\|^2)\nabla v \int_0^t g_2(t-s)(\nabla v(t)-\nabla v(s))dsdx \\
&- \int_\Omega\int_0^t g_2(t-s)\nabla v(s)ds\int_0^t g_2(t-s) \\
&\cdot (\nabla v(t)-\nabla v(s))dsdx + \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|\nabla \mathscr{Y}(x,1,\rho,t) \\
&\cdot \left(\int_0^t g_2(t-s)(\nabla v(t)-\nabla v(s))ds\right)d\rho dx \\
&+ \int_\Omega f_2(u,v)\int_0^t g_2(t-s)(v(t)-v(s))dsdx \\
&- \int_\Omega \nabla v_t \int_0^t g_2'(t-s)(\nabla v(t)-\nabla v(s))dsdx \\
&- \int_\Omega \frac{1}{l+1}|v_t|^l v_t \int_0^t g_2'(t-s)(v(t)-v(s))dsdx \\
&- \left(\|\nabla v_t\|^2 + \frac{1}{l+1}\|v_t\|_{l+2}^{l+2}\right)\left(\int_0^t g_2(s)ds\right)
\end{aligned} \tag{75}$$

Using Young's, Cauchy-Schwarz, Hölder's, and Poincaré's inequalities, and $l \le \gamma$, we obtain (70).

At this point, let us introduce the functional given by

Lemma 5. *The functional*

$$F_3(t) := \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} \rho e^{-\rho p}\left(|\mu_1(\rho)|\mathscr{Z}^2 + |\mu_2(\rho))\mathscr{Y}^2|\right)d\rho d\rho dx, \tag{76}$$

satisfies

$$F_3(t) \le \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} \rho\left(|\mu_1(\rho)|\mathscr{Z}^2 + |\mu_2(\rho))\mathscr{Y}^2|\right)d\rho d\rho dx, \tag{77}$$

$$\begin{aligned}
F_3'(t) &\le -\eta_1\int_0^1\int_{\tau_1}^{\tau_2}\rho\left(|\mu_1(\rho)|\|\mathscr{Z}\|^2 + |\mu_2(\rho)|\|\mathscr{Y}\|^2\right) \\
&\cdot d\rho d\rho + \lambda\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) - \eta_1\int_{\tau_1}^{\tau_2}(|\mu_1(\rho)| \\
&\cdot \|\mathscr{Z}(x,1,\rho,t)\|^2 + |\mu_2(\rho)|\|\mathscr{Y}(x,1,\rho,t)\|^2)d\rho d\rho,
\end{aligned} \tag{78}$$

where $\eta_1 > 0$.

Proof. By derivation of F_3, and using equations $(13)_3$ and $(13)_4$, we get

$$\begin{aligned}
F_3'(t) &= -2\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} e^{-\rho p}|\mu_1(\rho)|\nabla\mathscr{Z}\nabla\mathscr{Z}_\rho(x,\rho,\rho,t)d\rho d\rho dx \\
&- 2\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} e^{-\rho p}|\mu_2(\rho)|\nabla\mathscr{Y}\nabla\mathscr{Y}_\rho(x,\rho,\rho,t)d\rho d\rho dx \\
&= -\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\rho e^{-\rho p}|\mu_1(\rho)|\nabla\mathscr{Z}^2 d\rho d\rho dx \\
&- \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\rho)|\left[e^{-\rho}\nabla\mathscr{Z}^2(x,1,\rho,t) - \nabla\mathscr{Z}^2(x,0,\rho,t)\right] \\
&\cdot d\rho dx - \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\rho e^{-\rho p}|\mu_2(\rho)|\nabla\mathscr{Y}^2 d\rho d\rho dx \\
&- \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|\left[e^{-\rho}\nabla\mathscr{Y}^2(x,1,\rho,t)\right. \\
&\left.- \nabla\mathscr{Y}^2(x,0,\rho,t)\right]d\rho dx.
\end{aligned} \tag{79}$$

