

## Research Article

# L1-Multiscale Galerkin's Scheme with Multilevel Augmentation Algorithm for Solving Time Fractional Burgers' Equation

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In this paper, we consider the initial boundary value problem of the time fractional Burgers equation. A fully discrete scheme is proposed for the time fractional nonlinear Burgers equation with time discretized by L1-type formula and space discretized by the multiscale Galerkin method. The optimal convergence orders reach  $\mathcal{O}(\tau^{2-\alpha} + h^r)$  in the  $L^2$  norm and  $\mathcal{O}(\tau^{2-\alpha} + h^{r-1})$  in the  $H^1$  norm, respectively, in which  $\tau$  is the time step size,  $h$  is the space step size, and  $r$  is the order of piecewise polynomial space. Then, a fast multilevel augmentation method (MAM) is developed for solving the nonlinear algebraic equations resulting from the fully discrete scheme at each time step. We show that the MAM preserves the optimal convergence orders, and the computational cost is greatly reduced. Numerical experiments are presented to verify the theoretical analysis, and comparisons between MAM and Newton's method show the efficiency of our algorithm.

## 1. Introduction

In this paper, we consider the following time fractional Burgers equation [1–7]:

$${}_0^C D_t^\alpha u(x, t) + u(x, t)u_x(x, t) - u_{xx}(x, t) = f(x, t), \quad (x, t) \in \Omega, \quad (1)$$

with the initial and boundary conditions, given by

$$\begin{aligned} u(x, 0) &= u_0(x), & 0 \leq x \leq 1, \\ u(0, t) &= u(1, t) = 0, & 0 < t \leq T, \end{aligned} \quad (2)$$

where  $0 < \alpha < 1$ ,  $\Omega = \{(x, t) \mid 0 \leq x \leq 1, 0 < t \leq T\}$ ,  $u_0(x)$  and  $f(x, t)$  are given functions, and the notation  ${}_0^C D_t^\alpha$  denotes the

Caputo fractional partial derivative of order  $\alpha$ , defined by

$${}_0^C D_t^\alpha u(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha} \frac{\partial u(x, s)}{\partial s} ds, \quad \alpha \in (0, 1), \quad (3)$$

in which  $\Gamma(\cdot)$  represents the Gamma function.

The time fractional Burgers equation is a kind of nonlinear subdiffusion convection equation occurring in several physical problems such as unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe, propagation of weak shock, compressible turbulence shallow-water waves, shock waves in a viscous medium, waves in bubbly liquids, and electromagnetic waves [1, 4, 8]. Till now, there have been several analytical techniques developed to solve the time fractional Burgers equation. These methods include the Cole-Hopf transformation, Laplace transform, variable separation method [8], Adomian decomposition method [4], homotopy analysis method [6], and so on.

However, even if some fractional differential equations can be solved, the expressions of their exact solutions are often expressed by special functions, which are difficult to apply in practice. Moreover, due to the nonlocality of the fractional operators, analytical methods do not always work well on most fractional differential equations in real applications. Hence, it is of great importance to develop reliable and efficient numerical methods for solving fractional differential equations. Nowadays, the numerical methods cover the quadratic B-spline Galerkin method [2], cubic B-spline finite element method [3], finite difference methods [1, 9–13], and Fourier pseudospectral schemes [14]. There are also some other numerical methods (see, for example, [8, 15–17]).

In this paper, we first present a fully discrete scheme for solving the time fractional Burgers equation with the time approximated by the  $L1$ -type formula and the space discretization based on the multiscale Galerkin method. We give rigorous convergence analysis for the fully discrete scheme, which shows that the scheme enjoys the optimal convergence order  $\mathcal{O}(\tau^{2-\alpha} + h^r)$  in the  $L^2$  norm and  $\mathcal{O}(\tau^{2-\alpha} + h^{r-1})$  in the  $H^1$  norm, respectively, where  $\tau, h$ , and  $r$  are the time step size, space step size, and the order of piecewise polynomial space, respectively. Since the time fractional Burgers equation is a nonlinear differential equation, the fully discrete scheme results in a system of nonlinear algebraic equation at each time step. Iteration methods such as the Newton iteration method and the quasi-Newton iteration method are often employed to solve these nonlinear equations. In this case, a large amount of computational effort is demanded to compute and update the Jacobian matrix in each iteration process. The higher accuracy of the approximate solution is required, the larger dimension of the subspace is needed, and the longer computational time is consumed. To overcome this problem, we develop the multilevel augmentation method for solving the fully discrete scheme. The MAM solves a nonlinear equation at a high level consisting of two parts: solving the nonlinear equation only in a fixed initial subspace with the dimension much lower than that of the whole approximate subspace; compensating the error by matrix-vector multiplications at the high level. The MAM reduces the computational costs significantly and leads to a fast solution for the fully discrete scheme. We prove that the MAM preserves the same optimal convergence order as the original fully discrete scheme. The idea of MAM was first introduced in [18] for solving the linear Fredholm integral equations of the second kind. The theoretical setting of MAM was established by Chen et al. in [19] for solving operator equations covering both first kind and second kind equations; they further develop MAM for solving the nonlinear Hammerstein integral equation in [20]. We modified the framework and extended the idea of MAM to solve general nonlinear operator equations of the second kind and applied it to the Sine-Gordon equation in [21]. Readers are referred to [22–27] and the references therein for more applications of MAM.

This paper is organized in seven sections. In “Preliminaries,” some necessary notations, multiscale orthonormal bases in Sobolev space, and useful lemmas are introduced. In “ $L1$  Scheme for Discretization of Caputo Derivative in Time,”

we introduce the  $L1$ -formula for time discretization. In “Fully Discrete Scheme and Convergence,” a fully discrete scheme for time fractional Burgers equation is established, and the convergence analysis are given. The MAM and its convergence analysis are developed in “Multilevel Augmentation Method for Solving the Fully Discrete Scheme.” The numerical experiments are provided in “Numerical Experiments” to verify the theoretical estimates. Finally, a conclusion is included in “Conclusion.”

## 2. Preliminaries

Denote  $I = [0, 1]$ . Let  $(\cdot, \cdot)$  stand for the inner product on the space  $L^2(I)$  with the  $L^2$  norm  $\|\cdot\|_2$ . We denote by  $H_0^1(I)$  the Sobolev space of elements  $u$  satisfying the homogeneous boundary conditions that  $u(0) = u(1) = 0$ . The inner product and norm of  $H_0^1(I)$  are defined by

$$\langle u, v \rangle := \left( u', v' \right) = \int_0^1 u'(x) v'(x) dx, \quad u, v \in H_0^1(I), \quad (4)$$

$$\|u\|_1 := \sqrt{\langle u, u \rangle}, \quad u \in H_0^1(I),$$

respectively. Let  $n$  be a positive integer, we denote by  $\mathbb{X}_n$  the subspace of  $H_0^1(I)$  whose elements are the piecewise polynomials of order  $r$  with knots  $j/2^n, j-1 \in \mathbb{Z}_{2^{n-1}}$ , where the notation  $\mathbb{Z}_n := \{0, 1, 2, \dots, n-1\}$ . Obviously, the sequence of  $\mathbb{X}_n$  is nested, that is

$$\mathbb{X}_n \subset \mathbb{X}_{n+1}, \quad n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}, \quad (5)$$

which yields the following decomposition:

$$\mathbb{X}_n = \mathbb{X}_{n-1} \oplus {}^\perp \mathbb{W}_n = \mathbb{X}_0 \oplus {}^\perp \mathbb{W}_1 \oplus {}^\perp \mathbb{W}_2 \oplus {}^\perp \dots \oplus {}^\perp \mathbb{W}_n, \quad (6)$$

where  $\mathbb{W}_n$  is the orthogonal complement of  $\mathbb{X}_{n-1}$  in  $\mathbb{X}_n$ .

