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Research Article

Linearly Independent Solutions and Integral Representations for Certain Quadruple Hypergeometric Function

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Recently, hypergeometric functions of four variables are investigated by Bin-Saad and Younis. In this manuscript, our goal is to initiate a new quadruple hypergeometric function denoted by $X_{84}^{(4)}$, and then, we ensure the existence of solutions of systems of partial differential equations for this function. We also establish some integral representations involving the quadruple hypergeometric function $X_{84}^{(4)}$.

1. Introduction

Special functions, in recent years, are a piece of research that turned out to be very attractive to many scholars, hunting generalisations which are almost always evoked by applications. Hypergeometric functions of several variables have many applications in mathematical physics, statistical sciences, physics, dynamics, quantum mechanics, chemistry, and engineering (see, e.g., [1–7]). Multivariable hypergeometric functions occur in diverse areas of mathematics such as approximation theory, partition theory, representation theory, group theory, mirror symmetry, and algebraic geometry. They possess important properties such as recurrence and explicit relations, summation formulas, symmetric and convolution identities, and algebraic properties. Furthermore, multidimensional hypergeometric functions are used to solve boundary value problems (Dirichlet problem, Neumann problem, and Holmgren problem) for multidimensional degenerate differential equations (see [8–12]).

In [13], Bin-Saad and Younis introduced several integral representations of Euler type and Laplace type for new hypergeometric functions in four variables. The authors, in [14], defined four new quadruple hypergeometric functions, namely, $X_{80}^{(4)}, X_{81}^{(4)}, X_{82}^{(4)}$, and $X_{83}^{(4)}$, and they obtained fractional derivative formulas, integral representations, and operational formulas for these quadruple hypergeometric functions. More recently, Younis et al. [15] introduced and studied further quadruple hypergeometric functions denoted by $X_{85}^{(4)}, X_{86}^{(4)}, \cdots, X_{90}^{(4)}$. Each quadruple function in [13–15] can be expressed as

$$X^{(4)}(\cdot) = \sum_{m,n,p,q=0}^{\infty} \Theta(m,n,p,q) \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1}$$

where $\Theta(m,n,p,q)$ is a sequence of complex parameters and there are twelve parameters in every series $X^{(4)}(\cdot)$

(eight a's and four c's). The 1st, 2nd, 3rd, and 4th parameters in $X^{(4)}(\cdot)$ are connected with the integers m, n, p, and q, respectively. Every repeated parameter in the series $X^{(4)}(\cdot)$ points out a term with double parameters in $\Theta(m, n, p, q)$. Hence, it is possible to form various combinations of indices. It seems that there is no way to establish independently the number of distinct Gaussian hypergeometric series for each arbitrary integer $n \geq 2$ without giving explicitly all such series. Hence, in each situation with $n = 4$, one ought to start with actually building the set like the case $n = 3$ (refer to [16]).

Motivated by the works [13–15], we define here the following quadruple hypergeometric function:

$$\begin{aligned} &X_{84}^{(4)}(\ell_1, \ell_2, \ell_3, \ell_4; j_1, j_2, j_3; x, y, z, t) \\ &\quad = \sum_{m,n,p,q=0}^{\infty} \frac{(\ell_1)_{2m+n}(\ell_2)_{n+p}(\ell_3)_{p+q}(\ell_4)_q}{(j_1)_{m+p}(j_2)_n(j_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!} \\ &\quad\quad \cdot \left(|x| < \frac{1}{4}, |y| < 1 < |z| < 1 < |t| < 1\right), \end{aligned} \tag{2}$$

where $(\ell)_n$ is the well-known Pochhammer symbol given as

$$(\ell)_n := \begin{cases} 1, & n = 0, \\ \ell(\ell+1)\cdots(\ell+n-1), & n \in \mathbb{N} := \{1, 2, \cdots\}. \end{cases} \tag{3}$$

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}^-$, and $\mathbb{C}$ denote the sets of positive integers, negative integers, and complex numbers, respectively. Also,

$$\begin{aligned} \mathbb{N}_0 &:= \mathbb{N} \cup \{0\}, \\ \mathbb{Z}_0^- &:= \mathbb{Z}^- \cup \{0\}. \end{aligned} \tag{4}$$

Recently, various interesting hypergeometric functions in several variables have been investigated by many authors (see, e.g., [17–24]). In Section 2, we show how to find the linearly independent solutions of partial differential equations satisfied by the function $X_{84}^{(4)}$. Section 3 is aimed at presenting some integral representations of Euler type for our quadruple function.

