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Research Article

Exponential Stability of Swelling Porous Elastic with a Viscoelastic Damping and Distributed Delay Term

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In this paper, we consider a swelling porous elastic system with a viscoelastic damping and distributed delay terms in the second equation. The coupling gives new contributions to the theory associated with asymptotic behaviors of swelling porous elastic soils. The general decay result is established by the multiplier method.

1. Introduction and Preliminaries

In the late 19th century, Eringen [1] proposed a theory in which he presented a mixture of viscous liquids and elastic solids in addition to gas. And he also studied the equilibrium laws for all components of this mixture, and finally, you get the field equations for a heat conductive mixture (for more details, see [2]). In [3], the author has classified expansive (swelling) soils under the classification of porous media theory.

On the other hand, it contains clay minerals that attract and absorb water, which leads to an increase in pressure [4], and this is considered a harmful and dangerous problem in architecture and civil engineering in most countries of the world, especially in foundations, which leads to cracks in buildings and ripples in sidewalks and roads (see [5–8]). From there, studies began to eliminate or reduce the damage, as in ([9–13]), where the basic field equations of the linear theory of swelling porous elastic soils were presented by

$$\rho_u u_{tt} = P_{1x} + G_1 + H_1, \tag{1}$$

$$\rho_\phi \phi_{tt} = P_{2x} + G_2 + H_2, \tag{2}$$

where u, ϕ are the displacement of the fluid and the elastic solid material. And $\rho_u, \rho_\phi > 0$ are the densities of each constituent. The functions (P_1, G_1, H_1) represent the partial tension, internal body forces, and eternal forces acting on the displacement, respectively. Similarly (P_2, G_2, H_2), it works on the elastic solid. In addition, the constitutive equations of partial tensions are given by

$$\begin{pmatrix} P_1 \\ P_1 \end{pmatrix} = \underbrace{\begin{pmatrix} a_1, a_2 \\ a_2, a_3 \end{pmatrix}}_{A} \cdot \begin{pmatrix} u_x \\ \phi_x \end{pmatrix}, \tag{3}$$

where $a_1, a_3 > 0$ and $a_2 \neq 0$ is a real number. A is a matrix positive definite in the sense that $a_1 a_3 > a_2^2$.

Quintanilla [10] investigated (1) by taking

$$G_1 = G_2 = \xi(u_t - \phi_t),$$
$$H_1 = a_3 u_{xxt}, \tag{4}$$
$$H_2 = 0,$$

where $\xi > 0$; they obtained that the stability is exponential. Similarly, in [14], the authors considered (1) with different conditions

$$G_1 = G_2 = 0,$$
$$H_1 = -\rho_u \gamma(x) u_t,$$
$$H_2 = 0, \tag{5}$$

where $\gamma(x)$ is an internal viscous damping function with a positive mean. They established the exponential stability result (see ([10–20]) for some other interesting results on the swelling porous system).

Time delays arise in many applications because most phenomena naturally depend not only on the present state but also on some past occurrences.

In recent years, the control of PDEs with time delay effects has become an active area of research (see, for example, [15, 20–27]). In many cases, it was shown that delay is a source of instability unless additional condition or control terms are used; the stability issue of systems with delay is of theoretical and practical great importance.

A complement to these works, and by introducing the terms of memory and distributed delay, forms a new problem different from previous studies. Under appropriate assumptions and by using the energy method, we prove the stability results.

In this paper, we are interested in problem (1) with null internal body forces, but the eternal force acting only on the elastic solid is in the form of viscoelastic damping and distributed delay terms, that is,

$$G_1 = G_2 = H_1 = 0,$$
$$H_2 = -\int_0^t g(t-s)\phi_{xx}(x,s)ds - \beta_1 \phi_t - \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \phi_t(x, t-\sigma) d\sigma. \tag{6}$$

Remark 1. Regarding the problems of swelling porous elastic, we believe that there are no studies of viscoelasticity (the memory) and the distributed delay conditions that act as a simultaneous dissipation mechanism, and hence, our coupling constitutes a new contribution.