It is easily concluded from the definition of  $\mathbb{X}_n$  and  $\mathbb{W}_n$  that the dimensions of  $\mathbb{X}_n$  and  $\mathbb{W}_n$  are given by

$$x(n) := \dim(\mathbb{X}_n) = (r-1)2^n - 1 \quad \text{and} \quad w(n) := \dim(\mathbb{W}_n) = x(n) - x(n-1) = (r-1)2^{n-1}, \text{ respectively.}$$

Define two affine mappings on the interval  $I$  by  $\varphi_0(x) = x/2$  and  $\varphi_1(x) = x + 1/2, x \in I$ , which map the interval  $[0, 1]$  into  $[0, 1/2]$  and  $[1/2, 1]$ , respectively. Associated with the two mappings, we introduce two linear operators as follows:

$$\begin{aligned} (\mathcal{T}_0 u)(x) &:= \frac{\sqrt{2}}{2} u(\varphi_0^{-1}(x)) = \frac{\sqrt{2}}{2} u(2x), \\ (\mathcal{T}_1 u)(x) &:= \frac{\sqrt{2}}{2} u(\varphi_1^{-1}(x)) = \frac{\sqrt{2}}{2} u(2x-1). \end{aligned} \quad (7)$$

**Lemma 1** (see [26]). *Let  $w_{ij}, j \in \mathbb{Z}_{w(i)}$ , be an orthonormal basis of  $\mathbb{W}_i, i \geq 1$ . Then the functions  $\{\mathcal{T}_0 w_{ij}, \mathcal{T}_1 w_{ij} : j \in \mathbb{Z}_{w(i)}\}$  form an orthonormal basis for  $\mathbb{W}_{i+1}$ .*

Lemma 1 shows that the space  $\mathbb{W}_n$  can be recursively constructed by the linear operators  $\mathcal{T}_0$  and  $\mathcal{T}_1$  once  $\mathbb{W}_1$  has been given. Therefore, the basis of the space  $\mathbb{X}_n$  can be

Let  $k, m$  be two fixed positive integers and  $n := k + m$ .  
 Step 1: obtain the approximation of initial value function  
 $u_n^0 := \mathcal{P}_n u_0(x) = \sum_{(i,j) \in J_n} c_{ij} w_{ij}(x)$ ,  
 where  $c_{ij} = \langle u_0(x), w_{ij} \rangle_1, (i,j) \in J_n$ .  
 Step 2: for  $i = 1 : N$  ( $T = N\tau$ ), do the following:  
 (I): Solve  $u_k^i \in \mathbb{X}_k$  from (33) with  $n := k$ . Set  $u_{k,0}^i := u_k^i, l = 1$   
 (II): Compute  $u_{k,l}^{i,H} := (\mathcal{P}_{k+l} - \mathcal{P}_k)(\tilde{f}^i - \mathcal{K}u_{k,l-1}^i)$ , that is  
 $u_{k,l}^{i,H} = \sum_{(i,j) \in J_{k,m}} c_{ij} w_{ij}(x)$ ,  
 where  $J_{k,m} = J_{k+m} \setminus J_k, c_{ij} = \langle \tilde{f}^i - \mathcal{K}u_{k,l-1}^i, w_{ij} \rangle_1$  with  
 $\langle \tilde{f}^i, w_{ij} \rangle_1 := \langle f^i, w_{ij} \rangle + \mu(r_{k,m}^i, w_{ij})$ .  
 (III): solve  $u_{k,l}^{i,L} \in \mathbb{X}_k$  from the following equation  
 $\mathcal{P}_k(\mathcal{J} + \mathcal{K})(u_{k,l}^{i,L} + u_{k,l}^{i,H}) = \mathcal{P}_k \tilde{f}^i$ ,  
 that is  
 $\langle (\mathcal{J} + \mathcal{K})(u_{k,l}^{i,L} + u_{k,l}^{i,H}), w_{ij} \rangle_1 = \langle \tilde{f}^i, w_{ij} \rangle_1$ , for all  $(i,j) \in J_k$ .  
 (IV): Let  $u_{k,l}^i := u_{k,l}^{i,L} + u_{k,l}^{i,H}$ . Set  $l \leftarrow l + 1$  and go back to (III) until  $l = m \dots$

ALGORITHM 1: (MAM for time fractional Burgers equation).

constructed by Lemma 1 step by step. For the details of the construction and more, the readers can refer to [26].

Let  $\mathcal{P}_n$  be an orthogonal projection operator from  $H_0^1(I)$  into  $\mathbb{X}_n$  with respect to the inner product  $\langle \cdot, \cdot \rangle$ , that is, for all  $u \in H_0^1(I)$ ,

$$\langle \mathcal{P}_n u, v \rangle = \langle u, v \rangle, v \in \mathbb{X}_n, \quad (8)$$

or

$$(\partial_x \mathcal{P}_n u, \partial_x v) = (\partial_x u, \partial_x v), v \in \mathbb{X}_n. \quad (9)$$

The following approximation results on the operator  $\mathcal{P}_n$  will be used later. Throughout this paper, unless stated otherwise,  $c$  denotes a generic positive constant whose value may differ in different occurrences.

**Lemma 2** (see [28]). *If  $u \in H^r(I) \cap H_0^1(I)$ , then*

$$\begin{aligned} \|u - \mathcal{P}_n u\|_2 &\leq ch^r \|u\|_r, \\ |u - \mathcal{P}_n u|_1 &\leq ch^{r-1} \|u\|_r, \end{aligned} \quad (10)$$

where  $h := 2^{-n}$ .

### 3. L1 Scheme for Discretization of Caputo Derivative in Time

For a positive integer  $N$ , let  $\tau = T/N$  be the time step size and  $t_i = i\tau$  for  $i = 0, 1, \dots, N$ . Let  $u^i$  be the solution of  $u(x, t)$  on  $t = t_i$ .