2. Solving Systems of Partial Differential Equations

Following the theory of multiple hypergeometric functions [25], the system of partial differential equations for the quadruple hypergeometric function $X_{84}^{(4)}$ is given as follows:

$$\begin{cases} \left(j_1 + x\dfrac{\partial}{\partial x} + z\dfrac{\partial}{\partial z}\right)\left(x\dfrac{\partial}{\partial x} + 1\right)x^{-1}u - \left(\ell_1 + 2x\dfrac{\partial}{\partial x} + 2y\dfrac{\partial}{\partial y} + 1\right)\left(\ell_1 + 2x\dfrac{\partial}{\partial x} + 2y\dfrac{\partial}{\partial y}\right)u = 0, \\ \left(j_2 + y\dfrac{\partial}{\partial y}\right)\left(y\dfrac{\partial}{\partial y} + 1\right)y^{-1}u - \left(\ell_1 + 2x\dfrac{\partial}{\partial x} + 2y\dfrac{\partial}{\partial y}\right)\left(\ell_2 + 2x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}\right)u = 0, \\ \left(j_1 + x\dfrac{\partial}{\partial x} + z\dfrac{\partial}{\partial z}\right)\left(z\dfrac{\partial}{\partial z} + 1\right)z^{-1}u - \left(\ell_2 + y\dfrac{\partial}{\partial y} + z\dfrac{\partial}{\partial z}\right)\left(\ell_3 + z\dfrac{\partial}{\partial z} + t\dfrac{\partial}{\partial t}\right)u = 0, \\ \left(j_3 + t\dfrac{\partial}{\partial t}\right)\left(t\dfrac{\partial}{\partial t} + 1\right)t^{-1}u - \left(\ell_3 + z\dfrac{\partial}{\partial z} + t\dfrac{\partial}{\partial t}\right)\left(\ell_4 + t\dfrac{\partial}{\partial t}\right)u = 0, \end{cases} \tag{5}$$

where $u = X_{84}^{(4)}(\ell_1, \ell_2, \ell_3, \ell_4; j_1, j_2, j_3; x, y, z, t)$.

Starting from (5) and by making use of some elementary calculations, we define the system of second-order partial differential equations:

$$\begin{cases} x(1-4x)u_{xx} - 4xyu_{xy} + zu_{xz} - y^2u_{yy} + [j_1 - 2(2\ell_1 + 3)x]u_x - 2(\ell_1 + 1)yu_y - \ell_1(\ell_1 + 1)u = 0, \\ y(1-y)u_{yy} - 2xyu_{xy} - 2xzu_{xz} + yzu_{xz} - 2\ell_2 xu_x + [j_2 - (\ell_1 + \ell_2 + 1)y]u_y - \ell_1 zu_z - \ell_1\ell_2 u = 0, \\ z(1-z)u_{zz} - xu_{xz} - yzu_{yz} + ytu_{yt} - ztu_{zt} - \ell_3 yu_y + [j_1 - (\ell_3 + \ell_2 + 1)z]u_z - \ell_2 tu_t - \ell_2\ell_3 u = 0, \\ t(1-t)u_{tt} - ztu_{zt} - \ell_4 zu_z + [j_3 - (\ell_3 + \ell_4 + 1)t]u_t - \ell_3\ell_4 u = 0. \end{cases} \tag{6}$$