Applying the equality $\mathscr{Z}(x,0,\rho,t) = u_t(x,t)$, $\mathscr{Y}(x,0,\rho,t) = v_t(x,t)$, and $e^{-\rho} \le e^{-\rho p} \le 1$, for any $0 < \rho < 1$, we get

$$\begin{aligned}
F_3'(t) &= -\int_0^1\int_{\tau_1}^{\tau_2}\rho e^{-\rho p}\left(|\mu_1(\rho)|\|\nabla\mathscr{Z}\|^2 + |\mu_2(\rho)|\|\nabla\mathscr{Y}\|^2\right) \\
&\cdot d\rho d\rho - \int_{\tau_1}^{\tau_2} e^{-\rho}\left(|\mu_1(\rho)|\|\nabla\mathscr{Z}(x,1,\rho,t)\|^2\right. \\
&\left.+ |\mu_2(\rho)|\|\nabla\mathscr{Y}(x,1,\rho,t)\|^2\right)d\rho \\
&+ \left(\int_{\tau_1}^{\tau_2}|\mu_1(\rho)|d\rho\right)\|\nabla u_t\|^2 + \left(\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|d\rho\right) \\
&\cdot \|\nabla v_t\|^2.
\end{aligned} \tag{80}$$

As $-e^{-\rho}$ is an increasing function, we have $-e^{-\rho} \leq -e^{-\tau_2}$, for any $\rho \in [\tau_1, \tau_2]$.

Then, setting $\eta_1 = e^{-\tau_2}$, we find (78).

Theorem 6. *Assume (5)–(8) hold, then* $\exists \zeta_1, \zeta_2 > 0$ *such that the energy functional (16) satisfies*

$$E(t) \leq \zeta_2 e^{-\zeta_1 t}, \ \forall t \geq t_0. \tag{81}$$

Proof. We define the functional of Lyapunov

$$\mathcal{L}(t) := NE(t) + F_1(t) + N_2 F_2(t) + F_3(t), \tag{82}$$

where $N, N_2 > 0$.

First, if we let

$$\mathcal{K}(t) = F_1(t) + N_2 F_2(t) + F_3(t), \tag{83}$$

then, by (64), (69), and (77), we get

$$|\mathcal{K}(t)| \leq cE(t). \tag{84}$$

Consequently,

$$|\mathcal{K}(t)| = |\mathcal{L}(t) - NE(t)| \leq cE(t), \tag{85}$$

which yields

$$(N-c)E(t) \leq \mathcal{L}(t) \leq (N+c)E(t). \tag{86}$$

By derivation (82) and applying (17), (65), (70), (78), and (6), one gets

$$\begin{aligned}
\mathcal{L}'(t) \leq{}& \frac{1}{l+1}\{(1-h_0)+N_1\}\left[\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}\right] \\
&+ \{\lambda(1+N)+N_1+\varepsilon_2-h_0\}\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) \\
&+ \left\{\varepsilon_2 M_0\left((a-k) + \frac{(l+1)^{-1}}{l+2}(h_2 C_*)^{l+2} 2^{2(l+1)} + R_1\right)\right. \\
&+ N_1\left(\varepsilon_1(a-k+\lambda) - k + \left(\frac{b_1+b_2}{2}+\alpha\right)C_*^2\right) \\
&\left.+ \left(2\varepsilon_2(a-k)^2 + \frac{\alpha C_*^2}{2}\right)\right\}\left[\|\nabla u\|^2 + \|\nabla v\|^2\right] \\
&+ \left\{-\frac{1}{\xi}\left(\frac{M_0}{4\varepsilon_2} + \left(2\varepsilon_2 + \frac{\lambda}{4\varepsilon_2} + \frac{\alpha C_*^2}{2}\right)(a-k) + \frac{N_1}{4\varepsilon_1}\right)\right. \\
&\left.+ \frac{N}{2} - \frac{h_1}{4\varepsilon_2}\left(1 + \frac{(l+1)^{-1}}{l+2}(h_1)^l C_*^{l+2}\right)\right\} \\
&\cdot\left[\left(g_1'\circ\nabla u\right) + \left(g_2'\circ\nabla v\right)\right] - \eta_1\int_0^1\int_{\tau_1}^{\tau_2}\rho\left(|\mu_1(\rho)|\|\mathcal{Z}\|^2\right. \\
&\left.+ |\mu_2(\rho)|\|\mathcal{Y}\|^2\right)d\rho d\rho - \left\{\eta_1 + N\beta - \varepsilon_2 - \frac{N_1}{4\varepsilon_1}\right\} \\
&\cdot\int_{\tau_1}^{\tau_2}\left(|\mu_1(\rho)|\|\mathcal{Z}(x,1,\rho,t)\|^2\right. \\
&\left.+ |\mu_2(\rho)|\|\mathcal{Y}(x,1,\rho,t)\|^2\right)d\rho d\rho,
\end{aligned} \tag{87}$$

where $h_0 = \min\left(\int_0^{t_0} g_1(s)ds, \int_0^{t_0} g_2(s)ds\right)$, $M_0 = \max\left(M(\|\nabla u\|^2), M(\|\nabla v\|^{2\cdot})\right)$, $h_1 = \min\left(g_1(0), g_2(0)\right)$, $h_2 = \max\left(g_1(0), g_2(0)\right)$, $\xi = \max(\xi_1, \xi_2)$, and $R_1 = \min\left(b_1(C_*^{4(q+1)}/2) + b_2(C_*^{4q}/2), b_2(C_*^{4(p+1)}/2) + b_1(C_*^{4p}/2)\right)$.