Define

$$u^{i+\frac{1}{2}} := \frac{u^{i+1} + u^i}{2} \text{ and } \delta_t u^{i+\frac{1}{2}} := \frac{u^{i+1} - u^i}{\tau}. \quad (11)$$

For the approximation of fractional derivative  ${}_0^c D_t^\alpha g(t_i)$ , we use the following L1 scheme [29, 30]:

$${}_0^c D_t^\alpha g(t_i) \approx D_\tau^\alpha g(t_i) := \mu \left[ a_0 g(t_i) - \sum_{k=1}^{i-1} (a_{i-k-1} - a_{i-k}) g(t_k) - a_{i-1} g(t_0) \right], \quad (12)$$

where  $0 < \alpha < 1, \mu = \tau^{-\alpha}/\Gamma(2-\alpha)$  and  $a_k = (k+1)^{1-\alpha} - k^{1-\alpha}$ .

**Lemma 3** (see [30]). *If  $0 < \alpha < 1$  and  $a_k = (k+1)^{1-\alpha} - k^{1-\alpha}$ ,  $k \in \mathbb{N}_0$ , then*

$$\begin{aligned} I = a_0 > a_1 > a_2 > \dots > a_i > \dots \longrightarrow 0, \text{ as } k \longrightarrow \infty, \\ (1-\alpha)(k+1)^{-\alpha} < a_k < (1-\alpha)k^{-\alpha}. \end{aligned} \quad (13)$$

**Lemma 4** (see [30]). *Suppose  $0 < \alpha < 1, g(t) \in C^2[0, t_i]$ . Let*

$$R(g(t_i)) := {}_0^c D_t^\alpha g(t_i) - D_\tau^\alpha g(t_i). \quad (14)$$

Then

$$|R(g(t_i))| \leq \frac{1}{\Gamma(2-\alpha)} \left[ \frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1+2^{-\alpha}) \right] \max_{0 \leq t \leq t_i} |g''(t)| \tau^{2-\alpha}. \quad (15)$$

### 4. Fully Discrete Scheme and Convergence

In this section, we present a fully discrete scheme for the time fractional Burgers equation (1), and we derive the error estimates and convergence of the proposed fully discrete scheme. The Galerkin method associated with the multiscale basis introduced in ‘‘Preliminaries’’ is employed to discretize the spatial variable. The fully discrete scheme in weak formulation

TABLE 1: Errors and convergent orders of MAM in temporal direction for Example 1.

$\alpha$	$\tau$	Linear basis ( $k = 3, m = 7$ )		Quadratic basis ( $k = 2, m = 5$ )	
		$\ u^* - u_{3,7}\ _2$	Rate	$\ u^* - u_{2,5}\ _2$	Rate
1/3	1/4	3.8885e-4		3.8871e-4	
	1/8	1.3023e-4	1.5781	1.3012e-4	1.5789
	1/16	4.2962e-5	1.5999	4.2893e-5	1.6010
	1/32	1.3946e-5	1.6233	1.3985e-5	1.6168
	1/64	4.2759e-6	1.7055	4.5237e-6	1.6283
	1/128	1.3118e-6	1.7047	1.4543e-6	1.6372
1/2	1/4	7.8755e-4		7.8742e-4	
	1/8	2.8852e-4	1.4487	2.8850e-4	1.4486
	1/16	1.0421e-4	1.4692	1.0448e-4	1.4654
	1/32	3.7337e-5	1.4808	3.7550e-5	1.4763
	1/64	1.3340e-5	1.4848	1.3427e-5	1.4837
	1/128	4.5749e-6	1.5439	4.7858e-6	1.4883
3/4	1/4	1.8737e-3		1.8737e-3	
	1/8	7.9862e-4	1.2303	7.9879e-4	1.2300
	1/16	3.3825e-4	1.2394	3.3833e-4	1.2394
	1/32	1.4287e-4	1.2434	1.4285e-4	1.2440
	1/64	6.0343e-5	1.2434	6.0206e-5	1.2465
	1/128	2.5613e-5	1.2363	2.5352e-5	1.2478

for (1) reads as follows: for each  $t = t_i, i = 1, 2 \dots n$ , find  $u_n^i \in \mathbb{X}_n$  such that

$$\begin{cases} (D_\tau^\alpha u_n^i, v_n) + (u_{nx}^i, v_{nx}) + (u_n^i u_{nx}^i, v_n) = (f^i, v_n), v_n \in \mathbb{X}_n, \\ u_n^0 = \Pi_n u_0, \end{cases} \quad (16)$$

where  $f^i = f(x, t)|_{t=t_i}$  and  $\Pi_n$  denotes the interpolation operator.

We present an optimal error estimate of the fully discrete scheme (16) in the following theorem.

**Theorem 5.** Suppose that the problem (1)–(2) has a unique solution  $u \in C^2([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^r(\Omega))$ . Then

$$\|u^i - u_n^i\|_2 \leq c(\tau^{2-\alpha} + h^r), \quad (17)$$

where  $u^i = u(x, t)|_{t=t_i}$ .

*Proof.* Denote  $e_n^i := \mathcal{P}_n u^i - u_n^i, i = 0, 1, 2 \dots N$ . We conclude from (1) and (16) that  $e_n^i$  satisfies

$$\begin{aligned} (D_\tau^\alpha e_n^i, v_n) + (e_{nx}^i, v_{nx}) &= (D_\tau^\alpha (\mathcal{P}_n u^i - u^i), v_n) \\ &+ (D_\tau^\alpha u^i - D_{t_i}^\alpha u, v_n) + (u^i u_x^i - u_n^i u_{nx}^i, v_n). \end{aligned} \quad (18)$$

Taking  $v_n = e_n^i$  in (18), we have

$$\begin{aligned} (D_\tau^\alpha e_n^i, e_n^i) + |e_n^i|_1^2 &= ((\mathcal{P}_n - \mathcal{J})D_\tau^\alpha u^i, e_n^i) + (D_\tau^\alpha u^i - D_{t_i}^\alpha u, e_n^i) \\ &+ (u^i u_x^i - u_n^i u_{nx}^i, e_n^i). \end{aligned} \quad (19)$$

We estimate the terms of the right-hand side of (19) one by one. For the first term in right-hand side of (19), using the Cauchy-Schwarz inequality and Lemma 2, we have

$$|((\mathcal{P}_n - \mathcal{J})D_\tau^\alpha u^i, e_n^i)| \leq ch^r \|D_\tau^\alpha u^i\|_2 \|e_n^i\|_2 \leq ch^{2r} \|D_\tau^\alpha u^i\|_2^2 + \frac{1}{2} \|e_n^i\|_2^2. \quad (20)$$

To estimate the second term in the right-hand side of (19), we conclude from Lemma 4 that

$$\left| (D_\tau^\alpha u^i - D_{t_i}^\alpha u, e_n^i) \right| \leq \|D_\tau^\alpha u^i - D_{t_i}^\alpha u\|_2 \|e_n^i\|_2 \leq c\tau^{4-2r+\frac{1}{2}} \|e_n^i\|_2^2. \quad (21)$$

For the last term in the right-hand side of (19), using integration by parts and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |(u^i u_x^i - u_n^i u_{nx}^i, e_n^i)| &= \frac{1}{2} \left| \int_I (u^i - u_n^i) (u^i + u_n^i) e_{ix}^i dx \right| \\ &\leq M \|u^i - u_n^i\|_2 |e_n^i|_1 \leq \frac{M}{2} \|u^i - u_n^i\|_2^2 + |e_n^i|_1^2 \\ &\leq \frac{M}{2} \|u^i - \mathcal{P}_n u^i\|_2^2 + \frac{M}{2} \|e_n^i\|_2^2 + |e_n^i|_1^2 \leq ch^{2r} \\ &+ \frac{M}{2} \|e_n^i\|_2^2 + |e_n^i|_1^2, \end{aligned} \quad (22)$$

where  $\|u^i\| \leq M$  and  $\|u_n^i\| \leq M$ , due to the smoothness of  $u$  and the approximation  $u_n \in \mathbb{X}_n \subset H_0^1(I)$ .