It is noted that four equations of the system (6) are simultaneous. In fact, the hypergeometric function $X_{84}^{(4)}$ verifies the system. To find the linearly independent solutions of system (6), we will search the solutions in the form

$$u = x^{\alpha} y^{\beta} z^{\gamma} t^{\delta} w, \tag{7}$$

where w is an unknown function and α, β, γ, and δ are constants, which are to be determined. So, substituting $u = x^{\alpha} y^{\beta} z^{\gamma} t^{\delta} w$ into the system (6), we get

$$\begin{cases} x(1-4x)w_{xx} - 4xyw_{xy} + zw_{xz} - y^2 w_{yy} + \{j_1 + \gamma + 2\alpha - 2[2(l_1 + 2\alpha + \beta) + 3]x\}w_x - 2(\ell_1 + 2\alpha + \beta + 1)yw_y + \alpha x^{-1} z w_z - \left[-\alpha(j_1 + \alpha + \gamma - 1)x^{-1} + (\ell_1 + 2\alpha + \beta)(\ell_1 + 2\alpha + \beta + 1)\right]w = 0, \\ y(1-y)w_{yy} - 2xyw_{xy} - 2xzw_{xz} - yzw_{yz} - 2(\ell_2 + \alpha + \beta)xw_x + \{j_2 + 2\beta - [(\ell_1 + 2\alpha + \beta) + (\ell_2 + \beta + \gamma) + 1]y\}w_y - (\ell_1 + 2\alpha + \beta)zw_z - \left[-\beta(j_2 + \beta - 1)y^{-1} + (\ell_1 + 2\alpha + \beta)(\ell_2 + \beta + \gamma)\right]w = 0, \\ z(1-z)w_{zz} + xw_{xz} - yzw_{yz} - ytw_{yt} - ztw_{zt} + \gamma x z^{-1} w_x - (\ell_3 + \gamma + \delta)yw_y + \{j_1 + \alpha + 2\gamma - [(\ell_2 + \beta + \gamma) + (\ell_3 + \gamma + \delta) + 1]z\}w_z - (\ell_2 + \beta + \gamma)tw_t - \left\{-\gamma(j_1 + \alpha + \gamma - 1)z^{-1} + (\ell_2 + \beta + \gamma)(\ell_3 + \gamma + \delta)\right\}w = 0, \\ t(1-t)w_{tt} - ztw_{zt} - (\ell_4 + \delta)zw_z + \{j_3 + 2\delta - [(\ell_3 + \gamma + \delta) + (\ell_4 + \delta) + 1]t\}w_t - \left[-\delta(j_3 + \delta - 1)t^{1} + (\ell_3 + \gamma + \delta)(\ell_4 + \delta)\right]w = 0. \end{cases} \tag{8}$$

Systems (8) and (6) have the same structure and can therefore be approached with similar techniques. System (8) implies

$$\begin{cases} \alpha = 0, \\ \beta(\beta + j_2 - 1) = 0, \\ \gamma = 0, \\ \delta(\delta + j_3 - 1) = 0. \end{cases} \tag{9}$$

Therefore, system (9) has the following solutions:

$$\begin{array}{lcccc} & 1 & 2 & 3 & 4 \\ \alpha \coloneqq & 0 & 0 & 0 & 0 \\ \beta \coloneqq & 0 & 1 - j^2 & 0 & 1 - j^2 \\ \gamma \coloneqq & 0 & 0 & 0 & 0 \\ \delta \coloneqq & 0 & 0 & 1 - j^3 & 1 - j^3 \end{array} . \tag{10}$$

Finally, substituting two solutions of the system (10) into (8), we find the following linearly independent solutions of the system (6) at the origin:

$$\begin{aligned} u_1(x, y, z, t) &= X_{84}^{(4)}(\ell_1, \ell_2, \ell_3, \ell_4; j_1, j_2, j_3; x, y, z, t), \\ u_2(x, y, z, t) &= y^{1-j_2} X_{84}^{(4)}(\ell_1 + 1 - j_2, \ell_2 + 1 - j_2, \ell_3, \ell_4; j_1, 2 - j_2, j_3; x, y, z, t), \\ u_3(x, y, z, t) &= t^{1-j_3} X_{84}^{(4)}(\ell_1, \ell_2, \ell_3 + 1 - j_3, \ell_4 + 1 - j_3; j_1, j_2, 2 - j_3; x, y, z, t), \\ u_4(x, y, z, t) &= y^{1-j_2} t^{1-j_3} X_{84}^{(4)}(\ell_1 + 1 - j_2, \ell_2 + 1 - j_2, \ell_3 + 1 - j_3, \ell_4 + 1 - j_3; j_1, 2 - j_2, 2 - j_3; x, y, z, t). \end{aligned} \tag{11}$$