Thus, we are interested in the following problem:

$$\begin{cases} \rho_u u_{tt} - a_1 u_{xx} - a_2 \phi_{xx} = 0, \\ \rho_\phi \phi_{tt} - a_3 \phi_{xx} - a_2 u_{xx} + \int_0^t g(t-s)\phi_{xx}(x,s)ds + \beta_1 \phi_t + \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \phi_t(x, t-\sigma) d\sigma = 0, \end{cases} \tag{7}$$

where

$$(x, \sigma, t) \in \mathscr{H} = (0,1) \times (\tau_1, \tau_2) \times (0, \infty), \tag{8}$$

under the initial and boundary conditions

$$\begin{gathered} u(x,0) = u_0(x), u_t(x,0) = u_1(x), \\ \phi(x,0) = \phi_0(x), \phi_t(x,0) = \phi_1(x), x \in (0,1), \\ \phi_t(x,-t) = f_0(x,t), x \in (0,1) \times (0, \tau_2), \\ u(0,t) = u(1,t) = \phi(0,t) = \phi(1,t) = 0, t \geq 0. \end{gathered} \tag{9}$$

First, as in [27], taking the following new variable

$$\mathscr{Y}(x, \rho, \sigma, t) = \phi_t(x, t - \sigma\rho), \tag{10}$$

then we obtain

$$\begin{cases} \sigma \mathscr{Y}_t(x, \rho, \sigma, t) + \mathscr{Y}_\rho(x, \rho, \sigma, t) = 0, \\ \mathscr{Y}(x, 0, \sigma, t) = \phi_t(x, t). \end{cases} \tag{11}$$

Consequently, the problem is equivalent to

$$\begin{cases} \rho_u u_{tt} - a_1 u_{xx} - a_2 \phi_{xx} = 0, \\ \rho_\phi \phi_{tt} - a_3 \phi_{xx} - a_2 u_{xx} + \int_0^t g(t-s)\phi_{xx}(x,s)ds + \beta_1 \phi_t + \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \mathscr{Y}(x, 1, \sigma, t) d\sigma = 0, \\ \sigma \mathscr{Y}_t(x, \rho, \sigma, t) + \mathscr{Y}_\rho(x, \rho, \sigma, t) = 0, \end{cases} \tag{12}$$

where

$$(x, \rho, \sigma, t) \in (0,1) \times \mathscr{H}, \tag{13}$$

with the initial data

$$\begin{cases} u(x,0) = u_0(x), u_t(x,0) = u_1(x), \\ \phi(x,0) = \phi_0(x), \phi_t(x,0) = \phi_1(x), x \in (0,1), \\ \mathscr{Y}(x,\rho,\sigma,0) = f_0(x,\rho\sigma), (x,\rho,\sigma) \in (0,1) \times (0,1) \times (0,\tau_2), \end{cases} \tag{14}$$

and the boundary conditions

$$u(0,t) = u(1,t) = \phi(0,t) = \phi(1,t) = 0, t \geq 0. \tag{15}$$

Here, $\rho_u, \rho_\phi, a_1, a_3, \beta_1$ are positive constants and a_2 is a real number, with a_1, a_2, a_3 satisfying $a = a_3 - a_2^2/a_1 > 0$. The integrals represent the memory and the distributed delay terms with $\tau_1, \tau_2 > 0$ are a time delay, β_2 is an L^∞ function, and the kernel g is the relaxation function, under the following assumptions.

(H1) $g \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is a nonincreasing function satisfying

$$g(0) > 0, \; a - \int_0^\infty g(s)ds = l > 0, \tag{16}$$

where $a = a_3 - a_2^2/a_1 > 0$.

(H2) There exists a $\vartheta \in (\mathbb{R}_+, \mathbb{R}_+)$ positive nonincreasing differentiable function, such that

$$g'(t) \leq -\vartheta(t) g(t), \quad t \geq 0. \tag{17}$$

(H3) $\beta_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| d\sigma < \beta_1. \tag{18}$$

Remark 2. The results that we obtained in this work are also correct with other conditions, including

$$\begin{aligned} u_x(0,t) = u_x(1,t) = \phi_x(0,t) = \phi_x(1,t) = 0, \quad t \geq 0, \\ u(0,t) = u_x(1,t) = \phi(0,t) = \phi_x(1,t) = 0, \quad t \geq 0, \\ u_x(0,t) = u(1,t) = \phi_x(0,t) = \phi(1,t) = 0, \quad t \geq 0. \end{aligned} \tag{19}$$

Of course, there can be some difficulties with regard to the following boundary conditions:

$$\begin{aligned} u_x(0,t) = u_x(1,t) = \phi(0,t) = \phi(1,t) = 0, \quad t \geq 0, \\ u(0,t) = u(1,t) = \phi_x(0,t) = \phi_x(1,t) = 0, \quad t \geq 0, \end{aligned} \tag{20}$$

unless we assume

$$\int_0^1 u_0(x)dx = 0, \int_0^1 \phi_0(x)dx = 0, \tag{21}$$

respectively.

