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Research Article

Toeplitz Operators whose Symbols Are Borel Measures

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In this paper, we are concerned with Toeplitz operators whose symbols are complex Borel measures. When a complex Borel measure μ on the unit circle is given, we give a formal definition of a Toeplitz operator T_μ with symbol μ, as an unbounded linear operator on the Hardy space. We then study various properties of T_μ. Among them, there is a theorem that the domain of T_μ is represented by a trichotomy. Also, it was shown that if the domain of T_μ contains at least one polynomial, then T_μ is densely defined. In addition, we give evidence for the conjecture that T_μ with a singular measure μ reduces to a trivial linear operator.

1. Introduction

A classical Toeplitz operator is the compression of a multiplication operator on the Lebesgue space $L^2(\mathbb{T})$ of the unit circle $\mathbb{T}$ to the Hardy space $H^2(\mathbb{T})$. The study of Toeplitz operators seems to have originated from the paper of Toeplitz [1]. In the paper [2], he used Toeplitz matrices to characterize nonnegative continuous functions on the unit circle in terms of their Fourier coefficients. The remarkable paper of Brown and Halmos [3] started the systematic study of spectral properties of Toeplitz operators. Since then, the theory of Toeplitz operators has been studied in various ways. Recently, the theory of Toeplitz operators has been studied in a variety of settings and connections with other fields. One direction is to deal with Toeplitz operators on reproducing kernel spaces like Bergman spaces, Dirichlet spaces, or Fock spaces (cf. [4–8]). Another direction is to study Toeplitz operators with operator-valued symbols (cf. [9–11]). Also, truncated Toeplitz operators have attracted attention. A systematic approach on truncated Toeplitz operators can be found in the paper of Sarason in 2007 [12]. In that paper, he has used "compatible" measures to describe bounded truncated Toeplitz operators. The boundedness of infinite Hankel matrices is also related to the compatibility of measures: the infinite Hankel matrix of the moment of a nonnegative Carleson measure is bounded and vice versa [13]. (For related recent studies, see [14].) These works inspired us to consider Toeplitz operators whose symbols are measures. The Toeplitz operators whose symbols are measures have been studied in the setting of Bergman spaces and other spaces (cf. [15], chapter 7).

In this paper, we consider Toeplitz operators on the Hardy space, whose symbols are measures. In this study, unbounded Toeplitz operators arise naturally. When studying unbounded Toeplitz operators, it was usually considered that the symbols come from $L^2(\mathbb{T})$. In 2008, Sarason [16] treated not only the case of $L^2(\mathbb{T})$-symbols but the case of analytic functions on the open unit disk $\mathbb{D}$. It is natural to attempt to extend the symbols of Toeplitz operators to measures, because the initial reasearch for them was related to the moment problem. As mentioned before, Toeplitz and Hankel operators associated with measures can be seen in the papers [13] and [12]. In this paper, we provide an explicit definition of Toeplitz operators whose symbols are complex Borel measures and then consider their unbounded operator theory. As the study on Toeplitz operators whose symbols are functions shows the interplay between function theory and operator theory, the study on Toeplitz operators whose symbols are measures is also expected to show the interplay between measure theory and operator theory.

Our consideration for the symbol of a Toeplitz operator, denoted by T_μ, is a complex Borel measure μ on the unit circle. When we study an unbounded linear operator, we usually assume that its domain is dense, i.e., the operator is densely defined. Hence, one may ask if T_μ is densely defined, i.e., the domain is dense in H^2. Toeplitz operators with L^2-symbols are always densely defined. Unlike when the symbol is a function, it does not seem easy to answer the question. Nonetheless, we will show that the domain of T_μ is represented by a trichotomy (Theorem 8). In particular, we can show that if the domain of T_μ contains at least one polynomial, then T_μ is densely defined (Proposition 10). We also give evidence for the conjecture that the cases of singular measures induce trivial linear operators (Theorem 15).

The organization of this paper is as follows. In Section 2, we give notations, definitions, and preliminary facts, which will be used in the sequel. In Section 3, we give a formal definition of Toeplitz operators whose symbols are complex Borel measures on $\mathbb{T}$ and then investigate their properties in the viewpoint of unbounded linear operator theory.

2. Preliminaries

Let $\mathbb{T}$ be the unit circle in the complex plane. Let m be the normalized Lebesgue measure on $\mathbb{T}$, so that $m(\mathbb{T}) = 1$. For $1 \le p \le \infty$, we write $L^p(\mathbb{T}) = L^p(\mathbb{T}, m)$ for the Lebesgue space on $\mathbb{T}$ and $H^p(\mathbb{T})$ for the Hardy space on $\mathbb{T}$. Note that $H^p(\mathbb{T})$ is a closed subspace of $L^p(\mathbb{T})$.

Let $\mathbb{D}$ be the open unit disk and let $\bar{\mathbb{D}}$ be the closed unit disk in the complex plane. Let $C_A(\mathbb{D})$ denote the disk algebra, i.e., the set of all continuous functions on $\bar{\mathbb{D}}$ which is analytic on $\mathbb{D}$.

For $1 \le p \le \infty$, we write $H^p(\mathbb{D})$ for the Hardy space on $\mathbb{D}$. Two spaces $H^p(\mathbb{D})$ and $H^p(\mathbb{T})$ are identified via nontangential limits and Poisson integral. Thus, we often write H^p to denote the both of them. The norm in $L^p(\mathbb{T})$ (or $H^p(\mathbb{D})$) will be denoted by $\|\cdot\|_p$ and the inner product in $L^2(\mathbb{T})$ (or $H^2(\mathbb{D})$) will be denoted by $\langle \cdot, \cdot \rangle$. We refer the reader to the texts [17–19] and [20] for details of Hardy spaces.

The shift operator and its adjoint are one of the most interesting operators on the Hardy space. For convenience, we define them on $H(\mathbb{D})$, the class of all analytic functions on $\mathbb{D}$. For $f \in H(\mathbb{D})$, define

$$\begin{aligned} Sf(z) &= zf(z)(z \in \mathbb{D}), \\ S^*f(z) &= \frac{f(z) - f(0)}{z}(z \in \mathbb{D}). \end{aligned} \tag{1}$$

The operators S and S^* are often called the unilateral shift and the backward shift, respectively. We refer the reader to the text [21] which treats the shift operator in great detail.

One of the most remarkable theorems in analysis is Beurling's theorem (cf. [18, 20, 22]), which characterizes all S-invariant subspaces of H^2. (We use the term "subspace" for a closed linear subspace.) For a nonzero subspace M of H^2, M is S-invariant if and only if

$$M = \theta H^2 = \left\{ \theta f : f \in H^2 \right\}, \tag{2}$$

for some inner function $\theta \in H^\infty$. A bounded analytic function θ on $\mathbb{D}$ is called an inner function if its radial limit $\theta^*(e^{it}) = \lim_{r\to 1-}\theta(re^{it})$ has a unit modulus for almost all $e^{it} \in \mathbb{T}$. If an inner function has no zero in $\mathbb{D}$, we call it a singular inner function.

Let $M(\mathbb{T})$ be the set of all complex (finite) Borel measures on $\mathbb{T}$. Note that $M(\mathbb{T})$ is a Banach space with the total variation norm $\|\mu\| = |\mu|(\mathbb{T})$, where $|\mu|$ is the total variation measure of μ. We may regard the normalized Lebesgue measure m as a finite positive Borel measure. Hence, $m \in M(\mathbb{T})$. We write $\mathscr{B}_\mathbb{T}$ for the σ-algebra of all Borel sets in $\mathbb{T}$. We say μ is singular if $\mu \perp m$.

Suppose that $\mu \in M(\mathbb{T})$. For any function $f \in L^1(\mathbb{T}, |\mu|)$, let $f \cdot \mu$ denote the complex Borel measure on $\mathbb{T}$ defined by

$$(f \cdot \mu)(E) = \int_E f \, d\mu (E \in \mathscr{B}_\mathbb{T}). \tag{3}$$

Then, $|f \cdot \mu| = |f| \cdot |\mu|$. Hence, $\|f \cdot \mu\| = \|f\|_{L^1(\mathbb{T},|\mu|)}$. In particular, for every $f \in C(\mathbb{T})$, the measure $f \cdot \mu$ is defined and $\|f \cdot \mu\| \le \|f\|_\infty \|\mu\|$.