3. Integral Representations of Euler Type

Here, we give eight integral representations of Euler type for $X_{84}^{(4)}$ whose kernel contains the Gaussian hypergeometric function ${}_2F_1$ (see [16]), Appell function F_3 (see for details [16, 25]), the Exton triple functions X_{16}, X_{17}, X_{19} [26], Lauricella's function of three variables F_N [16], and the quadruple functions $X_4^{(4)}, X_{24}^{(4)}$ (see [20, 21]):

$$\begin{aligned} & X_{84}^{(4)}(\ell_1, \ell_2, \ell_3, \ell_4; j_1, j_2, j_3; x, y, z, t) \\ &= \frac{\Gamma(j_2)\Gamma(j_3)}{\Gamma(\ell_2)\Gamma(\ell_4)\Gamma(j_2 - \ell_2)\Gamma(j_3 - \ell_4)} \cdot \int_0^{\infty}\int_0^{\infty} (e^{-\alpha})^{\ell_2} (1 - e^{-\alpha})^{j_2 - \ell_2 - 1} \left(e^{-\beta}\right)^{\ell_4} \\ &\quad \times \left(1 - e^{-\beta}\right)^{j_3 - \ell_4 - 1} (1 - ye^{-\alpha})^{-\ell_1} \left(1 - te^{-\beta}\right)^{-\ell_3} \\ &\quad \times F_3\left(\frac{\ell_1}{2}, 1 + \ell_2 - j_2, \frac{\ell_1 + 1}{2}, \ell_3; j_1; \frac{4x}{(1 - ye^{-\alpha})^2}, \frac{-ze^{-\alpha}}{(1 - e^{-\alpha})(1 - te^{-\beta})}\right) d\alpha d\beta \, (\mathfrak{R}(\ell_2) \\ &\quad > 0, \mathfrak{R}(\ell_4) > 0, \mathfrak{R}(j_2 - \ell_2) > 0, \mathfrak{R}(j_3 - \ell_4) > 0), \end{aligned} \tag{12}$$

$$\begin{aligned} & X_{84}^{(4)}(\ell_1, \ell_2, \ell_3, \ell_4; j_1, j_2, j_3; x, y, z, t) \\ &= \frac{2M^{\ell_2}\Gamma(j_2)}{\Gamma(\ell_2)\Gamma(j_2 - \ell_2)} \int_0^{\infty} \frac{\cosh\alpha\left(\sinh^2\alpha\right)^{\ell_2 - 1/2}}{\left(1 + M\sinh^2\alpha\right)^{j_2 - \ell_1}} \\ &\quad \times \left[\left(1 + M\sinh^2\alpha\right) - My\sinh^2\alpha\right]^{-\ell_1} F_N \\ &\quad \cdot \left(\ell_4, \frac{\ell_1}{2}, 1 + \ell_2 - j_2, \ell_3, \frac{\ell_1 + 1}{2}, \ell_3; j_3, j_1, j_1; \cdot t, \frac{4x\left(1 + M\sinh^2\alpha\right)^2}{\left[\left(1 + M\sinh^2\alpha\right) - My\sinh^2\alpha\right]^2}, -zM\sinh^2\alpha\right) \\ &\quad \cdot d\alpha \, (\mathfrak{R}(\ell_2) > 0, \mathfrak{R}(j_2 - \ell_2) > 0, M > 0), \end{aligned} \tag{13}$$