In this paper, we consider $(u, \phi, \mathscr{Y})$ to be a solution of system (12)–(15) with the regularity needed to justify the calculations. In Section 2, we proved our decay result. And we symbolize that c is a positive constant.

2. Main Result

In this section, we prove our stability result for the energy of system (12)–(15).

We need the following lemmas.

Lemma 3. *The energy functional E, defined by*

$$\begin{aligned} E(t) = \frac{1}{2}\int_0^1 \Big[\rho_u u_t^2 + a_1 u_x^2 + \rho_\phi \phi_t^2 \\ + \left(a_3 - \int_0^t g(s)ds\right)\phi_x^2 + 2a_2 u_x \phi_x\Big] dx + \frac{1}{2} g \circ \phi_x \\ + \frac{1}{2}\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \sigma |\beta_2(\sigma)| \mathscr{Y}^2(x,\rho,\sigma,t) d\sigma d\rho dx, \end{aligned} \tag{22}$$

satisfies

$$\begin{aligned} E'(t) &\leq \frac{1}{2} g' \circ \phi_x - \frac{1}{2} g(t) \int_0^1 \phi_x^2 dx \\ &\quad - \left(\beta_1 - \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| d\sigma\right) \int_0^1 \phi_t^2 dx \\ &\leq \frac{1}{2} g' \circ \phi_x - \eta_0 \int_0^1 \phi_t^2 dx \leq 0, \end{aligned} \tag{23}$$

where $\eta_0 = \beta_1 - \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| d\sigma > 0$ *and*

$$(g \circ v_x)(t) = \int_0^1 \int_0^t g(t-s)(v_x(t) - v_x(s))^2 ds dx. \tag{24}$$

Proof. Multiplying $(12)_{1,2}$ by u_t and ϕ_t, then integration by parts over $(0,1)$, with (15), gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_u u_t^2 + a_1 u_x^2 + \rho_\phi \phi_t^2 + a_3 \phi_x^2 + 2a_2 u_x \phi_x\right] dx \\ + \beta_1 \int_0^1 \phi_t^2 dx + \int_0^1 \phi_t \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \mathscr{Y}(x,1,\sigma,t) d\sigma dx \\ - \int_0^1 \phi_{xt} \int_0^t g(t-s)\phi_x(s) ds dx = 0. \end{aligned} \tag{25}$$

The estimate of the last term in the LHS of (25) is as follows:

$$\begin{aligned}
-\int_0^1 \phi_{xt}\int_0^t g(t-s)\phi_x(s)dsdx
&= \int_0^1 \phi_{xt}\int_0^t g(t-s)(\phi_x(t)-\phi_x(s))dsdx \\
&\quad - \int_0^t g(s)ds\int_0^1 \phi_{xt}\phi_x dx \\
&= \frac{1}{2}\frac{d}{dt}g\circ\phi_x - \frac{1}{2}\frac{d}{dt}\int_0^t g(s)ds\int_0^1 \phi_x^2 dx \\
&\quad - \frac{1}{2}g'\circ\phi_x + \frac{1}{2}g(t)\int_0^1 \phi_x^2 dx.
\end{aligned} \tag{26}$$

Now, multiplying $((12))_3$ by $\mathscr{Y} \mid \beta_2(\sigma) \mid$, and by integration over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma|\beta_2(\sigma)|\mathscr{Y}^2(x,\rho,\sigma,t)d\sigma d\rho dx \\
&= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\mathscr{Y}\mathscr{Y}_\rho(x,\rho,\sigma,t)d\sigma d\rho dx \\
&= -\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\frac{d}{d\rho}\mathscr{Y}^2(x,\rho,\sigma,t)d\sigma d\rho dx \\
&= \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\left(\mathscr{Y}^2(x,0,\sigma,t)-\mathscr{Y}^2(x,1,\sigma,t)\right)d\sigma dx \\
&= \frac{1}{2}\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|d\sigma\int_0^1 \phi_t^2 dx - \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\sigma_2(\sigma)|\mathscr{Y}^2(x,1,\sigma,t)d\sigma dx.
\end{aligned} \tag{27}$$

