

Research Article

Hermite–Hadamard-Type Inequalities for Generalized Convex Functions via the Caputo–Fabrizio Fractional Integral Operator

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Due to applications in almost every area of mathematics, the theory of convex and nonconvex functions becomes a hot area of research for many mathematicians. In the present research, we generalize the Hermite–Hadamard-type inequalities for (p, h) -convex functions. Moreover, we establish some new inequalities via the Caputo–Fabrizio fractional integral operator for (p, h) -convex functions. Finally, the applications of our main findings are also given.

1. Introduction

In the last few decades, the subject of fractional calculus got attention of many researchers of different fields of pure and applied mathematics like mechanics, convex analysis, and relativity [1–3]. Nowadays, the researches in convex analysis cannot ignore the deep connectivity of both inequalities in convex analysis and fractional integral operator. Niels Henrik Abel gave birth to fractional calculus. The applications of fractional calculus can be seen in [4–9]. The first appearance of fractional derivative had been seen in a letter. The letter was written to Guillaume de l'Hopital by Gottfried Wilhelm Leibniz in 1695.

The fractional calculus techniques can be seen in many branches of science and engineering. Geometric and physical interpretation of fractional integration and fractional differentiation can be viewed in [10]. There are different fractional integral operators in which we use the integral inequalities (see, for example, [11–16]). The well-known inequality given by Hermite in 1881 can be stated as follows.

Theorem 1. Let $\zeta : L \rightarrow \mathbb{R}$ be a (p, h) -convex function defined on the interval L of real numbers and $c, d \in \mathbb{L}$ with $c < d$. Then, the following inequality holds:

$$\frac{1}{2h(1/2)} \zeta \left[\left(\frac{c^p + d^p}{2} \right) \right]^{1/p} \leq \frac{p}{d^p - c^p} \int_c^d x^{p-1} \zeta(x) dx \leq (\zeta(c) + \zeta(d)) \int_0^1 h(t) dt. \quad (1)$$

The fractional Hermite–Hadamard and Hermite–Hadamard inequalities via fractional integral can be seen in [17, 18]. For the history of Hermite–Hadamard-type inequalities, we refer to the readers [19, 20]. The outstanding applications of fractional calculus and fractional derivatives and integrals are given in [21]. Moreover, we refer the readers for a detailed study [22–27].

In the present article, we generalize the Hermite–Hadamard-type inequalities for (p, h) -convex functions. Moreover, we establish some new inequalities via the

Caputo-Fabrizio fractional integral operator for (p, h) -convex functions. Finally, the applications of our main findings are also given.

This paper is organized as follows: in Section 2, some preliminaries are given. In Section 3, we generalized Hermite–Hadamard via Caputo-Fabrizio for (p, h) -convex functions. In Section 4, we give some results related to Caputo-Fabrizio, and in Section 5, some applications to special means are given.

2. Preliminaries

We will start with some basic definitions related to our work.

Definition 2 (convex function) [28]. Let $\zeta : L \rightarrow \mathbb{R}$ be an extended real-valued function defined on a convex set $L \subseteq \mathbb{R}^n$. Then, the function ζ is convex on L if

$$\zeta(tc + (1-t)d) \leq t\zeta(c) + (1-t)\zeta(d), \quad (2)$$

for all $c, d \in L$ and $t \in (0, 1)$.

Definition 3 (p -convex function) [29]. A function $\zeta : L = [c, d] \subseteq \mathbb{R}$ is said to be a p -convex function if

$$\begin{aligned} \zeta(tc^p + (1-t)d^p)^{1/p} &\leq t\zeta(c) + (1-t)\zeta(d) \in L, \\ \forall c, d \in L, t \in [0, 1], p \neq 0. \end{aligned} \quad (3)$$

Definition 4 (h -convex function) [30]. Let $h : K \rightarrow \mathbb{R}$ be a nonnegative function. We say that $\zeta : L \rightarrow \mathbb{R}$ is an h -convex function or that $\zeta \in SX(h, L)$, if ζ is nonnegative, and for all $\forall c, d \in L, t \in [0, 1]$, we have

$$\zeta(tc + (1-t)d) \leq h(t)\zeta(c) + h(1-t)\zeta(d). \quad (4)$$

If inequality (4) is reversed, then ζ is said to be h -convex, i.e., $\zeta \in SV(h, L)$.

Remark 5.