At this stage, choosing two fixed numbers N, N_1, such that $N - c > 0$, and

$$\begin{aligned}
&h_1 - \lambda(1+N) - N_1 > 0, \\
&\alpha_1 = h_1 - 1 - N_1 > 0,
\end{aligned} \tag{88}$$

we choose ε_2 small enough such that

$$\alpha_2 = h_1 - \lambda(1+N) - N_1 - \varepsilon_2 > 0. \tag{89}$$

After that, we choose ε_1 small enough such that

$$\begin{aligned}
\alpha_3 ={}& \eta_1 + N\beta - \varepsilon_2 - \frac{N_1}{4\varepsilon_1} < 0, \\
\alpha_4 ={}& \left\{-\varepsilon_2 M_0\left((a-k) + \frac{(l+1)^{-1}}{l+2}(h_2 C_*)^{l+2} 2^{2(l+1)} + R_1\right)\right. \\
&+ N_1\left(k - \varepsilon_1(a-k+\lambda) - \left(\frac{b_1+b_2}{2}+\alpha\right)C_*^2\right) \\
&\left.- \left(2\varepsilon_2(a-k)^2 + \frac{\alpha C_*^2}{2}\right)\right\} > 0, \\
\alpha_5 ={}& \left\{\frac{1}{\xi}\left(\frac{N_1}{4\varepsilon_1} + \frac{M_0}{4\varepsilon_2} + \left(2\varepsilon_2 + \frac{\lambda}{4\varepsilon_2} + \frac{\alpha C_*^2}{2}\right)(a-k)\right)\right. \\
&\left.- \frac{N}{2} + \frac{h_1}{4\varepsilon_2}\left(1 + \frac{(l+1)^{-1}}{l+2}(h_1)^l C_*^{l+2}\right)\right\} > 0.
\end{aligned} \tag{90}$$

Thus, we get

$$\begin{aligned}
\mathcal{L}'(t) \leq{}& \frac{-1}{l+1}\alpha_1\left[\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}\right] - \alpha_2\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) \\
&- \alpha_4\left[\|\nabla u\|^2 + \|\nabla v\|^2\right] - \alpha_5\left[\left(g_1'\circ\nabla u\right) + \left(g_2'\circ\nabla v\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&- \eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} \rho\left(|\mu_1(\rho)| \|\mathscr{Z}\|^2 + |\mu_2(\rho)| \|\mathscr{Y}\|^2\right) d\rho d\rho \\
&+ \alpha_3 \int_{\tau_1}^{\tau_2} \left(|\mu_1(\rho)| \|\mathscr{Z}(x, 1, \rho, t)\|^2 + |\mu_2(\rho)|\right. \\
&\left. \cdot \|\mathscr{Y}(x, 1, \rho, t)\|^2\right) d\rho d\rho,
\end{aligned} \tag{91}
$$

$$
c_1 E(t) \le \mathscr{L}(t) \le c_2 E(t),\ \forall t \ge 0, \tag{92}
$$

using (16), estimates (91) and (86), respectively, we get

$$
\mathscr{L}'(t) \le -k_1 E(t) - k_2 E'(t), \forall t \ge t_0, \tag{93}
$$

for some $k_1, k_2, c_1, c_2 > 0$.

By the combination of (93) with (92), we obtain

$$
\mathscr{R}'(t) \le -\lambda_1 \mathscr{R}(t), \tag{94}
$$

where

$$
\mathscr{R}(t) = \mathscr{L}(t) + k_2 E(t) \sim E(t). \tag{95}
$$

Integrating the result (94) over (t_0, t), we find

$$
\mathscr{R}(t) \le \mathscr{R}(t_0) e^{-\zeta_1 (t - t_0)},\ \forall t_0 \ge t. \tag{96}
$$

It follows from (95) that (81) holds. This completes the proof.

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

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