Now, by substituting (26) into (25), and using Young's inequality, we have

$$\begin{aligned}
E'(t) &\le \frac{1}{2}g'\circ\phi_x - \frac{1}{2}g(t)\int_0^1 \phi_x^2 dx \\
&\quad - \left(\beta_1 - \int_{\tau_1}^{\tau_2} \mid \beta_2(\sigma) \mid d\sigma\right)\int_0^1 \phi_t^2 d \\
&\le \frac{1}{2}g'\circ\phi_x - \left(\beta_1 - \int_{\tau_1}^{\tau_2} \mid \beta_2(\sigma) \mid d\sigma\right)\int_0^1 \phi_t^2 dx,
\end{aligned} \tag{28}$$

then, by (18), there exists $\eta_0 > 0$ so that

$$E'(t) \le \frac{1}{2}g'\circ\phi_x - \eta_0\int_0^1 \phi_t^2 dx, \tag{29}$$

then we obtain (22) and (23) (E is a nonincreasing function).

Lemma 4. *The functional*

$$D_1(t) := \rho_\phi\int_0^1 \phi_t\phi dx - \frac{a_2}{a_1}\rho_u\int_0^1 \phi u_t dx + \frac{\beta_1}{2}\int_0^1 \phi^2 dx \tag{30}$$

satisfies

$$\begin{aligned}
D_1'(t) &\le -\frac{a_0}{2}\int_0^1 \phi_x^2 dx + \varepsilon_1\int_0^1 u_t^2 dx + c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 \phi_t^2 dx \\
&\quad + cg\circ\phi_x + c\int_0^1\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\mathscr{Y}^2(x,1,\sigma,t)d\sigma dx.
\end{aligned} \tag{31}$$

Proof. Direct computation using integration by parts and Young's inequality, for $\varepsilon_1 > 0$, yields

$$\begin{aligned}
D_1'(t) &= -a_3\int_0^1 \phi_x^2 dx + \rho_\phi\int_0^1 \phi_t^2 dx + \frac{a_2^2}{a_1}\int_0^1 \phi_x^2 dx \\
&\quad + \frac{a_2}{a_1}\rho_u\int_0^1 \phi_t u_t dx + \int_0^1 \phi_x\int_0^t g(t-s)\phi_x(s)dsdx \\
&\quad - \int_0^1 \phi\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\mathscr{Y}(x,1,\sigma,t)d\sigma dx \\
&\le -\left(a_3 - \frac{a_2^2}{a_1}\right)\int_0^1 \phi_x^2 dx + \rho_\phi\int_0^1 \phi_t^2 dx \\
&\quad + \frac{a_2}{a_1}\rho_u\int_0^1 \phi_t u_t d - \int_0^1 \phi\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\mathscr{Y}(x,1,\sigma,t)d\sigma dx \\
&\quad + \int_0^1 \phi_x\int_0^t g(t-s)\phi_x(s)dsdx.
\end{aligned} \tag{32}$$

The estimate of the two last terms in the RHS of (32) is as follows:

$$\begin{aligned}
&\int_0^1 \phi_x\int_0^t g(t-s)\phi_x(s)dsdx \\
&= \int_0^t g(s)ds\int_0^1 \phi_x^2 dx - \int_0^1 \phi_x\int_0^t g(t-s)(\phi_x(t)-\phi_x(s))dsdx \\
&\le \left(\delta_1 + \int_0^t g(s)ds\right)\int_0^1 \phi_x^2 dx + \frac{1}{4\delta_1}\left(\int_0^t g(s)ds\right)g\circ\phi_x,
\end{aligned} \tag{33}$$

$$\begin{aligned}
&-\int_0^1 \phi\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\mathscr{Y}(x,1,\sigma,t)d\sigma dx \\
&\le c\delta_2\left(\int_{\tau_1}^{\tau_2} \mid \beta_2(\sigma) \mid d\sigma\right)\int_0^1 \phi_x^2 dx \\
&\quad + \frac{1}{4\delta_2}\int_0^t\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\mathscr{Y}^2(x,1,\sigma,t)d\sigma dx,
\end{aligned} \tag{34}$$

where we have used Cauchy-Schwartz, Young's, and Poincaré's inequalities, for $\delta_1, \delta_2 > 0$, and (18).