For $\mu \in M(\mathbb{T})$, the nth Fourier–Stieltjes coefficient of μ is given by

$$\hat{\mu}(n) = \int_\mathbb{T} \bar{\zeta}^n \, d\mu(\zeta) (n \in \mathbb{Z}). \tag{4}$$

For any $\mu \in M(\mathbb{T})$, the bilateral sequence $\hat{\mu} = \{\hat{\mu}(n)\}_{n \in \mathbb{Z}}$ is bounded and the mapping $\mu \mapsto \hat{\mu}$ is a bounded linear transformation from $M(\mathbb{T})$ into $\ell^\infty(\mathbb{Z})$. Note that the mapping $\mu \mapsto \hat{\mu}$ is one-to-one, and hence, a measure $\mu \in M(\mathbb{T})$ is completely determined by its Fourier–Stieltjes coefficients. By the theorem of F. and M. Riesz, if $\mu \in M(\mathbb{T})$ is analytic, i.e., $\hat{\mu}(n) = 0$ for all $n \le 0$, then $\mu \ll m$ and $d\mu/dm \in H^1(\mathbb{T})$; in other words, $\mu = f \cdot m$ for some $f \in H^1(\mathbb{T})$.

For the definition of Toeplitz operators whose symbols are measures, we use the Cauchy transform as the "projection" of measures. For this reason, we use the notation $P\mu$ instead of $K\mu$ for the Cauchy transform of μ. We refer the reader to the text [23] for thorough treatments of the Cauchy transform. For $\mu \in M(\mathbb{T})$, the analytic function $P\mu$ on $\mathbb{D}$, given by

$$(P\mu)(z) = \int_\mathbb{T} \frac{1}{1 - \bar{\zeta} z} \, d\mu(\zeta) = \sum_{n=0}^\infty \hat{\mu}(n) z^n (z \in \mathbb{D}), \tag{5}$$

is called the Cauchy transform of μ. Clearly, the mapping P is a linear transformation from $M(\mathbb{T})$ into $H(\mathbb{D})$. We may regard $f \in L^1(\mathbb{T})$ as the absolutely continuous measure $f \cdot m \in M(\mathbb{T})$. Hence, we denote $P(f \cdot m)$ by Pf, i.e.,

$$(Pf)(z) = \int_\mathbb{T} \frac{f(\zeta)}{1 - \bar{\zeta} z} \, dm(\zeta) = \sum_{n=0}^\infty \hat{f}(n) z^n (z \in \mathbb{D}). \tag{6}$$

(Clearly, $\widehat{f \cdot m}(n) = \hat{f}(n)$.) As we have identified $H^2(\mathbb{D})$ with $H^2(\mathbb{T})$, the mapping P may be regarded as the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$ (the so-called Riesz projection).

Let $\varphi \in L^2(\mathbb{T})$. The *Toeplitz operator* T_φ *with symbol* φ is the linear operator on H^2 with domain

$$\mathscr{D}(T_\varphi) = \{f \in H^2(\mathbb{D}) \colon P(\varphi f) \in H^2(\mathbb{D})\}, \tag{7}$$

given by

$$T_\varphi f = P(\varphi f)\left(f \in \mathscr{D}(T_\varphi)\right). \tag{8}$$

(Recall that every function in $H^2(\mathbb{D})$ may be identified with its nontangential limit function which belongs to $H^2(\mathbb{T})$.) Clearly, $C_A(\mathbb{D}) \subseteq \mathscr{D}(T_\varphi)$. Hence, T_φ is densely defined. Also, T_φ is closed. Observe that

$${}_{\varphi}z^j, z^i = \varphi, z^{i-j} = \widehat{\varphi}(i-j), \tag{9}$$

for every $i, j \in \mathbb{N} \cup \{0\}$. Hence, the matrix representation of T_φ with respect to the orthonormal basis $\{1, z, z^2, \cdots\}$ is

$$\begin{bmatrix} \widehat{\varphi}(0) & \widehat{\varphi}(-1) & \widehat{\varphi}(-2) & \cdots \\ \widehat{\varphi}(1) & \widehat{\varphi}(0) & \widehat{\varphi}(-1) & \cdots \\ \widehat{\varphi}(2) & \widehat{\varphi}(1) & \widehat{\varphi}(0) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{10}$$

A matrix of this form is called a *Toeplitz matrix*; in other words, an infinite matrix $\{\alpha_{i,j}\}_{i,j\geq 0}$ is called a Toeplitz matrix if

$$\alpha_{i,j} = \alpha_{i+1,j+1}, \tag{11}$$

for every $i, j \in \mathbb{N} \cup \{0\}$.

For a bilateral sequence $s = \{s_n\}_{n\in\mathbb{Z}}$ of complex numbers, we denote by $T(s)$ the infinite Toeplitz matrix corresponding to s, i.e., $T(s)$ is the infinite matrix whose (i, j)-entry is s_{i-j}. Note that if $\varphi \in L^2(\mathbb{T})$, then the matrix representations of T_φ is $T(\widehat{\varphi})$. For $n \in \mathbb{N} \cup \{0\}$, we denote by $T_n(s)$ the $(n+1) \times (n+1)$ Toeplitz matrix corresponding to s, i.e.,

$$T_n(s) = \begin{bmatrix} s_0 & s_{-1} & \cdots & s_{-n} \\ s_1 & s_0 & \cdots & s_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n-1} & \cdots & s_0 \end{bmatrix}. \tag{12}$$

3. The Main Results

Let μ be a complex Borel measure on $\mathbb{T}$. For any function $f \in C_A(\mathbb{D})$, $f \cdot \mu$ is a complex Borel measure on $\mathbb{T}$, and hence, the Cauchy transform $P(f \cdot \mu)$ is an analytic function on $\mathbb{D}$. Define

$$\mathscr{D}(T_\mu) = \{f \in C_A(\mathbb{D}) \colon P(f \cdot \mu) \in H^2(\mathbb{D})\}. \tag{13}$$

It is easy to show that $\mathscr{D}(T_\mu)$ is a linear manifold of $H^2(\mathbb{D})$. Now define

$$T_\mu f = P(f \cdot \mu)\left(f \in \mathscr{D}(T_\mu)\right). \tag{14}$$

Then, T_μ is a linear operator on $H^2(\mathbb{D})$ with domain $\mathscr{D}(T_\mu)$.

Definition 1. The operator T_μ is called the Toeplitz operator with symbol μ.

We begin with the following:

Proposition 2. *Suppose that* $\mu \ll m$ *and the Radon–Nikodym derivative* $\varphi = d\mu/dm$ *belongs to* $L^2(\mathbb{T})$. *Then,* $\mathscr{D}(T_\mu) = C_A(\mathbb{D})$ *and*

$$T_\mu f = T_\varphi f, \tag{15}$$

for every $f \in C_A(\mathbb{D})$.

Proof. Suppose that $\mu = \varphi \cdot m$, where $\varphi \in L^2(\mathbb{T})$. Let f be an arbitrary function in $C_A(\mathbb{D})$. Then,

$$P(f \cdot \mu)(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \bar{\zeta} z}\, d\mu(\zeta) = \int_{\mathbb{T}} \frac{f(\zeta)\varphi(\zeta)}{1 - \bar{\zeta} z}\, dm(\zeta) = P(\varphi f)(z), \tag{16}$$

for every $z \in \mathbb{D}$, and so, $P(f \cdot \mu) = P(\varphi f)$. Since $\varphi f \in L^2(\mathbb{T})$, it follows that $P(f \cdot \mu) \in H^2(\mathbb{D})$. Hence, $f \in \mathscr{D}(T_\mu)$ and

$$T_\mu f = P(f \cdot \mu) = P(\varphi f) = T_\varphi f. \tag{17}$$

This completes the proof.

Proposition 2 shows that the notion of T_μ is a kind of generalization of the Toeplitz operators whose symbols are L^2-functions.

Remark 3.

(a) *Toeplitz operators with* L^1 *-symbols*: every function $\varphi \in L^1(\mathbb{T})$ would be regarded as the absolutely continuous measure $\varphi \cdot m \in M(\mathbb{T})$. Hence, we may use Definition 1 to define Toeplitz operators with L^1-symbols: if $\varphi \in L^1(\mathbb{T})$ and $\mu = \varphi \cdot m$, then

$$\mathscr{D}(T_\mu) = \{f \in C_A(\mathbb{D}) \colon P(\varphi f) \in H^2(\mathbb{D})\}, \tag{18}$$

$$T_\mu f = P(\varphi f), \tag{19}$$

for $f \in \mathscr{D}(T_\mu)$.