$$\begin{aligned}&X_{84}^{(4)}(\ell_1,\ell_2,\ell_3,\ell_4;j_1,j_2,j_3;x,y,z,t)\\&\quad=\frac{2(1+M)^{\ell_1}\Gamma(j_2)}{\Gamma(\ell_1)\Gamma(j_2-\ell_1)}\int_0^{\pi/2}\frac{(\sin^2\alpha)^{\ell_1-1/2}(\cos^2\alpha)^{j_2-\ell_1-1/2}}{(1+M\sin^2\alpha)^{j_2-\ell_2}}\\&\qquad\times\left[(1+M\sin^2\alpha)-(1+M)y\sin^2\alpha\right]^{-\ell_2}F_N\\&\qquad\cdot\left(\ell_4,\frac{1+\ell_1-j_2}{2},\ell_2,\ell_3,\frac{\ell_1-j_2}{2}\right.\\&\qquad+1,\ell_3;j_3,j_1,j_1;t,4x(1+M)^2\tan^4\alpha,\\&\qquad\left.\cdot\frac{z(1+M\sin^2\alpha)}{\left[(1+M\sin^2\alpha)-(1+M)y\sin^2\alpha\right]}\right)\\&\qquad\cdot d\alpha\,(\Re(\ell_1)>0,\Re(j_2-\ell_1)>0,M>-1),\end{aligned}\tag{14}$$

$$\begin{aligned}&X_{84}^{(4)}(\ell_1,\ell_2,\ell_3,\ell_4;j_1,j_2,j_3;x,y,z,t)\\&\quad=\frac{2M^{j_3-\ell_3}\Gamma(j_3)}{\Gamma(\ell_3)\Gamma(j_3-\ell_3)}\int_0^{\pi/2}\frac{(\sin^2\alpha)^{j_3-\ell_3-1/2}(\cos^2\alpha)^{\ell_3-1/2}}{(\cos^2\alpha+M\sin^2\alpha)^{j_3-\ell_4}}\\&\qquad\times\left[(\cos^2\alpha+M\sin^2\alpha)-t\cos^2\alpha\right]^{-\ell_4}X_{16}\\&\qquad\cdot\left(\ell_1,\ell_2,1+\ell_3-j_3;j_1,j_2;x,y,-\frac{z\cot^2\alpha}{M}\right)d\alpha\\&\qquad\cdot(\Re(\ell_3)>0,\Re(j_3-\ell_3)>0,M>-1),\end{aligned}\tag{15}$$

$$\begin{aligned}&X_{84}^{(4)}(\ell_1,\ell_2,\ell_3,\ell_4;j_1,j_2,j_3;x,y,z,t)\\&\quad=\frac{\Gamma(\ell_3+\ell_4)\Gamma(j_2)}{\Gamma(\ell_1)\Gamma(\ell_3)\Gamma(\ell_4)\Gamma(j_2-\ell_1)(W_1-V_1)^{j_2-\ell_2-1}(W_2-V_2)^{\ell_3+\ell_4-1}}\\&\qquad\cdot\int_{V_1}^{W_1}\int_{V_2}^{W_2}\times(\alpha-V_1)^{\ell_1-1}(W_1-\alpha)^{j_2-\ell_1-1}(\beta-V_2)^{\ell_3-1}\\&\qquad\cdot(W_2-\beta)^{\ell_4-1}\times[(W_1-V_1)-(\alpha-V_1)y]^{-\ell_2}X_{19}\\&\qquad\cdot\left(\ell_3+\ell_4,\ell_2,\frac{1+\ell_1-j_2}{2},\frac{\ell_1-j_2}{2}+1;j_3,j_1;\right.\\&\qquad\left.\cdot\frac{(\beta-V_2)(W_2-\beta)t}{(W_2-V_2)^2},\frac{(\beta-V_2)z}{(W_2-V_2)},4\left(\frac{\alpha-V_1}{W_1-\alpha}\right)^2x\right)\\&\qquad\cdot d\alpha d\beta\,(\Re(\ell_1)>0,\Re(\ell_3)>0,\Re(\ell_4)>0,\Re(j_2-\ell_1)\\&\quad>0,V_1<W_1,V_2<W_2),\end{aligned}\tag{16}$$