By substituting (33) and (34) into (32), we find

$$\begin{aligned} D_1'(t) \le & -\left(a_3 - \frac{a_2^2}{a_1} - \int_0^t g(s)ds - \delta_1 - \beta_1 c\delta_2\right)\int_0^1 \phi_x^2 dx \\ & + \varepsilon_1 \int_0^1 u_t^2 dx + c\left(1 + \frac{1}{\varepsilon_1}\right)\int_0^1 \phi_t^2 dx \\ & + \frac{1}{4\delta_1}\left(\int_0^t g(s)ds\right) g \circ \phi_x \\ & + \frac{1}{4\delta_2}\int_0^t \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \mathscr{Y}^2(x,1,\sigma,t) d\sigma dx. \end{aligned} \tag{35}$$

Bearing in mind that $a = a_3 - a_2^2/a_1 > 0$ and using (16), we get

$$\int_0^t g(s)ds < \int_0^\infty g(s)ds < a, \tag{36}$$

let $a_0 = (a_3 - a_2^2/a_1) - \int_0^t g(s)ds > 0$, and letting $\delta_1 = a_0/4$, $\delta_2 = a_0/4c\mu_1$, gives (31).

Lemma 5. *Assume that (16) hold. Then, the functional*

$$D_2(t) := a_2\left(\int_0^1 \phi_t u dx - \int_0^1 \phi u_t dx\right) \tag{37}$$

satisfies,

$$\begin{aligned} D_2'(t) \le & -\frac{a_2^2}{2\rho_\phi}\int_0^1 u_x^2 dx + c\int_0^1 \phi_x^2 dx + c\int_0^1 \phi_t^2 dx \\ & + cg \circ \phi_x + c\int_0^1 \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \mathscr{Y}^2(x,1,\sigma,t) d\sigma dx. \end{aligned} \tag{38}$$

Proof. By differentiating D_2, then using (12), integration by parts, and (15), we find

$$\begin{aligned} D_2'(t) = & -\frac{a_2^2}{\rho_\phi}\int_0^1 u_x^2 dx + \frac{a_2^2}{\rho_u}\int_0^1 \phi_x^2 dx \\ & - \left(\frac{a_2 a_3}{\rho_\phi} - \frac{a_1 a_2}{\rho_u}\right)\int_0^1 \phi_x u_x dx - \frac{a_2\beta_1}{\rho_\phi}\int_0^1 u\phi_t dx \\ & + \frac{a_2}{\rho_\phi}\int_0^1 u_x \int_0^t g(t-s)\phi_x(s) dsd \\ & - \frac{a_2}{\rho_\phi}\int_0^1 u\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \mathscr{Y}(x,1,\sigma,t) d\sigma dx. \end{aligned} \tag{39}$$

In what follows, we estimate the different terms in the RHS of (39); we use Young's, Cauchy-Schwartz, and Poincaré's inequalities. For $\delta_3, \delta_4, \delta_5 > 0$, we have

$$\begin{aligned} & -\left(\frac{a_2 a_3}{\rho_\phi} - \frac{a_1 a_2}{\rho_u}\right)\int_0^1 \phi_x u_x dx \\ & \quad \le \delta_3 \int_0^1 u_x^2 dx + \left(\frac{a_2 a_3}{\rho_\phi} - \frac{a_1 a_2}{\rho_u}\right)^2 \frac{1}{4\delta_3}\int_0^1 \phi^2 dx, \\ & \frac{a_2}{\rho_\phi}\int_0^1 u_x \int_0^t g(t-s)\phi_x(s) dsdx \\ & \quad \le 2\delta_4 \int_0^1 u_x^2 dx + \frac{c}{4\delta_4}\int_0^1 \phi_x^2 dx + \frac{c}{\delta_4} g \circ \phi_x, \\ & -\frac{a_2\beta_1}{\rho_\phi}\int_0^1 u\phi_t dx \le c\delta_5 \int_0^1 u_x^2 dx + \frac{c}{4\delta_5}\int_0^1 \phi_t^2 dx, \\ & -\frac{a_2}{\rho_\phi}\int_0^1 u\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \mathscr{Y}(x,1,\sigma,t) d\sigma dx \\ & \quad \le c\delta_6 \int_0^1 u_x^2 dx - \frac{c}{4\delta_6}\int_0^1 \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \mathscr{Y}^2(x,1,\sigma,t) d\sigma dx. \end{aligned} \tag{40}$$

By letting $\delta_3 = a_2/8\rho_\phi, \delta_4 = a_2/16\rho_\phi, \delta_5 = \delta_6 = a_2/8c\rho_\phi$ and substituting into (39), we get (38).