(b) *Toeplitz operators with H^1 -symbols*: let $\varphi \in H^1(\mathbb{T})$ and put $\mu = \varphi \cdot m \in M(\mathbb{T})$. For every $f \in C_A(\mathbb{D})$, $\varphi f \in H^1(\mathbb{T})$. Hence, $P(\varphi f) = \varphi f$ (if we view φ in the right-hand side as a function in $H^1(\mathbb{D})$). It follows that

$$\mathscr{D}(T_\mu) = \{f \in C_A(\mathbb{D}): \varphi f \in H^2(\mathbb{D})\}, \quad (20)$$

$$T_\mu f = \varphi f, \quad (21)$$

for $f \in \mathscr{D}(T_\mu)$. This shows that a Toeplitz operator with H^1-symbol behaves as a multiplication. Notice that the action of T_μ is the same as that of T_φ defined in ([16], Section 5). (In that paper, the domain of T_φ is given by $\mathscr{D}(T_\varphi) = \{f \in H^2(\mathbb{D}): \varphi f \in H^2(\mathbb{D})\}$.) Moreover, since φ is of Smirnov class, $\varphi = b/a$ for some $a, b \in H^\infty(\mathbb{D})$ such that a is an outer function, $a(0) > 0$, and $|a|^2 + |b|^2 = 1$ on $\mathbb{T}$. In this case, $\mathscr{D}(T_\varphi) = aH^2(\mathbb{D})$ (cf. [16]). It follows that

$$\mathscr{D}(T_\mu) = \mathscr{D}(T_\varphi) \cap C_A(\mathbb{D}) = aH^2(\mathbb{D}) \cap C_A(\mathbb{D}). \quad (22)$$

Since a is an outer function, it follows that $aH^2(\mathbb{D})$ is dense in $H^2(\mathbb{D})$.

Question: is $aH^2 \cap C_A(\mathbb{D})$ dense in H^2?

We give some concrete examples.

Example 4.

(a) Let φ be the analytic function on $\mathbb{D}$ such that $(\varphi(z))^2 = (1-z)^{-1}$ and $\varphi(0) = 1$. Then, $\varphi \in H^1(\mathbb{D})$ but $\varphi \notin H^2(\mathbb{D})$. Put $\mu = \varphi \cdot m$. By Remark 3, (b), we have

$$\mathscr{D}(T_\mu) = \{f \in C_A(\mathbb{D}): \varphi f \in H^2(\mathbb{D})\}. \quad (23)$$

How large is the domain $\mathscr{D}(T_\mu)$? Suppose that $g \in C_A(\mathbb{D})$ and $g(1) \neq 0$. Then, there exists a constant $c > 0$ such that $|g| \geq c$ on a neighborhood of $\zeta = 1$. It follows that $\varphi g \notin H^2(\mathbb{D})$. Hence, $g \notin \mathscr{D}(T_\mu)$. This shows that

$$\mathscr{D}(T_\mu) \subseteq \{f \in C_A(\mathbb{D}): f(1) = 0\}. \quad (24)$$

On the other hand, if $r > 0$ and if ψ_r is the function in $C_A(\mathbb{D})$ which satisfies $(\psi_r(z))^{1/r} = 1 - z$ and $\psi_r(0) = 1$, then, for every $g \in C_A(\mathbb{D})$,

$$\begin{aligned}\|\varphi\psi_r g\|_2^2 &= \int_{\mathbb{T}} |\varphi(\zeta)|^2 |\psi_r(\zeta)|^2 |g(\zeta)|^2 \, dm(\zeta) \\ &= \int_{\mathbb{T}} \frac{|1-\zeta|^{2r}}{|1-\zeta|} |g(\zeta)|^2 \, dm(\zeta) \\ &\leq \|g\|_\infty^2 \cdot \int_{\mathbb{T}} |1-\zeta|^{2r-1} \, dm(\zeta) \\ &= \frac{\|g\|_\infty^2}{2\pi} \int_{-\pi}^{\pi} \left|1 - e^{it}\right|^{2r-1} dt \\ &\leq \frac{\|g\|_\infty^2}{2\pi} \int_{-\pi}^{\pi} |t|^{2r-1} dt \\ &= \frac{\|g\|_\infty^2}{\pi} \frac{\pi^{2r}}{2r},\end{aligned} \quad (25)$$

and hence, $\varphi\psi_r g \in H^2(\mathbb{D})$, i.e., $\psi_r g \in \mathscr{D}(T_\mu)$. It follows that

$$\bigcup_{r>0} \psi_r C_A(\mathbb{D}) \subseteq \mathscr{D}(T_\mu). \quad (26)$$

Since $\psi_1 = 1 - z$, we have

$$(1-z) \cdot C_A(\mathbb{D}) \subseteq \mathscr{D}(T_\mu). \quad (27)$$

In particular, $\mathscr{D}(T_\mu)$ contains all polynomials vanishing at $\zeta = 1$.

(b) Let $\mu = \delta_1$ be the unit point mass concentrated at $\zeta = 1$. Note that the measure μ is discrete. Observe that, for $f \in C_A(\mathbb{D})$,

$$P(f \cdot \mu)(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \bar{\zeta} z} \, d\mu(\zeta) = \frac{f(1)}{1-z} (z \in \mathbb{D}). \quad (28)$$

Since $1/(1-z) = \sum_{n=0}^{\infty} z^n$, the function $1/(1-z)$ does not belong to $H^2(\mathbb{D})$. It follows that $P(f \cdot \mu) \in H^2(\mathbb{D})$ if and only if $f(1) = 0$. Therefore,

$$\mathscr{D}(T_\mu) = \{f \in C_A(\mathbb{D}): f(1) = 0\}. \quad (29)$$

Also, we have

$$T_\mu f = 0, \quad (30)$$

for all $f \in \mathscr{D}(T_\mu)$. Hence, T_μ is trivial, i.e., $T_\mu f = 0$ for all $f \in \mathscr{D}(T_\mu)$. Consequently, T_μ is bounded (on $\mathscr{D}(T_\mu)$). Notice that $\mathscr{D}(T_\mu)$ does not contain the constant function 1. We show later (see Remark 11) that $\mathscr{D}(T_\mu)$ is dense in $H^2(\mathbb{D})$.

(c) *The Cantor middle-third measure*: let C denote the Cantor ternary set and let φ be the Cantor function, i.e., for $x = \sum_{j=1}^{\infty} (a_j/3^j) \in C$,

$$\varphi(x)=\sum_{j=1}^{\infty}\frac{a_j/2}{2^j}, \quad (31)$$

and $\varphi(x)=\sup\{\varphi(y): y<x, y\in C\}$ for $x\notin C$. Then, φ is continuous and monotonically increasing. Hence, there exists a positive Borel measure μ on $\mathbb{T}$ such that

$$\mu\left(\left\{e^{2\pi i\theta}: 0\le\theta<t\right\}\right)=\varphi(t)(0\le t\le 1). \quad (32)$$

The measure μ (the so-called Cantor middle-third measure) is a typical example of a singular continuous measure. We refer the reader to the papers [24] and [25] which treat measures of the Cantor type. It is known that

$$\widehat{\mu}(n)=(-1)^n\prod_{j=1}^{\infty}\cos\frac{2\pi n}{3^j}\ (n\in\mathbb{Z}). \quad (33)$$

Hence,

$$|\mu\wedge(n)|^2=\prod_{j=1}^{\infty}\left(1-\sin^2\frac{2\pi n}{3^j}\right)(n\in\mathbb{Z}). \quad (34)$$

Since $0\le\sin^2(2\pi n/3^j)<1$ for each j and $\sum_{j=1}^{\infty}\sin^2(2\pi n/3^j)<\infty$, it follows that $\widehat{\mu}(n)\neq 0$. Note also that $\widehat{\mu}(-n)=\widehat{\mu}(n)$ and $\widehat{\mu}(3n)=\widehat{\mu}(n)$ for every $n\in\mathbb{Z}$. We may here ask the following questions:

(a) What is $\mathscr{D}(T_\mu)$? Is $\mathscr{D}(T_\mu)$ dense in $H^2(\mathbb{D})$?

(b) What is T_μ? Is T_μ trivial?

We next ask: *when is the domain $\mathscr{D}(T_\mu)$ dense in $H^2(\mathbb{D})$?* It does not seem easy to answer this question in general. The following lemma is used to derive some properties of $\mathscr{D}(T_\mu)$ which are helpful to determine the density of $\mathscr{D}(T_\mu)$ in $H^2(\mathbb{D})$. Recall that S is the shift operator on $H(\mathbb{D})$, i.e., if $f\in H(\mathbb{D})$, then $Sf(z)=zf(z)$ for $z\in\mathbb{D}$.