On the other hand

$$\begin{aligned} (D_\tau^\alpha e_n^i, e_n^i) &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left( a_0 e_n^i - \sum_{k=1}^{i-1} (a_{i-k-1} - a_{i-k}) e_n^k - a_{i-1} e_n^0, e_n^i \right), \\ &= \frac{a_0 \tau^{-\alpha}}{\Gamma(2-\alpha)} \|e_n^i\|_2^2 - \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left( \sum_{k=1}^{i-1} (a_{i-k-1} - a_{i-k}) e_n^k + a_{i-1} e_n^0, e_n^i \right). \end{aligned} \quad (23)$$

Substituting (20)–(23) into (19) and noting that  $1 = a_0 > a_1 > \dots > a_i > \dots$ , we deduce that

TABLE 2: Errors and convergent orders of MAM and DNM via linear basis in spatial direction with  $k = 3, \tau = 0.001$ , and  $\alpha = 0.25$  for Example 1.

$m$	$x(n)$	MAM				DNM			
		$ u^* - u_{3,m} _1$	Rate	$\ u^* - u_{3,m}\ _2$	Rate	$ u^* - u_n _1$	Rate	$\ u^* - u_n\ _2$	Rate
0	7	3.1264e-2		1.2161e-3		3.1264e-2		1.2161e-3	
1	15	1.7726e-2	0.8186	3.1564e-4	1.9461	1.6011e-2	0.9654	3.1282e-4	1.9590
2	31	8.9080e-3	0.9927	7.7686e-5	2.0225	8.0530e-3	0.9915	7.8743e-5	1.9901
3	63	4.4596e-3	0.9982	1.9377e-5	2.0033	4.0324e-3	0.9979	1.9714e-5	1.9979
4	127	2.2305e-3	0.9996	4.8450e-6	1.9998	2.0169e-3	0.9995	4.9263e-6	2.0006
5	255	1.1153e-3	0.9999	1.2132e-6	1.9976	1.0086e-3	0.9999	1.2342e-6	1.9970
6	511	5.5768e-4	1.0000	2.8891e-7	2.0702	5.0429e-4	1.0000	3.3678e-7	1.8737

TABLE 3: Comparison of CPU time between MAM and DNM via linear basis with  $k = 3, \tau = 0.001$ , and  $\alpha = 0.25$  for Example 1.

$n = k + m$	$x(n)$	$T_{\text{MAM}}$	$T_{\text{DNM}}$
$3 = 3 + 0$	7	1.33	1.28
$4 = 3 + 1$	15	3.21	3.70
$5 = 3 + 2$	31	7.09	12.22
$6 = 3 + 3$	63	15.45	41.96
$7 = 3 + 4$	127	41.47	152.71
$8 = 3 + 5$	255	112.52	570.63
$9 = 3 + 6$	511	340.87	4581.64

$$\begin{aligned}
\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \|e_n^i\|_2^2 &\leq \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left( \sum_{k=1}^{i-1} (a_{i-k-1} - a_{i-k}) \frac{\|e_n^k\|_2^2 + \|e_n^i\|_2^2}{2} + a_{i-1} \frac{\|e_n^0\|_2^2 + \|e_n^i\|_2^2}{2} \right) \\
&\quad + \left( \frac{M}{2} + 1 \right) \|e_n^i\|_2^2 + c(\tau^{2-\alpha} + h^r)^2 \\
&= \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)} \left( \|e_n^i\|_2^2 + \sum_{k=1}^{i-1} \|e_n^k\|_2^2 + \|e_n^0\|_2^2 \right) \\
&\quad + \left( \frac{M}{2} + 1 \right) \|e_n^i\|_2^2 + c(\tau^{2-\alpha} + h^r)^2.
\end{aligned} \tag{24}$$

Choose  $\tau$  such that  $\tau^{-\alpha}/2\Gamma(2-\alpha) > ((M/2) + 1)$ , and denote  $\sigma = \tau^{-\alpha}/2\Gamma(2-\alpha)$ ,  $\lambda = \tau^{-\alpha}/2\Gamma(2-\alpha) - ((M/2) + 1)$ ; then, we have

$$\|e_n^i\|_2^2 \leq \frac{\sigma}{\lambda} \sum_{k=1}^{i-1} \|e_n^k\|_2^2 + \frac{\sigma}{\lambda} \|e_n^0\|_2^2 + c(\tau^{2-\alpha} + h^r)^2. \tag{25}$$

By Gronwall's inequality, we have

$$\|e_n^i\|_2 \leq c \|e_n^0\|_2 + c(\tau^{2-\alpha} + h^r), \tag{26}$$

which, together with Lemma 2 and the initial error estimate, yields that

$$\|u^i - u_n^i\|_2 \leq \|u^i - \mathcal{P}_n u^i\|_2 + \|e_n^i\|_2 \leq c(\tau^{2-\alpha} + h^r). \tag{27}$$

This completes the proof.

*Remark 6.* If we choose  $v_n = D_\tau^\alpha e_n^i$  in (19) and make a similar analysis as the above Theorem 5, we can obtain the optimal convergence order in  $H^1$  norm

$$|u^i - u_n^i|_1 \leq c(\tau^{2-\alpha} + h^{r-1}). \tag{28}$$

## 5. Multilevel Augmentation Method for Solving the Fully Discrete Scheme

At each time step, the fully discrete scheme (16) leads to a nonlinear system, which makes the computational cost expensive. We present a fast multilevel augmentation method in this section to solve these nonlinear systems. To this end, we rewrite (16) into

$$(u_{nx}^i, v_{nx}) + (\mu u_n^i + u_n^i u_{nx}^i, v_n) = (f^i + \mu r_n^i, v_n), \tag{29}$$

where  $r_n^i = \sum_{k=1}^{i-1} (a_{i-k-1} - a_{i-k}) u_n^k + a_{i-1} u_n^0$  and  $\mu = \tau^{-\alpha}/\Gamma(2-\alpha)$ . Define a nonlinear operator  $\mathcal{K} : \mathbb{X} \rightarrow \mathbb{X}$  as follows:

$$\langle \mathcal{K}(u), v \rangle_1 := (\mu u + u u_x, v), \quad v \in \mathbb{X}. \tag{30}$$

Similar to the proof of Lemma 3 in [23], we applied the Riesz representation theorem to the right-hand side of (29); there exists a element  $\tilde{f}^i \in \mathbb{X}$ , such that

$$\langle \tilde{f}^i, v \rangle_1 = (f^i + \mu r_n^i, v), \quad \forall v \in \mathbb{X}_n. \tag{31}$$

Then, Equation (29) can be reformulated as

$$\langle u_n^i, v_n \rangle_1 + \langle \mathcal{K} u_n^i, v_n \rangle_1 = \langle \tilde{f}^i, v_n \rangle_1, \quad v_n \in \mathbb{X}_n, \tag{32}$$

or equivalently

$$(\mathcal{F} + \mathcal{P}_n \mathcal{K}) u_n^i = \mathcal{P}_n \tilde{f}^i. \tag{33}$$

Since Equation (16) has been reformulated as a nonlinear operator equation of the second kind (33), and  $\mathcal{K}$  has the properties (P1) and (P2) described in [23], then MAM developed in [21] is applicable.