Lemma 6. *The functional*

$$D_3(t) := -\rho_u \int_0^1 u_t u dx \tag{41}$$

satisfies

$$D_3'(t) \le -\rho_u \int_0^1 u_t^2 dx + 2a_1 \int_0^1 u_x^2 dx + \frac{a_3}{4}\int_0^1 \phi_x^2 dx. \tag{42}$$

Proof. Direct computations give

$$D_3'(t) = -\rho_u \int_0^1 u_t^2 dx + a_1 \int_0^1 u_x^2 dx + a_2 \int_0^1 u_x \phi_x dx. \tag{43}$$

Estimate (42) easily follows by using Young's inequality and $a_1 a_3 > a_2^2$.

Now, let us introduce the following functional used.

Lemma 7. *The functional*

$$D_4(t) := \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \sigma e^{-\sigma\rho} |\beta_2(\sigma)| \mathscr{Y}^2(x,\rho,\sigma,t) d\sigma d\rho dx \tag{44}$$

satisfies

$$D_4'(t) \le -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \sigma |\beta_2(\sigma)| \mathscr{Y}^2(x,\rho,\sigma,t)\, d\sigma\, d\rho\, dx + \beta_1 \int_0^1 \phi_t^2 dx - \eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \mathscr{Y}^2(x,1,\sigma,t)\, d\sigma\, dx, \tag{45}$$

where $\eta_1 > 0$.

Proof. By differentiating D_4, with respect to t and using the last equation in (12), we have

$$\begin{aligned} D_4'(t) &= -2\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\sigma\rho} |\beta_2(\sigma)| \mathscr{Y}\mathscr{Y}_\rho(x,\rho,\sigma,t)\, d\sigma\, d\rho\, dx \\ &= -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \sigma e^{-\sigma\rho} |\beta_2(\sigma)| \mathscr{Y}^2(x,\rho,\sigma,t)\, d\sigma\, d\rho\, dx \\ &\quad - \int_0^1 \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \left[e^{-\sigma} \mathscr{Y}^2(x,1,\sigma,t) - \mathscr{Y}^2(x,0,\sigma,t)\right] d\sigma\, dx. \end{aligned} \tag{46}$$

By using $\mathscr{Y}(x,0,\sigma,t) = \phi_t(x,t)$, and $e^{-\sigma} \le e^{-\sigma\rho} \le 1$, for all $0 < \rho < 1$, we find

$$\begin{aligned} D_4'(t) &= -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \sigma |\beta_2(\sigma)| \mathscr{Y}^2(x,\rho,\sigma,t)\, d\sigma\, d\rho\, dx \\ &\quad - \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\sigma} |\beta_2(\sigma)| \mathscr{Y}^2(x,1,\sigma,t)\, d\sigma\, dx \\ &\quad + \left(\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\, d\sigma \right) \int_0^1 \phi_t^2 dx. \end{aligned} \tag{47}$$

Because $-e^{-\sigma}$ is an increasing function, we have $-e^{-\sigma} \le -e^{-\tau_2}$, for all $\sigma \in [\tau_1, \tau_2]$.

Finally, setting $\eta_1 = e^{-\tau_2}$ and recalling (18) give (45). We are now ready to prove the main result.

Theorem 8. *Assume (16)–(18) hold.*

Then, $\forall t_0 > 0$, *there exist* $\lambda_1, \lambda_2 > 0$ *such that the energy functional given by (22) satisfies*

$$E(t) \le \lambda_1 e^{-\lambda_2 \int_{t_0}^t \vartheta(s) ds}, \quad \forall t \ge t_0. \tag{48}$$

Proof. We define the functional of Lyapunov

$$\mathscr{L}(t) := NE(t) + N_1 D_1(t) + N_2 D_2(t) + D_3(t) + N_4 D_4(t), \tag{49}$$

where $N, N_1, N_2, N_4 > 0$ to be selected later.

By differentiating (49) and using (22), (31), (38), (42), and (45), we have

$$\begin{aligned} \mathscr{L}'(t) &\le -\left[\frac{a_0 N_1}{2} - cN_2 - \frac{a_3}{4} \right] \int_0^1 \phi_x^2 dx \\ &\quad - [\rho_u - N_1 \varepsilon_1] \int_0^1 u_t^2 dx - \left[\frac{a_2^2 N_2}{2\rho_\phi} - 2a_1 \right] \int_0^1 u_x^2 dx \\ &\quad + c[N_1 + N_2] g \circ \phi_x + \frac{N}{2} g' \circ \phi_x \\ &\quad - \left[\eta_0 N - cN_1 \left(1 + \frac{1}{\varepsilon_1} \right) - N_2 c - \beta_1 N_4 \right] \int_0^1 \phi_t^2 dx \\ &\quad - [N_4 \eta_1 - cN_1 - cN_2] \int_0^1 \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \mathscr{Y}^2 \\ &\quad \cdot (x,1,\sigma,t)\, d\sigma\, dx - N_4 \eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \sigma |\beta_2(\sigma)| \mathscr{Y}^2 \\ &\quad \cdot (x,\rho,\sigma,t)\, d\sigma\, d\rho\, dx. \end{aligned} \tag{50}$$