We then have the following:

Lemma 5. *For every $\mu\in M(\mathbb{T})$ and $f\in C_A(\mathbb{D})$,*

$$P(Sf\cdot\mu)=SP(f\cdot\mu)+P(Sf\cdot\mu)(0). \quad (35)$$

Proof. For each $z\in\mathbb{D}$,

$$\begin{aligned}P(Sf\cdot\mu)(z)-P(Sf\cdot\mu)(0)&=\int_{\mathbb{T}}\frac{\zeta f(\zeta)}{1-\bar{\zeta}z}\,d\mu(\zeta)-\int_{\mathbb{T}}\zeta f(\zeta)\,d\mu(\zeta)\\&=\int_{\mathbb{T}}\frac{\bar{\zeta}z}{1-\bar{\zeta}z}\zeta f(\zeta)\,d\mu(\zeta)\\&=z\int_{\mathbb{T}}\frac{f(\zeta)}{1-\bar{\zeta}z}\,d\mu(\zeta)\\&=zP(f\cdot\mu)(z)\\&=SP(f\cdot\mu)(z).\end{aligned} \quad (36)$$

The following proposition gives an important information for the domain of T_μ.

Proposition 6. *Let $\mu\in M(\mathbb{T})$ and let α be a complex number such that $|\alpha|\neq 1$. Then, the following statements hold:*

(a) For $f\in C_A(\mathbb{D})$, $f\in\mathscr{D}(T_\mu)$ if and only if $(S-\alpha)f\in\mathscr{D}(T_\mu)$

(b) For $f\in H^2(\mathbb{D})$, $f\in cl_{H^2}(\mathscr{D}(T_\mu))$ if and only if $(S-\alpha)f\in cl_{H^2}(\mathscr{D}(T_\mu))$

Proof. (a) Suppose that $f\in C_A(\mathbb{D})$. Then, by Lemma 5,

$$\begin{aligned}P((S-\alpha)f\cdot\mu)&=P(Sf\cdot\mu)-P(\alpha f\cdot\mu)\\&=SP(f\cdot\mu)+P(Sf\cdot\mu)(0)-\alpha P(f\cdot\mu)\\&=(S-\alpha)P(f\cdot\mu)+P(Sf\cdot\mu)(0).\end{aligned} \quad (37)$$

Hence, $P((S-\alpha)f\cdot\mu)\in H^2(\mathbb{D})$ if and only if $(S-\alpha)P(f\cdot\mu)\in H^2(\mathbb{D})$. Since $P(f\cdot\mu)\in H(\mathbb{D})$ and $|\alpha|\neq 1$, it follows that $P(f\cdot\mu)\in H^2(\mathbb{D})$ if and only if $(S-\alpha)P(f\cdot\mu)\in H^2(\mathbb{D})$. Therefore, $f\in\mathscr{D}(T_\mu)$ if and only if $(S-\alpha)f\in\mathscr{D}(T_\mu)$. This proves (a).

(b) Suppose that $f\in H^2(\mathbb{D})$ and $f\in cl_{H^2}(\mathscr{D}(T_\mu))$. Then, there exists a sequence $\{f_j\}$ in $\mathscr{D}(T_\mu)$ such that $\|f-f_j\|_2\longrightarrow 0$. Since $S-\alpha$ is a bounded operator on $H^2(\mathbb{D})$, we have

$$\|(S-\alpha)f-(S-\alpha)f_j\|_2=\|(S-\alpha)\left(f-f_j\right)\|_2\longrightarrow 0. \quad (38)$$

By (a), each $(S-\alpha)f_j$ belongs to $\mathscr{D}(T_\mu)$. It follows that $(S-\alpha)f\in cl_{H^2}(\mathscr{D}(T_\mu))$.

Conversely, suppose that $f\in H^2(\mathbb{D})$ and $(S-\alpha)f\in cl_{H^2}(\mathscr{D}(T_\mu))$. Then, there exists a sequence $\{g_j\}$ in $\mathscr{D}(T_\mu)$ such that

$$\|(S-\alpha)f-g_j\|_2\longrightarrow 0. \quad (39)$$

We want to show that $f\in cl_{H^2}(\mathscr{D}(T_\mu))$. To see this we consider two cases.

Case 1. ($|\alpha|<1$). Assume first that $g_j(\alpha)=0$ for all j. Then,

$$g_j=(S-\alpha)f_j, \quad (40)$$

where $f_j\in C_A(\mathbb{D})$. Since $g_j\in\mathscr{D}(T_\mu)$, it follows from (a) that $f_j\in\mathscr{D}(T_\mu)$. Note that the approximate point spectrum of the operator S on $H^2(\mathbb{D})$ is $\sigma_{\text{ap}}(S)=\mathbb{T}$ (cf. [26]). Since α does not belong to $\mathbb{T}$, the operator $S-\alpha$ is bounded below on $H^2(\mathbb{D})$. It follows that there exists a constant $c>0$ such that

$$\|(S-\alpha)f-g_j\|_2=\|(S-\alpha)\left(f-f_j\right)\|_2\ge c\cdot\|f-f_j\|_2 \quad (41)$$

for all j. This implies that $\|f-f_j\|_2\to 0$. Therefore, $f\in cl_{H^2}(\mathscr{D}(T_\mu))$.

In the case that $g_j(\alpha) \neq 0$ for some j, we may assume that $g_1(\alpha) \neq 0$. Note that $g_j \longrightarrow (S-\alpha)f$ weakly. Hence, $g_j(z) \longrightarrow ((S-\alpha)f)(z)$ for each $z \in \mathbb{D}$. In particular, we have

$$g_j(\alpha) \longrightarrow 0. \quad (42)$$

Now put

$$h_j = g_j - \frac{g_j(\alpha)}{g_1(\alpha)} g_1 \quad (j = 1, 2, 3, \cdots). \quad (43)$$

Then, $h_j \in \mathscr{D}(T_\mu)$ and $h_j(\alpha) = 0$ for all j. Observe that

$$\|(S-\alpha)f - h_j\|_2 \leq \|(S-\alpha)f - g_j\|_2 + \left|\frac{g_j(\alpha)}{g_1(\alpha)}\right| \|g_1\|_2. \quad (44)$$

It follows that

$$\|(S-\alpha)f - h_j\|_2 \to 0. \quad (45)$$

Hence, by the preceding paragraph, we conclude that $f \in cl_{H^2}(\mathscr{D}(T_\mu))$.

Case 2. ($|\alpha| > 1$). The operator $S - \alpha$ on $H^2(\mathbb{D})$ is invertible. Hence,

$$\|f - (S-\alpha)^{-1} g_j\|_2 \longrightarrow 0. \quad (46)$$

Since $(S-\alpha)^{-1} = -\sum_{n=0}^{\infty} S^n/\alpha^{n+1}$ and $\mathscr{D}(T_\mu)$ is S-invariant by (a), each $(S-\alpha)^{-1} g_j$ belongs to $cl_{H^2}(\mathscr{D}(T_\mu))$. It follows that $f \in cl_{H^2}(\mathscr{D}(T_\mu))$, and the proof is complete.

Remark 7. If we take $\alpha = 0$ in Proposition 6, then the linear subspaces $\mathscr{D}(T_\mu)$ and its closure $cl_{H^2}(\mathscr{D}(T_\mu))$ are S-invariant. Also, the equality in Lemma 5 can be rewritten as $S^* P(Sf \cdot \mu) = P(f \cdot \mu)$. Consequently, we have $S^* T_\mu S f = T_\mu f$ for every $f \in \mathscr{D}(T_\mu)$.

As a consequence of Proposition 6, we derive the following theorem which describes the domain $\mathscr{D}(T_\mu)$. Recall that an inner function is said to be singular if it has no zero in the unit disk.

Theorem 8. *Let $\mu \in M(\mathbb{T})$. Then, one of the following holds:*

(i) $\mathscr{D}(T_\mu) = \{0\}$

(ii) $\mathscr{D}(T_\mu)$ *is dense in* $H^2(\mathbb{D})$

(iii) $cl_{H^2}(\mathscr{D}(T_\mu)) = \theta H^2(\mathbb{D})$, *where θ is a singular inner function*

Proof. By Proposition 6, $cl_{H^2}(\mathscr{D}(T_\mu))$ is an S-invariant subspace of $H^2(\mathbb{D})$. It follows from Beurling's theorem that

$$cl_{H^2}(\mathscr{D}(T_\mu)) = \theta H^2(\mathbb{D}), \quad (47)$$

where θ is an inner function or $\theta = 0$. If $\theta = 0$, then the case (i) occures. If θ is a nonzero constant function, case (ii) occurs. Now, suppose that θ is nonconstant. We show that θ has no zero in $\mathbb{D}$. To see this, choose any nonzero function f in $\mathscr{D}(T_\mu)$. Fix an arbitrary point α of $\mathbb{D}$ and let n be the multiplicity of the zero of f at α. Then,

$$f(z) = (z-\alpha)^n g(z) (z \in \mathbb{D}), \quad (48)$$

where $g \in C_A(\mathbb{D})$ and $g(\alpha) \neq 0$. Hence, by a repeated application of Proposition 6(a), we have

$$g \in \mathscr{D}(T_\mu) \subseteq \theta H^2(\mathbb{D}). \quad (49)$$

It follows that $g = \theta h$ for some $h \in H^2(\mathbb{D})$. Thus, $\theta(\alpha)$ cannot be 0. Since α was arbitrary, we conclude that θ has no zero in $\mathbb{D}$. Therefore θ is a singular inner function.