TABLE 4: Errors and convergent orders of MAM and DNM via quadratic basis in spatial direction with  $k=2, \tau=0.001$ , and  $\alpha=0.1$  for Example 1.

$m$	$x(n)$	MAM				DNM			
		$ u^* - u_{2,m} _1$	Rate	$\ u^* - u_{2,m}\ _2$	Rate	$ u^* - u_n _1$	Rate	$\ u^* - u_n\ _2$	Rate
0	7	1.5670e-2		6.0316e-4		1.5670e-2		6.0316e-4	
1	15	4.0566e-3	1.9496	8.1280e-5	2.8916	4.0054e-3	1.9679	7.7216e-5	2.9656
2	31	1.0196e-3	1.9922	1.0218e-5	2.9918	1.0068e-3	1.9922	9.7080e-6	2.9917
3	63	2.5528e-4	1.9979	1.2796e-6	2.9973	2.5203e-4	1.9981	1.2151e-6	2.9980
4	127	6.3843e-5	1.9995	1.5919e-7	3.0070	6.3030e-5	1.9995	1.5096e-7	3.0089
5	255	1.5962e-5	1.9999	2.1784e-8	2.8694	1.5759e-5	1.9999	1.7198e-8	3.1339

TABLE 5: Comparison of CPU time between MAM and DNM via quadratic basis with  $k=2, \tau=0.001$ , and  $\alpha=0.1$  for Example 1.

$n = k + m$	$x(n)$	$T_{\text{MAM}}$	$T_{\text{DNM}}$
$2 = 2 + 0$	7	2.44	2.48
$3 = 2 + 1$	15	5.18	8.91
$4 = 2 + 2$	31	8.49	31.60
$5 = 2 + 3$	63	14.04	128.97
$6 = 2 + 4$	127	23.16	518.63
$7 = 2 + 5$	255	43.77	2506.75

We now briefly describe the MAM for solving (33). As we presented in ‘‘Preliminaries,’’ the approximation subspace sequence is nested, for a fixed positive integer  $k, n = k + m$ ,  $m$  is any nonnegative integer, and we have the following decomposition:

$$\mathbb{X}_{k+m} := \mathbb{X}_k \oplus {}^\perp \mathbb{W}_{k,m}, \text{ with } \mathbb{W}_{k,m} := \mathbb{W}_{k+1} \oplus {}^\perp \mathbb{W}_{k+2} \oplus {}^\perp \dots \oplus {}^\perp \mathbb{W}_{k+m}. \quad (34)$$

Now, we are in a position to solve (33) with  $n = k + m$ , and  $k$  is fixed and smaller than  $n$ . Firstly, we solve (33) with  $n = k$  exactly and obtain  $u_k^i$ . Next, we obtain an approximation of  $u_{k+1}^i$  of (33) with  $n = k + 1$ . To this end, we decompose

$$u_{k+1}^i = u_{k+1}^{i,L} + u_{k+1}^{i,H}, \text{ with } u_{k+1}^{i,L} \in \mathbb{X}_k, u_{k+1}^{i,H} \in \mathbb{W}_{k+1}.$$

With the help of (34), Equation (33) with  $n = k + 1$  can be rewritten as an equivalent form as

$$(\mathcal{P}_{k+1} - \mathcal{P}_k)(u_{k+1}^{i,L} + u_{k+1}^{i,H}) - (\mathcal{P}_{k+1} - \mathcal{P}_k)\mathcal{K}u_{k+1}^i = (\mathcal{P}_{k+1} - \mathcal{P}_k)\tilde{f}^i, \quad (35)$$

$$\mathcal{P}_k(\mathcal{F} - \mathcal{K})(u_{k+1}^{i,L} + u_{k+1}^{i,H}) = \mathcal{P}_k\tilde{f}^i. \quad (36)$$

Note that

$$(\mathcal{P}_{k+1} - \mathcal{P}_k)(u_{k+1}^{i,L} + u_{k+1}^{i,H}) = u_{k+1}^{i,H}. \quad (37)$$

Equation (35) becomes

$$u_{k+1}^{i,H} = (\mathcal{P}_{k+1} - \mathcal{P}_k)(\tilde{f}^i + \mathcal{K}u_{k+1}^i). \quad (38)$$

The  $u_{k+1}^i$  in the right-hand side can be approximated by the previous level solution  $u_{k,0}^i := u_k^i$ . We compute

$$u_{k,1}^{i,H} = (\mathcal{P}_{k+1} - \mathcal{P}_k)(\tilde{f}^i + \mathcal{K}u_{k,0}^i) \in \mathbb{W}_{k+1}. \quad (39)$$

Replace  $u_{k+1}^{i,H}$  in (36) by  $u_{k,1}^{i,H}$ , and solve  $u_{k,1}^{i,L} \in \mathbb{X}_k$  from

$$\mathcal{P}_k(\mathcal{F} - \mathcal{K})(u_{k,1}^{i,L} + u_{k,1}^{i,H}) = \mathcal{P}_k\tilde{f}^i. \quad (40)$$

Let

$$u_{k,1}^i := u_{k,1}^{i,L} + u_{k,1}^{i,H}, \quad (41)$$

which is an approximation to the solution  $u_{k+1}^i$ .