$$\begin{aligned}&X_{84}^{(4)}(\ell_1,\ell_2,\ell_3,\ell_4;j_1,j_2,j_3;x,y,z,t)\\&\quad=\frac{\Gamma(\ell_3+b)\Gamma(j_1)}{2^{\ell_3+b+j_1-2}\Gamma(\ell_3)\Gamma(a)\Gamma(b)\Gamma(j_1-a)}\int_{-1}^1\int_{-1}^1(1+\alpha)^{\ell_3-1}\\&\qquad\cdot(1-\alpha)^{b-1}\times(1+\beta)^{a-1}(1-\beta)^{j_1-a-1}X_{17}\left(\ell_1,\ell_2,\ell_3\right.\\&\qquad\left.+b;a,j_2,j_1-a;\frac{(1+\beta)x}{2},y,\frac{(1+\alpha)(1-\beta)z}{4}\right)_2\\&\qquad\cdot F_1\left(\ell_4,1-b;j_3,\left(\frac{1+\alpha}{\alpha-1}\right)t\right)d\alpha d\beta\,(\Re(\ell_3)\\&\quad>0,\Re(a)>0,\Re(b)>0,\Re(j_1-a)>0),\end{aligned}\tag{17}$$

$$\begin{aligned}&X_{84}^{(4)}(\ell_1,\ell_2,\ell_3,\ell_4;j_1,j_2,j_3;x,y,z,t)\\&\quad=\frac{4\Gamma(\ell_1+\ell_4)\Gamma(\ell_2+\ell_3)}{\Gamma(\ell_1)\Gamma(\ell_2)\Gamma(\ell_3)\Gamma(\ell_4)}\int_0^{\pi/2}\int_0^{\pi/2}(\sin^2\alpha)^{\ell_1-1/2}\\&\qquad\cdot(\cos^2\alpha)^{\ell_4-1/2}(\sin^2\beta)^{\ell_2-1/2}\times(\cos^2\beta)^{\ell_3-1/2}X_4^{(4)}\\&\qquad\cdot\left(\ell_1,\ell_1,\ell_2,\ell_1,\ell_1,\ell_2,\ell_2,\ell_2;j_1,j_2,j_1,j_3;\right.\\&\qquad\left.\cdot x\sin^4\alpha,y\sin^2\alpha\sin^2\beta,\frac{z\sin^2 2\beta}{4},t\cos^2\alpha\cos^2\beta\right)\\&\qquad\cdot d\alpha d\beta\,(\Re(\ell_i)>0\,(i=1,2,3,4)),\end{aligned}\tag{18}$$

$$\begin{aligned}&X_{84}^{(4)}(\ell_1,\ell_2,\ell_3,\ell_4;j_1,j_2,j_3;x,y,z,t)\\&\quad=\frac{(1+M)^{\ell_1}\Gamma(\ell_1+\ell_4)}{\Gamma(\ell_1)\Gamma(\ell_4)}\int_0^1\frac{\alpha^{\ell_1-1}(1-\alpha)^{\ell_4-1}}{(1+M\alpha)^{\ell_1+\ell_4}}\times X_{24}^{(4)}\\&\qquad\cdot\left(\ell_1,\ell_1,\ell_2,\ell_1,\ell_1,\ell_2,\ell_3,\ell_4;j_1,j_2,j_1,j_3;\right.\\&\qquad\left.\cdot\frac{(1+M)^2\alpha^2x}{(1+M\alpha)^2},\frac{(1+M)\alpha y}{(1+M\alpha)},z,\frac{(1-\alpha)t}{(1+M\alpha)}\right)\\&\qquad\cdot d\alpha\,(\Re(\ell_1)>0,\Re(\ell_4)>0,M>-1),\end{aligned}\tag{19}$$

where the Gaussian hypergeometric function ${}_2F_1$, Appell function F_3, Lauricella triple hypergeometric function F_N, Exton hypergeometric functions X_{16}, X_{17}, X_{19}, and the quadruple functions $X_4^{(4)}, X_{24}^{(4)}$ are defined, respectively, by