Remark 9. Unfortunately, we cannot find a concrete example for the third case. It would be possible that the third case never occurs.

The following proposition is another consequence of Proposition 6 which gives a sufficient condition for the domain $\mathscr{D}(T_\mu)$ to be dense in $H^2(\mathbb{D})$.

Proposition 10. *If $cl_{H^2}(\mathscr{D}(T_\mu))$ contains a polynomial, then $\mathscr{D}(T_\mu)$ is dense in $H^2(\mathbb{D})$.*

Proof. Suppose that $cl_{H^2}(\mathscr{D}(T_\mu))$ contains a polynomial. Then, by Proposition 6, (b), there exists a polynomial $p \in cl_{H^2}(\mathscr{D}(T_\mu))$, all of whose zeros are in $\mathbb{T}$, such that $p(0) = 1$. Let $\zeta_1, \cdots, \zeta_N \in \mathbb{T}$ be the zeros of p, listed according to their multiplicities. Then,

$$p(z) = \left(1 - \bar{\zeta}_1 z\right) \cdots \left(1 - \bar{\zeta}_N z\right). \quad (50)$$

Choose a sequence $\{k_n\}$ in $\mathbb{N}$ such that $k_{n+1} > N k_n$ (e.g., $k_n = (N+1)^n$). For each $n \in \mathbb{N}$, define

$$p_n(z) = \frac{1}{n} \sum_{j=1}^{n} \left(1 - \left(\bar{\zeta}_1 z\right)^{k_j}\right) \cdots \left(1 - \left(\bar{\zeta}_N z\right)^{k_j}\right). \quad (51)$$

All of them are polynomials, divisible by p. Since $cl_{H^2}(\mathscr{D}(T_\mu))$ is S-invariant, the polynomials p_n belong to $\mathscr{D}(T_\mu)$. It follows by a direct computation that

$$\|1 - p_n\|_2^2 \leq \frac{n}{n^2} \left[\begin{pmatrix} N \\ 1 \end{pmatrix} \begin{pmatrix} N \\ 1 \end{pmatrix} + \cdots + \begin{pmatrix} N \\ N \end{pmatrix} \begin{pmatrix} N \\ N \end{pmatrix} \right], \quad (52)$$

for every $n \in \mathbb{N}$. This implies that $p_n \longrightarrow 1$ in $H^2(\mathbb{D})$.

Therefore, the constant function 1 belongs to $cl_{H^2}(\mathscr{D}(T_\mu))$. Since $cl_{H^2}(\mathscr{D}(T_\mu))$ is S-invariant, we conclude that $cl_{H^2}(\mathscr{D}(T_\mu)) = H^2(\mathbb{D})$; in other words, $\mathscr{D}(T_\mu)$ is dense in $H^2(\mathbb{D})$.

Remark 11. Proposition 10 shows that the domains $\mathscr{D}(T_\mu)$, presented in (a) and (b) of Example 4, are dense in $H^2(\mathbb{D})$, because they contain the polynomial $p(z) = 1 - z$. The proof of Proposition 10 shows that every polynomial, all of whose zeros are in $\mathbb{T}$, is an outer function.

In order to consider the matrix representation of a linear operator on $H^2(\mathbb{D})$, it is necessary that its domain contains all polynomials. Let us interpret the condition that $\mathscr{D}(T_\mu)$ contains all polynomials. Note that this is equivalent to the condition that $\mathscr{D}(T_\mu)$ contains any polynomial which does not vanish on $\mathbb{T}$, by Proposition 6, (a).

Lemma 12. *Let $\mu \in M(\mathbb{T})$. Then, the following are equivalent:*

(i) $\mathscr{D}(T_\mu)$ contains all polynomials, or equivalently, $\mathscr{D}(T_\mu)$ contains the constant function 1

(ii) $\mu \ll m$ and $d\mu/dm \in H^2(\mathbb{T}) + \overline{H_0^1}(\mathbb{T})$

Proof. (i) $\Rightarrow$ (ii): suppose that the constant function 1 belongs to $\mathscr{D}(T_\mu)$. Then, $P\mu = P(1 \cdot \mu) \in H^2(\mathbb{D})$. Let ψ denote the nontangential limit function of $P\mu$. Since $P\mu = \sum_{n=0}^{\infty} \widehat{\mu}(n) z^n$, it follows that $\widehat{\psi}(n) = \widehat{\mu}(n)$ for all $n \in \mathbb{N} \cup \{0\}$. Put $\nu = \mu - \psi \cdot m$. Then, $\nu \in M(\mathbb{T})$ and

$$\widehat{\nu}(n) = \widehat{\mu}(n) - \widehat{\psi}(n) = 0, \qquad (53)$$

for all $n \in \mathbb{N} \cup \{0\}$. It follows from the F. and M. Riesz theorem that $\nu \ll m$ and $\nu = \chi \cdot m$ for some $\chi \in \overline{H_0^1}(\mathbb{T})$. Thus, we have $\mu = \nu + \psi \cdot m = (\chi + \psi) \cdot m$. This proves (ii).

(ii) $\Rightarrow$ (i): suppose that (ii) holds so that $\mu = (\psi + \chi) \cdot m$ for some $\psi \in H^2(\mathbb{T})$ and $\chi \in \overline{H_0^1}(\mathbb{T})$. Then, $\widehat{\mu}(n) = \widehat{\psi}(n)$ for all $n \in \mathbb{N} \cup \{0\}$. Hence, we have

$$\sum_{n=0}^{\infty} |\mu\wedge(n)|^2 < \infty. \qquad (54)$$

Since $P\mu = \sum_{n=0}^{\infty} \widehat{\mu}(n) z^n$, it follows that $P(1 \cdot \mu) = P\mu \in H^2(\mathbb{D})$. Clearly, the constant function 1 belongs to $C_A(\mathbb{D})$. Therefore, $1 \in \mathscr{D}(T_\mu)$. Now, Proposition 6, (a), implies that $\mathscr{D}(T_\mu)$ contains all polynomials.

Corollary 13. *Let $\mu \in M(\mathbb{T})$ be a real measure. Then, $\mathscr{D}(T_\mu) = C_A(\mathbb{D})$ if and only if $\mu \ll m$ and $d\mu/dm \in L^2(\mathbb{T})$.*

Proof. Suppose that $\mathscr{D}(T_\mu) = C_A(\mathbb{D})$. Then, $\mu \ll m$ and $\mu = (\psi + \chi) \cdot m$ for some $\psi \in H^2(\mathbb{T})$ and $\chi \in \overline{H_0^1}(\mathbb{T})$ by Lemma 12. Since μ is a real measure, we have

$$\widehat{\mu}(-n) = \int_{\mathbb{T}} \bar{z}^{-n}\, d\mu = \overline{\int_{\mathbb{T}} \bar{z}^{n}\, d\mu} = \overline{\widehat{\mu}(n)}, \qquad (55)$$

for every $n \in \mathbb{Z}$. Thus, $\widehat{\chi}(-n) = \overline{\widehat{\psi}(n)}$ for every $n \in \mathbb{N}$. Since $\psi \in H^2(\mathbb{T})$, we have

$$\sum_{n=-\infty}^{-1} |\chi\wedge(n)|^2 = \sum_{n=1}^{\infty} |\psi\wedge(n)|^2 < \infty. \qquad (56)$$

It follows that $\chi \in \overline{H_0^2}(\mathbb{T})$. Therefore, $d\mu/dm = \psi + \chi \in L^2(\mathbb{T})$.

The converse is a part of Proposition 2.