This procedure is repeated  $m$  times to obtain an approximation  $u_{k,m}^i$  of the solution  $u_{k+m}^i$  of (33) with  $n = k + m$ . The solution  $u_{k,m}^i$  is called a multilevel augmentation solution. Since at any step  $l = 0, 1, \dots, m$ , we only need to invert the same nonlinear operator  $\mathcal{P}_k(\mathcal{F} - \mathcal{K})$  with a fixed small  $k$  instead of the nonlinear operator  $\mathcal{P}_{k+l}(\mathcal{F} - \mathcal{K})$ . This means the algorithm has a high computational efficiency. At every time step, the fully discrete scheme (33) is solved by the MAM, and the whole process can be summarized as the following algorithm:

**Theorem 7.** Let  $u$  be the exact solution of (1) and  $u_{k,m}^i$  be the approximation solution obtained by Algorithm 1. Suppose that the solution of Equation (33) belongs to  $H^r(I)$  for  $i = 1, 2, \dots, T/\tau$ . Then, there exist a positive integer  $N$  such that for all  $k \geq N$  and  $m \in \mathbb{N}$

$$\|u^i - u_{k,m}^i\|_2 \leq C(\tau^{2-\alpha} + h^r). \quad (42)$$

*Proof.* As stated in [20, 21, 23],  $u_{k,m}^i$  is the solution of the equation:

$$(\mathcal{F} + \mathcal{P}_k\mathcal{K})u_{k,m}^i = \mathcal{P}_{k+m}\tilde{f}^i - (\mathcal{P}_{k+m} - \mathcal{P}_k)\mathcal{K}u_{k,l-1}^i, \quad (43)$$



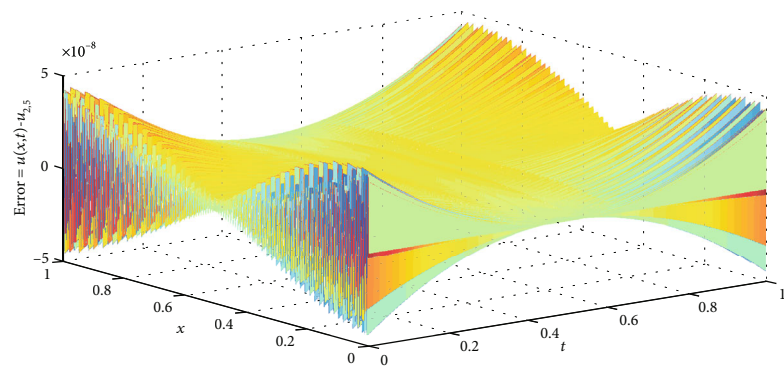


FIGURE 1: The graph of error function  $u(x,t) - u_{2,5}(x,t)$  for Example 1.

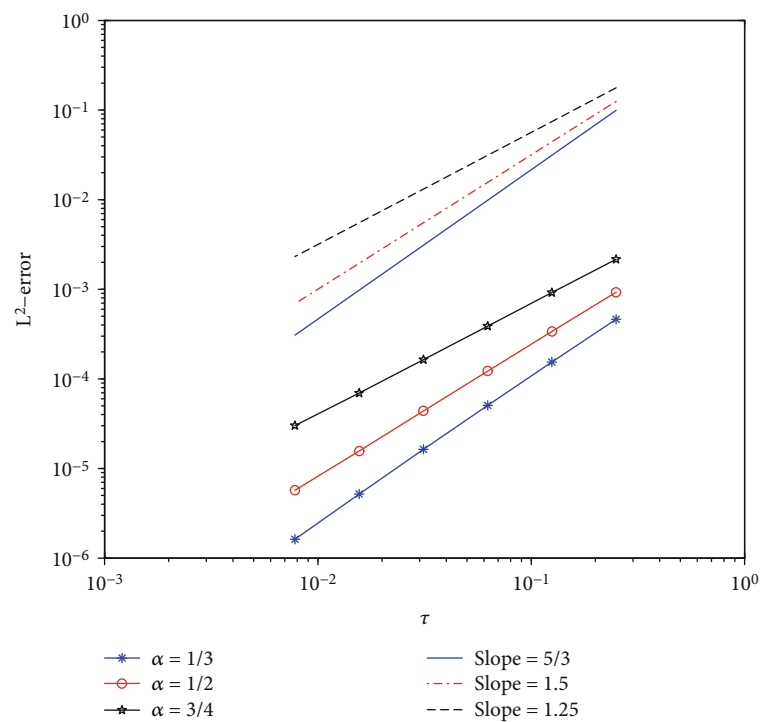


FIGURE 2: The temporal convergence order of MAM for Example 2.

TABLE 6: Errors and convergent orders of MAM and DNM via linear basis in spatial direction with  $k=3, \tau=0.005$ , and  $\alpha=0.25$  for Example 2.

$m$	$x(n)$	MAM				DNM			
		$ u^* - u_{3,m} _1$	Rate	$\ u^* - u_{3,m}\ _2$	Rate	$ u^* - u_n _1$	Rate	$\ u^* - u_n\ _2$	Rate
0	7	9.9718e-1		3.8317e-2		9.9718e-1		3.8317e-2	
1	15	5.1444e-1	0.9548	7.8333e-3	2.2903	5.0239e-1	0.9891	9.6573e-3	1.9883
2	31	2.5795e-1	0.9959	1.9554e-3	2.0021	2.5167e-1	0.9973	2.4190e-3	1.9972
3	63	1.2907e-1	0.9990	4.8547e-4	2.0100	1.2589e-1	0.9993	6.0479e-4	1.9999
4	127	6.4546e-2	0.9998	1.2093e-4	2.0052	6.2955e-2	0.9998	1.5096e-4	2.0022
5	255	3.2274e-2	0.9999	2.9983e-5	2.0120	3.1478e-2	1.0000	3.7490e-5	2.0096
6	511	1.6137e-2	1.0000	7.2066e-6	2.0568	1.5739e-2	1.0000	9.1214e-6	2.0392
7	1023	8.0687e-3	1.0000	1.5895e-6	2.1808	7.8696e-3	1.0000	2.0338e-6	2.1651

TABLE 7: Comparison of CPU time between MAM and DNM via linear basis with  $k=3, \tau=0.005$ , and  $\alpha=0.25$  for Example 2.

$n = k + m$	$x(n)$	$T_{\text{MAM}}$	$T_{\text{DNM}}$
$3 = 3 + 0$	7	0.29	0.51
$4 = 3 + 1$	15	0.78	1.08
$5 = 3 + 2$	31	1.67	3.75
$6 = 3 + 3$	63	3.86	13.18
$7 = 3 + 4$	127	9.82	48.16
$8 = 3 + 5$	255	27.78	179.33
$9 = 3 + 6$	511	87.87	716.02
$10 = 3 + 7$	1023	289.36	7004.77

which is equivalent to the following discrete form:

$$\begin{aligned} (\mu u_{k,m,x}^i, v_x) + (\mu u_{k,m}^i + u_{k,m}^i u_{k,m,x}^i, v) &= \mu(u_{k,m}^i - u_{k,m-1}^i, v_2) \\ &+ (f^i, v) + \mu(r_{k,m}^i, v) + (u_{k,m}^i u_{k,m,x}^i - u_{k,m-1}^i u_{k,m-1,x}^i, v_2), \end{aligned} \quad (44)$$

where  $v \in \mathbb{X}_{k+m}, v_2 = (\mathcal{P}_{k+m} - \mathcal{P}_k)v$ . Rearranging the terms, we have

$$\begin{aligned} (D_\tau^\alpha u_{k,m}^i, v) + (u_{k,m,x}^i, v_x) &= -(u_{k,m}^i u_{k,m,x}^i, v) \\ &+ \mu(u_{k,m}^i - u_{k,m-1}^i, v_2) + (u_{k,m}^i u_{k,m,x}^i - u_{k,m-1}^i u_{k,m-1,x}^i, v_2) \\ &+ (f^i, v). \end{aligned} \quad (45)$$