By setting

$$\varepsilon_1 = \frac{\rho_u}{2N_1}, \tag{51}$$

we obtain

$$\begin{aligned} \mathscr{L}'(t) &\le -\left[\frac{a_0 N_1}{2} - cN_2 - \frac{a_3}{4} \right] \int_0^1 \phi_x^2 dx - \left[\frac{\rho_u}{2} \right] \int_0^1 u_t^2 dx \\ &\quad - \left[\frac{a_2^2 N_2}{2\rho_\phi} - 2a_1 \right] \int_0^1 u_x^2 dx + c[N_1 + N_2] g \circ \phi_x \\ &\quad + \frac{N}{2} g' \circ \phi_x - [\eta_0 N - cN_1 - N_2 c - \beta_1 N_4] \int_0^1 \phi_t^2 dx \\ &\quad - [N_4 \eta_1 - cN_1 - cN_2] \int_0^1 \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \mathscr{Y}^2 \\ &\quad \cdot (x,1,\sigma,t)\, d\sigma\, dx - N_4 \eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \sigma |\beta_2(\sigma)| \mathscr{Y}^2 \\ &\quad \cdot (x,\rho,\sigma,t)\, d\sigma\, d\rho\, dx. \end{aligned} \tag{52}$$

At this stage, we choose our different constants. First, choosing N_2 large enough such that

$$\alpha_1 = \frac{a_2^2 N_2}{2\rho_\phi} - 2a_1 > 0, \tag{53}$$

then we pick N_1 large enough such that

$$\alpha_2 = \frac{a_0 N_1}{2} - cN_2 - \frac{a_3}{4} > 0, \tag{54}$$

then we select N_4 large enough such that

$$\alpha_3 = N_4\eta_1 - cN_1 - cN_2 > 0. \quad (55)$$

Thus, we arrive at

$$\begin{aligned}\mathscr{L}'(t) \le & -\alpha_2\int_0^1 \phi_x^2 dx - \frac{\rho_u}{2}\int_0^1 u_t^2 dx - \alpha_1\int_0^1 u_x^2 dx \\ & - [\eta_0 N - c]\int_0^1 \phi_t^2 dx + \alpha_6 g' \circ \phi_x + \alpha_7 g \circ \phi_x \\ & - \alpha_3\int_0^1\int_{\tau_1}^{\tau_2} |\beta_2(\sigma)|\mathscr{Y}^2(x,1,\sigma,t)d\sigma d \\ & - \alpha_5\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma|\beta_2(\sigma)|\mathscr{Y}^2(x,\rho,\sigma,t)d\sigma d\rho dx,\end{aligned} \quad (56)$$

where $\alpha_5 = \eta_1 N_4$, $\alpha_6 = N/2$, $\alpha_7 = c[N_1 + N_2]$.

On the other hand, if we let

$$\mathfrak{L}(t) = N_1 D_1(t) + N_2 D_2(t) + D_3(t) + N_4 D_4(t), \quad (57)$$

then

$$\begin{aligned}|\mathfrak{L}(t)| \le & N_1\rho_\phi\int_0^1 |\phi\phi_t|dx + N_1\frac{a_2}{a_1}\rho_u\int_0^1 |\phi u_t|dx \\ & + N_1\frac{\mu_1}{2}\int_0^1 \phi^2 dx + N_2 a_2\int_0^1 |\phi u_t - u\phi_t|dx \\ & + \rho_u\int_0^1 |u_t u|\mathrm{d}x + N_4\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma e^{-\sigma\rho}|\beta_2(\sigma)|\mathscr{Y}^2 \\ & \cdot (x,\rho,\sigma,t)d\sigma d\rho dx.\end{aligned} \quad (58)$$

Exploiting Young's, Cauchy-Schwartz, and Poincaré inequalities, we obtain

$$\begin{aligned}|\mathfrak{L}(t)| \le & c\int_0^1 \left(u_t^2 + \phi_t^2 + \phi_x^2 + u_x^2\right)dx + cg \circ \phi_x \\ & + c\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \sigma|\beta_2(\sigma)|\mathscr{Y}^2(x,\rho,\sigma,t)d\sigma d\rho.\end{aligned} \quad (59)$$