On the other hand, we would like to conjecture the following:

Conjecture 14. *Every Toeplitz operator with a singular symbol is trivial.*

We give evidence for Conjecture 14 by using the known fact about the Cauchy transform. Let E be a closed subset of $\mathbb{T}$ and let

$$F(E) = \{g \in H^2(\mathbb{D}) \colon g = P\mu \text{ for some } \mu \in M(E)\}. \qquad (57)$$

Then, it is known that $F(E) = \{0\}$ if and only if $m(E) = 0$ (cf. [23], Theorem 5.5.2).

We then have the following:

Theorem 15. *If $\mu \in M(\mathbb{T})$ is singular and $m(\sup p\mu) = 0$, then T_μ is trivial.*

Proof. Let $E = \sup p\mu$. By assumption, $m(E) = 0$. Thus, $F(E) = \{0\}$. Suppose that $f \in \mathscr{D}(T_\mu)$, i.e., $f \in C_A(\mathbb{D})$ and $P(f \cdot \mu) \in H^2(\mathbb{D})$. Note that $\sup p(f \cdot \mu) \subseteq \sup p\mu = E$. Hence, $f \cdot \mu \in M(E)$. So the function $P(f \cdot \mu) \in H^2(\mathbb{D})$ belongs to $F(E) = \{0\}$. It follows that $P(f \cdot \mu) = 0$. We have shown that $P(f \cdot \mu) \in H^2(\mathbb{D})$ implies $P(f \cdot \mu) = 0$. In other words,

$$f \in \mathscr{D}(T_\mu)\, T_\mu f = 0. \qquad (58)$$

Therefore T_μ is trivial (on its domain).

Remark 16. Conjecture 14 seems to be known when μ is a positive singular measure. Indeed, if μ is a positive singular measure, then its Poisson integral is the real part of $(1+\theta)/(1-\theta)$ for some inner function θ (cf. [23], Remark 9.1.4). Now, if $f \in C_A(\mathbb{D})$ and $P(f \cdot \mu) \in H^2(\mathbb{D})$, then the function

$g = (1-\theta)P(f\cdot\mu)$ belongs to $H^2(\mathbb{D})!\theta H^2(\mathbb{D})$ (cf. [27], Chapter III), and hence, $\theta\bar{g} \in zH^2(\mathbb{D})$. Since $1-\theta$ is the outer H^2-function, it follows that

$$P(\bar{f}\cdot\mu) = \frac{\bar{g}}{1\theta} = \frac{\bar{g}}{1-\bar{\theta}} = -\frac{\theta\bar{g}}{1-\theta}, \tag{59}$$

which implies that $P(\bar{f}\cdot\mu) \in zH^2(\mathbb{D})$. Therefore, $P(f\cdot\mu) = 0$.

The Cantor-middle-third measure μ in Example 4, (c), is a singular continuous measure, and its support is the Cantor set (in $\mathbb{T}$) whose Lebesgue measure is 0. Hence, Theorem 15 implies that T_μ is trivial.

We have seen that the Toeplitz operator T_μ in Example 4, (b), is a densely defined trivial linear operator. This result can be extended to the case that μ has a finite support. In this case, the fact that T_μ is trivial may follow from Theorem 15. However, we give a direct proof and also show that T_μ is densely defined.

Proposition 17. *Let $\mu \in M(\mathbb{T})$ be a discrete measure whose support is a finite set. Then, the Toeplitz operator T_μ is a densely defined trivial linear operator with domain*

$$\mathscr{D}(T_\mu) = \{f \in C_A(\mathbb{D}) : f(\zeta) = 0 \textit{ for every } \zeta \in \sup p\mu\}. \tag{60}$$

Proof. Suppose that $\sup p\mu$ consists of N distinct points $\zeta_1, \cdots, \zeta_N$ of $\mathbb{T}$. Then,

$$\mu = c_1\delta_{\zeta_1} + \cdots + c_N\delta_{\zeta_N}, \tag{61}$$

where $c_1, \cdots, c_N$ are nonzero complex numbers and δ_ζ is the unit point mass concentrated at ζ.

We first show that

$$\mathscr{D}(T_\mu) = \{f \in C_A(\mathbb{D}) : f(\zeta_1) = \cdots = f(\zeta_N) = 0\}. \tag{62}$$

For any $f \in C_A(\mathbb{D})$,

$$P(f\cdot\mu)(z) = \sum_{j=1}^N c_j P\left(f\cdot\delta_{\zeta_j}\right)(z) = \sum_{j=1}^N \frac{c_j f(\zeta_j)}{1-\bar{\zeta}_j z}\,(z \in \mathbb{D}). \tag{63}$$

It follows that

$$\{f \in C_A(\mathbb{D}) : f(\zeta_1) = \cdots = f(\zeta_N) = 0\} \subseteq \mathscr{D}(T_\mu). \tag{64}$$

Conversely, let $f \in \mathscr{D}(T_\mu)$. Then, $P(f\cdot\mu) \in H^2(\mathbb{D})$. For each j, put

$$F_j(\zeta) = \frac{c_j f(\zeta_j)}{1-\bar{\zeta}_j\zeta}\,(\zeta \in \mathbb{T}). \tag{65}$$

Then, $F = \sum_{j=1}^N F_j$ is the nontangential limit function of $P(f\cdot\mu)$. Thus, $F \in H^2(\mathbb{T})$. Choose disjoint open arcs $I_j \subseteq \mathbb{T}$ with $\zeta_j \in I_j$. Fix an index j_0 and let χ denote the characteristic function of I_{j_0}. Then, $\chi\cdot F \in L^2(\mathbb{T})$. Also, $\chi\cdot F_j \in L^\infty(\mathbb{T})$ for each $j \neq j_0$. Hence,

$$\chi\cdot F_{j_0} = \chi\cdot F - \sum_{j\neq j_0}\left(\chi\cdot F_j\right) \in L^2(\mathbb{T}). \tag{66}$$

Since $(1-\chi)\cdot F_{j_0} \in L^\infty(\mathbb{T})$, it follows that

$$F_{j_0} = \chi\cdot F_{j_0} + (1-\chi)\cdot F_{j_0} \in L^2(\mathbb{T}). \tag{67}$$

This implies that $f(\zeta_{j_0}) = 0$, because otherwise, $F_{j_0} \notin L^2(\mathbb{T})$. Since j_0 was arbitrary, we have $f(\zeta_j) = 0$ for each j. It follows that

$$\mathscr{D}(T_\mu) \subseteq \{f \in C_A(\mathbb{D}) : f(\zeta_1) = \cdots = f(\zeta_N) = 0\}. \tag{68}$$

This proves (62). In particular, $\mathscr{D}(T_\mu)$ contains the polynomial $p(z) = (z-\zeta_1)\cdots(z-\zeta_N)$. Hence, by Proposition 10, $\mathscr{D}(T_\mu)$ is dense in $H^2(\mathbb{D})$.

Equations (62) and (63) imply that $T_\mu f = 0$ for all $f \in \mathscr{D}(T_\mu)$, i.e., T_μ is trivial. This completes the proof.

Example 18. Let $\mu \in M(\mathbb{T})$ be a discrete measure whose support has only finitely many limit points, for example,

$$\mu = \sum_{n=1}^\infty \frac{1}{2^n}\delta_{\zeta_n}, \tag{69}$$

where $\zeta_n = e^{\pi i/2^n}$. By an argument similar to the proof of Proposition 17, we may show that

$$\mathscr{D}(T_\mu) = \{f \in C_A(\mathbb{D}) : f(\zeta) = 0 \text{ for every } \zeta \in \sup p\mu\}, \tag{70}$$

and $T_\mu f = 0$ for all $f \in \mathscr{D}(T_\mu)$. Hence, T_μ is trivial. Note that every polynomial has only finitely many zeros. It follows that $\mathscr{D}(T_\mu)$ cannot contain any polynomial. Nevertheless, $\mathscr{D}(T_\mu)$ contains a nonzero function by Fatou's theorem for $C_A(\mathbb{D})$, which says that, for any given closed set $K \subseteq \mathbb{T}$ with $m(K) = 0$, there exists a function in $C_A(\mathbb{D})$ which vanishes precisely on K (cf. [19]). Hence by Theorem 8, $\mathscr{D}(T_\mu)$ is dense in $H^2(\mathbb{D})$ or $cl_{H^2}(\mathscr{D}(T_\mu)) = \theta H^2(\mathbb{D})$ for some singular inner function θ. But it does not seem easy to determine whether $\mathscr{D}(T_\mu)$ is dense in $H^2(\mathbb{D})$ or not.