$${}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}\;(|x|<1),\tag{20}$$

$$\begin{aligned}&F_3(a,b,c,d;e;x,y)\\&\quad=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_n(c)_m(d)_n}{(e)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}\;(\max\{|x|,|y|\}<1),\end{aligned}\tag{21}$$

$$\begin{aligned}&F_N(\ell_1,\ell_2,\ell_3,b_1,b_2,b_1;j_1,j_2,j_2;x,y,z)\\&\quad=\sum_{m,n,p=0}^{\infty}\frac{(\ell_1)_m(\ell_2)_n(\ell_3)_p(b_1)_{m+p}(b_2)_n}{(j_1)_m(j_2)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p}\\&\qquad\cdot(r+s<1\wedge v<1,|x|\le r,|y|\le s,|z|\le v),\end{aligned}\tag{22}$$

$$\begin{aligned}&X_{16}(\ell_1,\ell_2,\ell_3;j_1,j_2;x,y,z)\\&\quad=\sum_{m,n,p=0}^{\infty}\frac{(\ell_1)_{2m+n}(\ell_2)_{n+p}(\ell_3)_p}{(j_1)_{m+p}(j_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\\&\qquad\cdot\left(s<1\wedge v\le T^v(s)\wedge r<\frac{1}{4}(1-s)^2\right)\\&\quad\cup\left(s<1\wedge T^v(s)<v<1-s\wedge r<\frac{v}{1-v}\ell_2\left(\frac{s}{1-v}\right)\right),\end{aligned}\tag{23}$$

$$T^{v}(s)=\begin{cases}\left(\dfrac{1-s}{1-1/2s}\right)^{2}, & 0<s<\dfrac{2}{3},\\ \dfrac{1-s}{2s}, & \dfrac{2}{3}\le s<1\end{cases}\quad(|x|\le r,|y|\le s,|z|\le v),\tag{24}$$

$$\begin{aligned}&X_{17}(\ell_1,\ell_2,\ell_3;j_1,j_2,j_3;x,y,z)\\&\quad=\sum_{m,n,p=0}^{\infty}\frac{(\ell_1)_{2m+n}(\ell_2)_{n+p}(\ell_3)_p}{(j_1)_m(j_2)_n(j_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\\&\qquad\cdot\left(r<\frac{1}{4}\wedge v<1\wedge s<\left(1-2\sqrt{r}\right)(1-v),|x|\le r,|y|\le s,|z|\le v\right),\end{aligned}\tag{25}$$

$$\begin{aligned}&X_{19}(\ell_1,\ell_2,\ell_3,\ell_4;j_1,j_2;x,y,z)\\&\quad=\sum_{m,n,p=0}^{\infty}\frac{(\ell_1)_{2m+n}(\ell_2)_n(\ell_3)_p(\ell_4)_p}{(j_1)_m(j_2)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\\&\qquad\cdot\left(s+2\sqrt{r}<1\wedge v<1,|x|\le r,|y|\le s,|z|\le v\right),\end{aligned}\tag{26}$$

$$\begin{aligned}&X_4^{(4)}(\ell_1,\ell_1,\ell_2,\ell_1,\ell_1,\ell_2,\ell_2,\ell_2;j_1,j_2,j_1,j_3;x,y,z,t)\\&\quad=\sum_{m,n,p,q=0}^{\infty}\frac{(\ell_1)_{2m+n+q}(\ell_2)_{q+n+2p}}{(j_1)_{m+p}(j_2)_n(j_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}\\&\qquad\cdot\left(|x|<\frac{1}{4},|y|<1,|z|<\frac{1}{4},|t|<1\right),\end{aligned}\tag{27}$$

$$\begin{aligned}&X_{24}^{(4)}(\ell_1,\ell_1,\ell_2,\ell_1,\ell_1,\ell_2,\ell_3,\ell_3;j_1,j_2,j_1,j_3;x,y,z,t)\\&\quad=\sum_{m,n,p,q=0}^{\infty}\frac{(\ell_1)_{2m+n+q}(\ell_2)_{n+p}(\ell_3)_{p+q}}{(j_1)_{m+p}(j_2)_n(j_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}\\&\qquad\cdot\left(|x|<\frac{1}{4},|y|<1,|z|<1,|t|<1\right).\end{aligned}\tag{28}$$