On the other hand, from (22), we can write

$$\begin{aligned}a_1 u_x^2 + 2a_2\phi_x u_x + a_4\phi_x^2 = & \frac{1}{2}\left[a_1\left(u_x + \frac{a_2}{a_1}\phi_x\right)^2 + a_4\left(\phi_x + \frac{a_2}{a_4}u_x\right)^2\right. \\ & \left. + \left(a_1 - \frac{a_2^2}{a_4}\right)u_x^2 + \left(a_4 - \frac{a_2^2}{a_1}\right)\phi_x^2\right],\end{aligned} \quad (60)$$

where

$$a_4 = a_3 - \int_0^t g(s)ds. \quad (61)$$

Since $a_1 a_3 > a_2^2$ and (16), we deduce that

$$a_1 u_x^2 + 2a_2\phi_x u_x + a_4\phi_x^2 > \frac{1}{2}\left[\left(a_1 - \frac{a_2^2}{a_4}\right)u_x^2 + \left(a_4 - \frac{a_2^2}{a_1}\right)\phi_x^2\right]. \quad (62)$$

Consequently, we find

$$|\mathfrak{L}(t)| = |\mathscr{L}(t) - NE(t)| \le cE(t), \quad (63)$$

that is,

$$(N - c)E(t) \le \mathscr{L}(t) \le (N + c)E(t). \quad (64)$$

At this point, we choose N large enough such that

$$N - c > 0, N\eta_0 - c > 0, \quad (65)$$

and exploiting (22), estimates (56) and (64), respectively, leads to

$$c_2 E(t) \le \mathscr{L}(t) \le c_3 E(t), \quad \forall t \ge 0, \quad (66)$$

$$\mathscr{L}'(t) \le -k_1 E(t) + k_2 g \circ \phi_x, \forall t \ge t_0, \quad (67)$$

for some $k_1, k_2, c_2, c_3 > 0$.

By multiplying (67) by $\vartheta(t)$, we get

$$\vartheta(t)\mathscr{L}'(t) \le -k_1\vartheta(t)E(t) + k_2\vartheta(t)g \circ \phi_x, \forall t \ge t_0. \quad (68)$$

Now, by using (17), we have the following estimate:

$$\begin{aligned}\vartheta(t)g \circ \phi_x = & \vartheta(t)\int_0^1\int_0^t g(t-s)(\phi_x(t) - \phi_x(s))^2 ds dx \\ \le & \int_0^1\int_0^t \vartheta(\mathrm{t}-s)g(t-s)(\phi_x(t) - \phi_x(s))^2 ds d \\ \le & -\int_0^1\int_0^t g'(t-s)(\phi_x(t) - \phi_x(s))^2 ds dx \\ = & -g' \circ \phi_x \le -2E'(t).\end{aligned} \quad (69)$$

Thus, (68) becomes

$$\vartheta(t)\mathscr{L}'(t) \le -k_1\vartheta(t)E(t) - 2k_2 E'(t), \quad \forall t \ge t_0, \quad (70)$$

which can be rewritten as

$$(\vartheta(t)\mathscr{L}(t) + 2k_2 E(t))' - \vartheta'(t)\mathscr{L}(t) \le -k_1\vartheta(t)E(t), \quad \forall t \ge t_0. \quad (71)$$

By using $\vartheta'(t) \le 0, \forall t \ge 0$, we have

$$(\vartheta(t)\mathscr{L}(t) + 2k_2 E(t))' \le -k_1\vartheta(t)E(t), \quad \forall t \ge t_0. \quad (72)$$

By exploiting (66), we notice that

$$\mathscr{K}(t) = \vartheta(t)\mathscr{L}(t) + 2k_2 E(t) \sim E(t). \quad (73)$$

Consequently, for $\kappa > 0$, we get

$$\mathscr{K}'(t) \le -\kappa \mathscr{K}(t)\vartheta(t), \quad \forall t \ge t_0. \tag{74}$$

Integrating (74) over (t_0, t) gives

$$\mathscr{K}(t) \le \mathscr{K}(t_0)e^{-\kappa\int_{t_0}^{t}\vartheta(s)ds}, \quad \forall t \ge t_0. \tag{75}$$

Consequently, (48) is established by virtue of (66) and (75).

Remark 9. The estimate (48) also remains valid for $t \in [0, t_0]$, thanks to the boundedness and continuity of E and ϑ.

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

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