To each Toeplitz operator T_μ, there corresponds an infinite Toeplitz matrix $T(\widehat{\mu})$. In general, however, it is a bit awkward to call $T(\widehat{\mu})$ as the matrix representation of T_μ, because the domain $\mathscr{D}(T_\mu)$ may not contain the monomials z^n. Nevertheless, often, information about T_μ gives information about $T(\widehat{\mu})$. The following is one of such example.

Corollary 19. *Let $\mu \in M(\mathbb{T})$ be a discrete measure whose support consists of N points of $\mathbb{T}$. Then,*

$$\det T_n(\widehat{\mu}) = 0, \tag{71}$$

for all $n \geq N$.

Proof. Suppose that μ is the discrete measure given by (61). Then, the domain $\mathscr{D}(T_\mu)$ is given by (62). Choose any polynomial p in $\mathscr{D}(T_\mu)$ whose degree is N (e.g., $p(z) = (z - \zeta_1) \cdots (z - \zeta_N)$). Write $p = \sum_{k=0}^{N} a_k z^k$. Since $T_\mu z^k = \sum_{n=0}^{\infty} \widehat{\mu}(n-k) z^n$, it follows that

$$\begin{aligned} 0 = T_\mu p &= \sum_{k=0}^{N} a_k T_\mu z^k = \sum_{k=0}^{N} a_k \sum_{n=0}^{\infty} \widehat{\mu}(n-k) z^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{N} a_k \widehat{\mu}(n-k) \right) z^n. \end{aligned} \tag{72}$$

Hence, we have

$$\sum_{k=0}^{N} a_k \widehat{\mu}(n-k) = 0, \tag{73}$$

for all $n \geq 0$. Now, let $n \geq N$ and put

$$x = [a_0 \ \cdots \ a_N \ 0 \ \cdots \ 0]^T \in \mathbb{C}^{n+1}. \tag{74}$$

Then, by (73), $T_n(\widehat{\mu})x = 0$, i.e., $x \in \ker T_n(\widehat{\mu})$. Since $x \neq 0$, the square matrix $T_n(\widehat{\mu})$ is not invertible, or equivalently, $\det T_n(\widehat{\mu}) = 0$.

Lastly, we may ask: what is the adjoint of T_μ ? To answer this question, we need the following:

Lemma 20. *Let $\mu \in M(\mathbb{T})$. Then,*

$$\langle T_\mu f, g \rangle = \int_{\mathbb{T}} f \bar{g} \, d\mu, \tag{75}$$

for every $f \in \mathscr{D}(T_\mu)$ and $g \in C_A(\mathbb{D})$.

Proof. Suppose that $f \in \mathscr{D}(T_\mu)$ and $g \in C_A(\mathbb{D})$. Then, $T_\mu f \in H^2(\mathbb{D})$. Write $T_\mu f = \sum_{n=0}^{\infty} a_n z^n$ and $g = \sum_{n=0}^{\infty} b_n z^n$. Then,

$$\langle T_\mu f, g \rangle = \sum_{n=0}^{\infty} a_n \bar{b}_n. \tag{76}$$

Observe that, for each $z \in \mathbb{D}$,

$$\begin{aligned} (T_\mu f)(z) &= \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \bar{\zeta} z} \, d\mu(\zeta) = \int_{\mathbb{T}} f(\zeta) \sum_{n=0}^{\infty} \bar{\zeta}^n z^n \, d\mu(\zeta) \\ &= \sum_{n=0}^{\infty} \left[\int_{\mathbb{T}} f(\zeta) \bar{\zeta}^n \, d\mu(\zeta) \right] z^n. \end{aligned} \tag{77}$$

Hence, we have

$$a_n = \int_{\mathbb{T}} f(\zeta) \bar{\zeta}^n \, d\mu(\zeta). \tag{78}$$

Observe that, for each $0 < r < 1$,

$$g_r = \sum_{n=0}^{\infty} b_n r^n z^n \in C_A(\mathbb{D}). \tag{79}$$

It follows that

$$\begin{aligned} \langle T_\mu f, g_r \rangle &= \sum_{n=0}^{\infty} a_n \bar{b}_n r^n = \sum_{n=0}^{\infty} \int_{\mathbb{T}} f(\zeta) \bar{\zeta}^n \bar{b}_n r^n \, d\mu(\zeta) \\ &= \int_{\mathbb{T}} f(\zeta) \overline{\sum_{n=0}^{\infty} b_n r^n \zeta^n} \, d\mu(\zeta) = \int_{\mathbb{T}} f \bar{g}_r \, d\mu. \end{aligned} \tag{80}$$

If we let $r \longrightarrow 1$, then $\|g - g_r\|_\infty \longrightarrow 0$, and hence, $\langle T_\mu f, g_r \rangle \longrightarrow \langle T_\mu f, g \rangle$ and $\int_{\mathbb{T}} f \bar{g}_r \, d\mu \rightarrow \int_{\mathbb{T}} f \bar{g} \, d\mu$. This proves (75).

Assume that $\mu \in M(\mathbb{T})$ and $\mathscr{D}(T_\mu)$ is dense in $H^2(\mathbb{D})$. Then, the adjoint T_μ^* of T_μ can be defined; the domain of T_μ^* is

$$\mathscr{D}\left(T_\mu^*\right) = \left\{ g \in H^2(\mathbb{D}) \colon \exists h \in H^2(\mathbb{D}) \text{s.t.} \langle T_\mu f, g \rangle = \langle f, h \rangle \forall f \in \mathscr{D}(T_\mu) \right\}, \tag{81}$$

and, for each $g \in \mathscr{D}(T_\mu^*)$, $T_\mu^* g$ is the (unique) element of $H^2(\mathbb{D})$ such that

$$\langle T_\mu f, g \rangle = \left\langle f, T_\mu^* g \right\rangle, \tag{82}$$

for every $f \in \mathscr{D}(T_\mu)$.

If $\varphi \in L^\infty(\mathbb{T})$, then $T_\varphi^* = T_{\bar{\varphi}}$. Hence, it is reasonable to expect that the adjoint of T_μ is the Toeplitz operator induced by the "complex conjugation" of μ. For $\mu \in M(\mathbb{T})$, define

$$\bar{\mu}(E) = \overline{\mu(\bar{E})} (E \in \mathscr{B}_{\mathbb{T}}). \tag{83}$$

Then, $\bar{\mu} \in M(\mathbb{T})$. Of course, $\mu \in M(\mathbb{T})$ is a real measure if and only if $\bar{\mu} = \mu$. Note that

$$\widehat{\bar{\mu}}(n) = \overline{\widehat{\mu}(-n)}, \tag{84}$$

for every $n \in \mathbb{Z}$.

We now have the following:

Proposition 21. *Let $\mu \in M(\mathbb{T})$. Assume that $\mathscr{D}(T_\mu)$ is dense in $H^2(\mathbb{D})$. Then,*

$$T_{\bar{\mu}} \subseteq T_\mu^*, \tag{85}$$

that is $\mathscr{D}(T_{\bar{\mu}}) \subseteq \mathscr{D}(T_\mu^*)$ and $T_{\bar{\mu}} = T_\mu^*$ on $\mathscr{D}(T_{\bar{\mu}})$.

Proof. Let $g \in \mathscr{D}(T_{\bar{\mu}})$. By Lemma 20, it follows that

$$\langle T_{\mu} f, g \rangle = \int_{\mathbb{T}} f \bar{g} \, d\mu = \overline{\int_{\mathbb{T}} g \bar{f} \, d\bar{\mu}} = \left\langle f, T_{\bar{\mu}} g \right\rangle, \quad (86)$$

for every $f \in \mathscr{D}(T_{\mu})$. It follows that $g \in \mathscr{D}(T_{\mu}^*)$ and $T_{\mu}^* g = T_{\bar{\mu}} g$. Therefore, we conclude that

$$\mathscr{D}\left(T_{\bar{\mu}}\right) \subseteq \mathscr{D}\left(T_{\mu}^*\right), \quad (87)$$

and $T_{\mu}^* g = T_{\bar{\mu}} g$ for every $g \in \mathscr{D}(T_{\bar{\mu}})$. This completes the proof.

If $\mu \in M(\mathbb{T})$, and T is the restriction of the Toeplitz operator T_{μ} to $cl_{H^2}(\mathscr{D}(T_{\mu}))$, then T is a densely defined linear operator. In this case, T^* is a linear operator from $H^2(\mathbb{D})$ onto $cl_{H^2}(\mathscr{D}(T_{\mu}))$. By the same argument as the proof of Proposition 21, we have $\mathscr{D}(T_{\bar{\mu}}) \subseteq \mathscr{D}(T^*)$ and $T^* g = T_{\bar{\mu}} g$ for $g \in \mathscr{D}(T_{\bar{\mu}})$.