Noting that the exact solution  $u$  at  $t = t_i$  satisfies

$$(D_\tau^\alpha u^i, v) + (u_x^i, v_x) = -(u^i u_x^i, v) + (f^i, v) + (D_\tau^\alpha u^{i-c} D_{t_i}^\alpha u, v). \quad (46)$$

Subtracting (45) from (46), we obtain that for all  $v \in \mathbb{X}_{k+m}, v_2 = (\mathcal{P}_{k+m} - \mathcal{P}_k)v$

$$\begin{aligned} (D_\tau^\alpha (u^i - u_{k,m}^i), v) + (u_x^i - u_{k,m,x}^i, v_x) &= -(u^i u_x^i - u_{k,m}^i u_{k,m,x}^i, v) \\ &+ (D_\tau^\alpha u^{i-c} D_{t_i}^\alpha u, v) - \mu(u_{k,m}^i - u_{k,m-1}^i, v_2) \\ &- (u_{k,m}^i u_{k,m,x}^i - u_{k,m-1}^i u_{k,m-1,x}^i, v_2). \end{aligned} \quad (47)$$

Denote  $\rho^i := u^i - \mathcal{P}_{k+m} u^i$  and  $e^i := \mathcal{P}_{k+m} u^i - u_{k,m}^i$ , then  $u^i - u_{k,m}^i = \rho^i + e^i$ . Using these notations and noting that  $(\rho_x^i, v_x) = 0$ , we derive the error equation as follows:

$$\begin{aligned} (D_\tau^\alpha e^i, v) + (e_x^i, v_x) &= (D_\tau^\alpha \rho^i, v) + (D_\tau^\alpha u^{i-c} D_{t_i}^\alpha u, v) \\ &- (u^i u_x^i - u_{k,m}^i u_{k,m,x}^i, v) - \mu(u_{k,m}^i - u_{k,m-1}^i, v_2) \\ &- (u_{k,m}^i u_{k,m,x}^i - u_{k,m-1}^i u_{k,m-1,x}^i, v_2). \end{aligned} \quad (48)$$

Let  $M_1 = \max \{\|u_{k,m}^i\|_\infty, i \in \mathbb{Z}_N\}$  and  $M' = \max \{M_1, M\}$ , where  $M$  is the positive constant appearing in (22); then,  $\|u^i\|_\infty \leq M'$ ,  $\|u_{k,m}^i\|_\infty \leq M'$ . We take  $v = e^i$  in (48) and estimate the terms in the right-hand side of (48).

For the first three terms in the right-hand side of (48), similar to the analysis of (20)–(22), we have

$$|(D_\tau^\alpha \rho^i, e^i)| = |((\mathcal{P}_{k+m} - \mathcal{P}) D_\tau^\alpha u^i, e^i)| \leq ch^{2r} \|D_\tau^\alpha u^i\|_2^2 + \frac{1}{2} \|e^i\|_2^2, \quad (49)$$

$$|(D_\tau^\alpha u^{i-c} D_{t_i}^\alpha u, e^i)| \leq c\tau^{4-2r} + \frac{1}{2} \|e^i\|_2^2, \quad (50)$$

$$|(u^i u_x^i - u_{k,m}^i u_{k,m,x}^i, e^i)| \leq ch^{2r} + M' \|e^i\|_2^2 + \frac{1}{2} |e^i|_1^2. \quad (51)$$

By the Cauchy-Schwartz inequality, Young's inequality, and noting that  $\|v_2\|_2 \leq \|v\|_2$ , we have

$$|-\mu(u_{k,m}^i - u_{k,m-1}^i, v_2)| \leq \frac{\mu^2}{\sigma_1} \|u_{k,m}^i - u_{k,m-1}^i\|_2^2 + \frac{\sigma_1}{4} \|e^i\|_2^2. \quad (52)$$

For the last term in the right-hand side of (48), it follows from integration by parts, the Cauchy-Schwartz inequality, and the Young inequality that

$$\begin{aligned} |(u_{k,m}^i u_{k,m,x}^i - u_{k,m-1}^i u_{k,m-1,x}^i, v_2)| &\leq 2M' \|u_{k,m}^i - u_{k,m-1}^i\|_2 |e^i|_1 \\ &\leq M'^2 \|u_{k,m}^i - u_{k,m-1}^i\|_2^2 + \frac{1}{2} |e^i|_1^2. \end{aligned} \quad (53)$$

On the other hand side, as presented in (23), we have

$$(D_\tau^\alpha e^i, e^i) \leq \frac{\tau^\alpha}{\Gamma(2-\alpha)} \|e^i\|_2^2 - \frac{\tau^\alpha}{\Gamma(2-\alpha)} \left( \sum_{k=1}^{i-1} (a_{i-k-1} - a_{i-k}) e^k + a_{i-1} e^0, e^i \right). \quad (54)$$

Combining (49)–(54) and  $(e_x^i, e_x^i) = |e^i|_1^2$ , we have

$$\begin{aligned} \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \|e^i\|_2^2 &\leq \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left( \sum_{k=1}^{i-1} (a_{i-k-1} - a_{i-k}) \frac{\|e^k\|_2^2 + \|e^i\|_2^2}{2} + a_{i-1} \frac{\|e^0\|_2^2 + \|e^i\|_2^2}{2} \right) \\ &+ \left( 1 + M' + \frac{\sigma_1}{4} \right) \|e^i\|_2^2 + \left( \frac{\mu^2}{\sigma_1} \right) \|u_{k,m}^i - u_{k,m-1}^i\|_2^2 + c(\tau^{2-\alpha} + h^r)^2 \\ &= \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)} \left( \|e^i\|_2^2 + \sum_{k=1}^{i-1} \|e^k\|_2^2 + \|e^0\|_2^2 \right) \\ &+ \left( 1 + M' + \frac{\sigma_1}{4} \right) \|e^i\|_2^2 + \left( \frac{\mu^2}{\sigma_1} \right) \|u_{k,m}^i - u_{k,m-1}^i\|_2^2 + c(\tau^{2-\alpha} + h^r)^2. \end{aligned} \quad (55)$$

Choose  $\tau$  such that  $\tau^{-\alpha}/2\Gamma(2-\alpha) > 1 + M' + (\sigma_1/4)$ , and denote  $\bar{\sigma} = \tau^{-\alpha}/2\Gamma(2-\alpha), \bar{\mu} := \bar{\sigma} - (1 + M' + (\sigma_1/4))$ . Then, we have



TABLE 8: Errors and convergent orders of MAM and DNM via quadratic basis in spatial direction with  $k = 2, \tau = 0.005$ , and  $\alpha = 0.1$  for Example 2.