Proof. We begin by recalling the following integral representations of the beta function (see, for example, [27, 28]):

$$B(a,b)=\begin{cases}\displaystyle\int_0^1\alpha^{\alpha-1}(1-\alpha)^{b-1}d\alpha & (\Re(a)>0,\Re(b)>0),\\ \dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} & (a,b\mathbb{C}/Z_0^-),\end{cases}\tag{29}$$

$$\begin{aligned}B(a,b)&=\int_0^1\alpha^{a-1}(1-\alpha)^{b-1}d\alpha\\&=(W-V)^{1-a-b}\int_V^W(\alpha-V)^{a-1}(W-\alpha)^{b-1}d\alpha\\&\quad\cdot(\Re(a)>0,\Re(b)>0,V<W),\end{aligned}\tag{30}$$

$$\begin{aligned}B(a,b)&=2\int_0^{\pi/2}(\sin\alpha)^{2a-1}(\cos\alpha)^{2b-1}d\alpha\\&=\int_0^{\infty}(e^{-\alpha})^a(1-e^{-\alpha})^{b-1}d\alpha\,(\Re(a)>0,\Re(b)>0),\end{aligned}\tag{31}$$

$$\begin{aligned}B(a,b)&=2^{1-a-b}\int_{-1}^1(1+\alpha)^{a-1}(1-\alpha)^{b-1}d\alpha\\&=2M^a\int_0^{\infty}\frac{\cosh\alpha(\sinh\alpha)^{2a-1}}{\left(1+M\sinh^2\alpha\right)^{a+b}}d\alpha\,(\Re(a)\\&\quad>0,\Re(b)>0,M>0).\end{aligned}\tag{32}$$

For convenience, let $\mho$ denote the right-hand side of relation (12). Then, by substituting the expression of F_3 from definition (21) into the right-hand side of (12) and using (31), we have

$$\begin{aligned}\mho&=\sum_{m,n,p,q=0}^{\infty}\frac{(\ell_1)_{2m}(\ell_1+2m)_n(1+\ell_2-j_2)_p(\ell_3)_p(\ell_3+p)_q(-1)^p}{(j_1)_{m+p}}\\&\quad\times\frac{\Gamma(j_2)}{\Gamma(\ell_2)\Gamma(j_2-\ell_2)}\int_0^{\infty}(e^{-\alpha})^{\ell_2+n+p}(1-e^{-\alpha})^{j_2-\ell_2-p-1}d\alpha\\&\quad\times\frac{\Gamma(j_3)}{\Gamma(\ell_4)\Gamma(j_3-\ell_4)}\int_0^{\infty}\left(e^{-\beta}\right)^{\ell_4+q}\left(1-e^{-\beta}\right)^{j_3-\ell_4-1}d\beta\\&\quad\times\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}=\sum_{m,n,p,q=0}^{\infty}\frac{(1+\ell_2-j_2)_p\Gamma(j_2-\ell_2-p)(-1)^p}{\Gamma(j_2-\ell_2)}\\&\quad\times\frac{(\ell_1)_{2m+n}(\ell_2)_{n+p}(\ell_3)_{p+q}(\ell_4)_q}{(j_1)_{m+p}(j_2)_n(j_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}\\&=\sum_{m,n,p,q=0}^{\infty}\frac{(\ell_1)_{2m+n}(\ell_2)_{n+p}(\ell_3)_{p+q}(\ell_4)_q}{(j_1)_{m+p}(j_2)_n(j_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}\\&=X_{84}^{(4)}(\ell_1,\ell_2,\ell_3,\ell_4;j_1,j_2,j_3;x,y,z,t);\end{aligned}\tag{33}$$

we are led to the desired result. A similar argument in the proof of relation (12) will be able to establish the results (13)–(19). So, details of the proof are omitted. ☐

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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