We also have the following:

Proposition 22. *Let $\mu \in M(\mathbb{T})$ be positive. Then, the following hold:*

(a) T_{μ} is positive, i.e., $\langle T_{\mu} f, f \rangle \geq 0$ for all $f \in \mathscr{D}(T_{\mu})$

(b) $\ker T_{\mu} = \{f \in C_A(\mathbb{D}) : f(\zeta) = 0$ for every $\zeta \in \operatorname{supp} \mu\}$

Proof. (a) Suppose that $\mu \geq 0$. Then, by Lemma 20, we have

$$\langle T_{\mu} f, f \rangle = \int_{\mathbb{T}} |f|^2 \, d\mu \geq 0, \quad (88)$$

for every $f \in \mathscr{D}(T_{\mu})$.

(b) Suppose that $\mu \in M(\mathbb{T})$ is positive. If $f \in \ker T_{\mu}$, then $\int_{\mathbb{T}} |f|^2 \, d\mu = \langle T_{\mu} f, f \rangle = 0$. Hence, $f = 0 \mu$-a.e. on $\mathbb{T}$. We show that $f = 0$ on supp μ. Assume to the contrary that $f(\zeta_0) \neq 0$ for some $\zeta_0 \in$ sup $p\mu$. Since $f \in C_A(\mathbb{D})$, there exist a constant $\varepsilon > 0$ and an open arc $I \subseteq \mathbb{T}$ with center ζ_0 such that $|f(\zeta)| \geq \varepsilon$ for all $\zeta \in I$. Since $\zeta_0 \in \operatorname{supp} \mu$, we have $\mu(I) > 0$. It follows that

$$\int_{\mathbb{T}} |f|^2 \, d\mu \geq \int_I |f|^2 \, d\mu \geq \varepsilon \cdot \mu(I) > 0, \quad (89)$$

which is a contradiction. Hence, $f(\zeta) = 0$ for all $\zeta \in \operatorname{supp} \mu$. Therefore,

$$\ker T_{\mu} \subseteq \{f \in C_A(\mathbb{D}) : f = 0 \text{ on } \operatorname{supp} \mu\}. \quad (90)$$

The reverse inclusion is trivial.

The operator T_{μ} may be positive even though μ is complex. For example, for any complex number α, the measure $\alpha \cdot \delta_1$ is trivial, and hence, it is positive.

We conclude with a remark on the boundedness of T_{μ}. It is well known (cf. [3]) that for $\varphi \in L^2(\mathbb{T})$, T_{φ} is bounded if and only if $\varphi \in L^{\infty}(\mathbb{T})$, in which case, $\|T_{\varphi}\| = \|\varphi\|_{\infty}$. If $\mu \geq 0$ and T_{μ} is bounded, then

$$\int_{\mathbb{T}} |f|^2 \, d\mu \leq c \cdot \|f\|_2^2 \quad \left(f \in \mathscr{D}\left(T_{\mu}\right)\right). \quad (91)$$

Let us call a positive measure $\mu \in M(\mathbb{T})$ a compatible measure if μ satisfies (91) for all $f \in C_A(\mathbb{D})$. The word "compatible" comes from the paper [12]. One can show that the following statements are equivalent:

(i) μ is a compatible measure

(ii) $\mu \ll m$ and $d\mu/dm \in L^{\infty}(\mathbb{T})$

(iii) $\mathscr{D}(T_{\mu})$ contains all polynomials and T_{μ} is bounded

If these conditions are satisfied and if $\varphi = d\mu/dm$, then $\mathscr{D}(T_{\mu}) = C_A(\mathbb{D})$ and

$$T_{\mu} f = T_{\varphi} f, \quad (92)$$

for every $f \in C_A(\mathbb{D})$. In (iii), we cannot reduce the condition that $\mathscr{D}(T_{\mu})$ contains all polynomials to the condition that $\mathscr{D}(T_{\mu})$ is dense in $H^2(\mathbb{D})$: there is a measure $\mu \in M(\mathbb{T})$ which is not compatible such that T_{μ} is densely defined and bounded (see Example 4, (b)).

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares there are no conflicts of interest.

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References

[1] O. Toeplitz, "Zur theorie der quadratischen formen von unendlichvielen veränderlichen," *Nachrichten von der Gesellschaft der Wissenschaften zu Göttinger, Mathematisch-Physikalische Klasse*, vol. 1910, pp. 489–506, 1910.

[2] O. Toeplitz, "Über die Fourier'sche entwickelung positiver funktionen," *Rendiconti del Circolo Matematico di Palermo*, vol. 32, no. 1, pp. 191-192, 1911.

[3] A. Brown and P. R. Halmos, "Algebraic properties of Toeplitz operators," *Journal fur die Reine und Angewandte Mathematik*, vol. 213, pp. 89–102, 1964.

[4] S. Axler, J. B. Conway, and G. McDonald, "Toeplitz operators on Bergman spaces," *Canadian Journal of Mathematics*, vol. 34, no. 2, pp. 466–483, 1982.

[5] R. Rochberg and Z. J. Wu, "Toeplitz operators on Dirichlet spaces," *Integral Equations and Operator Theory*, vol. 15, no. 2, pp. 325–342, 1992.

[6] G. Cao, "Fredholm properties of Toeplitz operators on Dirichlet spaces," *Pacific Journal of Mathematics*, vol. 188, no. 2, pp. 209–223, 1999.

[7] J. J. Duistermaat and Y. J. Lee, "Toeplitz operators on the Dirichlet space," *Journal of Mathematical Analysis and Applications*, vol. 300, no. 1, pp. 54–67, 2004.

[8] K. Stroethoff, "Hankel and Toeplitz operators on the Fock space," *The Michigan Mathematical Journal*, vol. 39, no. 1, pp. 3–16, 1992.

[9] R. E. Curto, I. S. Hwang, and W. Y. Lee, "Hyponormality and subnormality of block Toeplitz operators," *Advances in Mathematics*, vol. 230, no. 4-6, pp. 2094–2151, 2012.

[10] R. E. Curto, I. S. Hwang, and W. Y. Lee, "Hyponormality of bounded-type Toeplitz operators," *Mathematische Nachrichten*, vol. 287, no. 11–12, pp. 1207–1222, 2014.

[11] R. E. Curto, I. S. Hwang, and W. Y. Lee, "Matrix functions of bounded type: an interplay between function theory and operator theory," *Memoirs of the American Mathematical Society*, vol. 260, no. 1253, p. 0, 2019.

[12] D. Sarason, "Algebraic properties of truncated Toeplitz operators," *Operators and Matrices*, vol. 1, no. 4, pp. 491–526, 2007.

[13] H. Widom, "Hankel matrices," *Transactions of the American Mathematical Society*, vol. 121, no. 1, pp. 1–35, 1966.

[14] D. Girela and N. Merchán, "A Hankel matrix acting on spaces of analytic functions," *Integral Equations and Operator Theory*, vol. 89, no. 4, pp. 581–594, 2017.

[15] K. Zhu, *Operator Theory in Function Spaces*, AMS, Providence, RI, 2nd edition, 2007.

[16] D. Sarason, "Unbounded Toeplitz operators," *Integral Equations and Operator Theory*, vol. 61, no. 2, pp. 281–298, 2008.

[17] P. L. Duren, *Theory of H^p Spaces*, Academic Press, New York-London, 1970.

[18] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York-London, 1964.

[19] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.

[20] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 3rd edition, 1987.

[21] N. K. Nikol'ski, *Treatise on the Shift Operator*, Springer-Verlag, Berlin, 1986.

[22] A. Beurling, "On two problems concerning linear transformations in Hilbert space," *Acta Mathematica*, vol. 81, pp. 239–255, 1949.

[23] J. A. Cima, A. L. Matheson, and W. T. Ross, *The Cauchy Transform*, AMS, Providence, RI, 2006.

[24] E. Hille and J. D. Tamarkin, "Remarks on a known example of a monotone continuous function," *The American Mathematical Monthly*, vol. 36, no. 5, pp. 255–264, 1929.

[25] J. R. Blum and B. Epstein, "On the Fourier-Stieltjes coefficients of Cantor-type distributions," *Israel Journal of Mathematics*, vol. 17, no. 1, pp. 35–45, 1974.

[26] P. R. Halmos, *A Hilbert Space Problem Book*, Springer-Verlag, New York-Berlin, 2nd edition, 1982.

[27] D. Sarason, *Sub-Hardy Hilbert spaces in the unit disk*, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1994, University of Arkansas Lecture Notes in the Mathematical Sciences, 10.