$m$	$x(n)$	MAM				DNM			
		$ u^* - u_{2,m} _1$	Rate	$\ u^* - u_{2,m}\ _2$	Rate	$ u^* - u_n _1$	Rate	$\ u^* - u_n\ _2$	Rate
0	7	3.9445e-1		1.5119e-2		3.9445e-1		1.5119e-2	
1	15	1.0367e-1	1.9279	2.0779e-3	2.8631	1.0124e-1	1.9620	1.9499e-3	2.9549
2	31	2.9991e-2	1.7893	3.8616e-4	2.4279	2.5478e-2	1.9905	2.4561e-4	2.9889
3	63	7.5607e-3	1.9879	5.0534e-5	2.9339	6.3800e-3	1.9976	3.0758e-5	2.9973
4	127	1.8938e-3	1.9972	6.3432e-6	2.9940	1.5957e-3	1.9994	3.8482e-6	2.9987
5	255	4.7367e-4	1.9993	8.1030e-7	2.9687	3.9896e-4	1.9999	5.0488e-7	2.9301

TABLE 9: Comparison of CPU time between MAM and DNM with  $k = 2, \tau = 0.005$ , and  $\alpha = 0.1$  for Example 2.

$n = k + m$	$x(n)$	$T_{\text{MAM}}$	$T_{\text{DNM}}$
$2 = 2 + 0$	7	0.57	0.57
$3 = 2 + 1$	15	1.33	2.02
$4 = 2 + 2$	31	2.16	7.64
$5 = 2 + 3$	63	3.43	32.32
$6 = 2 + 4$	127	5.89	129.80
$7 = 2 + 5$	255	10.34	543.35

$$\begin{aligned} \|e^i\|_2^2 &\leq \frac{\bar{\sigma}}{\bar{\mu}} \left( \sum_{k=1}^{i-1} \|e^k\|_2^2 + \|e^0\|_2^2 \right) \\ &\quad + \left( \frac{\tau^{4-2\alpha}}{4\bar{\mu}\Gamma^2(2-\alpha)} + \frac{2M'^2}{\bar{\mu}} \right) \|u_{k,m}^i - u_{k,m-1}^i\|_2^2 + c(\tau^{2-\alpha} + h^r)^2. \end{aligned} \quad (56)$$

When the exact solution of (33) belongs to  $H^r(I)$ , there exists a positive integer  $N$ , for all  $k \geq N$  and any  $m \in \mathbb{N}$  (see [21, 23]):

$$\|u_{k,m}^i - u_{k,m-1}^i\| \leq ch^r. \quad (57)$$

Combining (56) and (57), we conclude from Gronwall's inequality that

$$\begin{aligned} \|e^i\|_2 &\leq \|e^0\|_2 + c(\tau^{2-\alpha}h^r + h^r) + c(\tau^{2-\alpha} + h^r) \\ &\leq \|e^0\|_2 + c(\tau^{2-\alpha} + h^r). \end{aligned} \quad (58)$$

Noting that  $\|e^0\|_2 \leq ch^r$ , then

$$\|u^i - u_{k,m}^i\|_2 \leq \|u^i - \mathcal{P}_{k+m}u^i\|_2 + \|e^i\|_2 \leq C(\tau^{2-\alpha} + h^r). \quad (59)$$

This completes the proof.

*Remark 8.* If we choose  $v = D_\tau^\alpha e^i$  in (48) and make a similar analysis as the above Theorem 7, we can obtain the optimal convergence order in  $H^1$  norm:

$$|u^i - u_{k,m}^i|_1 \leq c(\tau^{2-\alpha} + h^{r-1}). \quad (60)$$

## 6. Numerical Experiments

We present in this section numerical examples to illustrate the efficiency and accuracy of our proposed method. The computer programs are run on a personal computer with 2.5G CPU and 8G memory.

*Example 1.* We consider the time fractional Burgers equation (1) with the exact solution:

$$u(x, t) = (4t^2 - 4t + 1)x^2(x - 1)^2. \quad (61)$$

The corresponding initial condition and forcing term are

$$\begin{aligned} u(x, 0) &= u_0(x) = x^2(x - 1)^2 \\ f(x, t) &= \left( \frac{8t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{4t^{1-\alpha}}{\Gamma(2-\alpha)} \right) x^2(x - 1)^2 \\ &\quad + 2(4t^2 - 4t + 1)^2 x^3(x - 1)^3(2x - 1) \\ &\quad - 2(4t^2 - 4t + 1)(6x^2 - 6x + 1). \end{aligned} \quad (62)$$

Both piecewise linear ( $r = 2$ ) and quadratic ( $r = 3$ ) multi-scale orthonormal bases introduced in ‘‘Preliminaries’’ are employed in our numerical approximation. The numerical results are reported in Tables 1–5.  $k$  and  $m$  stand for the numbers of initial level and augmentation level used in the MAM, respectively.  $x(n)$  denotes the dimension of approximation subspace  $\mathbb{X}_n$  with  $n = k + m$ . Table 1 shows the  $L^2$  errors and temporal convergence rates for different  $\alpha$  using the MAM with  $(k, m) = (3, 7)$  for linear basis and  $(k, m) = (2, 5)$  for quadratic basis. It is seen that our numerical scheme has an accuracy of  $2 - \alpha$ , which is in agreement with our theoretical analysis. In the spatial direction, we illustrate the accuracy, convergence order, and computational efficiency of the MAM, with a comparison to those of the direct Newton's method (DNM) for solving the fully discrete scheme (16). The numerical results listed in Tables 2 and 3 are linear basis cases, and Tables 4 and 5 are quadratic basis cases. We can easily see from these tables that both MAM and DNM have the optimal convergence orders in the  $H^1$  norm (1 for the linear case and 2 for the quadratic

case) and in the  $L^2$  norm (2 for the linear case and 3 for the quadratic case). We also observe that MAM and DNM have nearly the same accuracy, while MAM takes significantly less time than DNM. To intuitively show the approximation effect, we plot in Figure 1 the absolute error surface of the approximation solution  $u_{2,5}$  obtained by MAM.

**Example 2.** We consider the time fractional Burgers equation (1) with initial condition:

$$u_0(x) = 0, \\ f(x, t) = \frac{2t^{2-\alpha} \sin(2\pi x)}{\Gamma(3-\alpha)} + \pi t^4 \sin(4\pi x) + 4\pi^2 t^2 \sin(2\pi x). \quad (63)$$

The exact solution of this problem is

$$u(x, t) = t^2 \sin(2\pi x). \quad (64)$$

The numerical results are presented in Figure 2 and Tables 6–9, where Figure 2 displays the convergence orders in temporal direction with different  $\alpha$ , and Tables 6–9 show the accuracy, convergence order, and computing time for the spatial direction. All the numerical results verify our theoretical analysis and also show the efficiency of the proposed algorithm.

## 7. Conclusion

In this article, the  $L1$ -discretization formula and the multi-scale Galerkin method are adopted to discrete the Caputo fractional derivative and spatial variable, respectively, and the multilevel augmentation algorithm is proposed for solving the resulting fully discrete scheme which is a nonlinear system at each time step. The MAM only needs to solve nonlinear systems in a fixed subspace with much lower dimension than that for the whole approximation subspace and compensate the error by multiplications of matrices and vectors at the high level. Therefore, the computational cost is greatly reduced. Numerical experiments are presented to confirm our theoretical results. Compared with the DNM, the proposed MAM has substantial advantages in computing time and is suitable for solving large-scale and high-accuracy problems.

## Data Availability

The data of numerical simulation used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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