- (1) If we take $h(t) = t$, then (4) reduces to (2)
- (2) If the function h has the property: $h(t) \geq t$ for all $t \in [0, 1]$, then any nonnegative convex function ζ belongs to the class $SX(h, L)$
- (3) If the function h has the property: $h(t) \leq t$ for all $t \in [0, 1]$, then any nonnegative convex function ζ belongs to the class $SV(h, L)$

Definition 6 ((p, h) -convex function) [31]. A function $\zeta : L = [c, d] \rightarrow \mathbb{R}$ is called a (p, h) -convex function if

$$\begin{aligned} \zeta(tc^p + (1-t)d^p)^{1/p} &\leq h(t)\zeta(c) + h(1-t)\zeta(d), \\ \forall c, d \in L, t \in [0, 1]. \end{aligned} \quad (5)$$

Definition 7 (Caputo-Fabrizio fractional time derivative) [32]. The usual Caputo fractional time derivative (UFD $_t$) of order γ is given by

$$D_t^\gamma \zeta(t) = \frac{1}{\alpha(1-\gamma)} \int_c^t \frac{\zeta'(x)}{(t-x)^\gamma} dx, \quad (6)$$

with $\gamma \in [0, 1]$ and $d \in [-\infty, t)$, $\zeta \in H^1(c, d)$, $c < d$. By changing the Kernel $(t-x)^{-\gamma}$ with the function $\exp(-\gamma(t-x)^\gamma)/(1-\gamma)$ and $1/\alpha(1-\gamma)$ with $B(\gamma)/(1-\gamma)$, we obtain the new definition of fractional time derivative:

$$D_t^\gamma \zeta(t) = \frac{B(\gamma)}{1-\gamma} \int_c^t \zeta'(x) \exp\left(\frac{-\gamma(t-x)^\gamma}{1-\gamma}\right) dx. \quad (7)$$

Definition 8 (Caputo-Fabrizio fractional integral) [32]. Let $\zeta \in H^1(c, d)$, $c < d$, $\gamma \in [0, 1]$; then, the definition of the left fractional derivative in the sense of Caputo and Fabrizio becomes

$$({}_c^{\text{CFC}}D^\gamma \zeta)(t) = \frac{B(\gamma)}{1-\gamma} \int_c^t \zeta'(x) \exp\left(\frac{-\gamma(t-x)^\gamma}{1-\gamma}\right) dx, \quad (8)$$

and the associated fractional integral is

$$({}_c^{\text{CF}}I^\gamma \zeta)(t) = \frac{1-\gamma}{B(\gamma)} \zeta(t) + \frac{\gamma}{B(\gamma)} \int_c^t \zeta(x) dx, \quad (9)$$

where $B(\gamma) > 0$ is a normalization function satisfying $B(\gamma) = B(1) = 1$.

In the right case, we have

$$({}_c^{\text{CFC}}D^\gamma \zeta)(t) = \frac{B(\gamma)}{1-\gamma} \int_t^d \zeta'(x) \exp\left(\frac{-\gamma(t-x)^\gamma}{1-\gamma}\right) dx, \quad (10)$$

and the associated fractional integral is

$$({}_c^{\text{CF}}I^\gamma \zeta)(t) = \frac{1-\gamma}{B(\gamma)} \zeta(t) + \frac{\gamma}{B(\gamma)} \int_t^d \zeta(x) dx. \quad (11)$$

Lemma 9. Let $\zeta : L \rightarrow \mathbb{R}$ be a differentiable mapping on L , $c, d \in L$ with $c < d$. If $\zeta' \in L_1[c, d]$, then the following inequality holds:

$$\begin{aligned} \frac{\zeta(c) + \zeta(d)}{2} - \frac{1}{d^p - c^p} \int_c^d \frac{\zeta(x)}{x^{1-p}} dx \\ = \frac{d^p - c^p}{2p} \int_0^1 \left(M_p^{-1}(c, d; t) \right) (1-2t) \zeta'(M_p(c, d; t)) dt, \end{aligned} \quad (12)$$

where $M_p^{-1}(c, d; t) = [tc^p + (1-t)d^p]^{(1/p)-1}$.

3. A Generalized Hermite–Hadamard-Type Inequality via the Caputo-Fabrizio Fractional Operator for a (p, h) -Convex Function

The double inequality named as Hermite–Hadamard inequality is considered one of the fundamental inequalities for convex functions.

Theorem 10. *Let a function $\zeta : [c, d] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a (p, h) -convex function on $[c, d]$ and $\zeta \in L_1[c, d]$ if $\gamma \in [0, 1]$; then, the following double inequality holds:*

$$\begin{aligned} \frac{1}{2h(1/2)} \zeta \left[\left(\frac{c^p + d^p}{2} \right) \right]^{1/p} &\leq \frac{2pB(\gamma)}{\gamma(d^p - c^p)} \left[({}^{CF}I^\gamma \eta)(k) \right. \\ &\quad \left. + ({}^{CF}I^\gamma_d \eta)(k) - \frac{2(1-\gamma)}{B(\gamma)} \eta(k) \right] \\ &\leq (\zeta(c) + \zeta(d)) \int_0^1 h(t) dt, \end{aligned} \tag{13}$$

where $\eta(x) = x^{1-p} \zeta(x)$.

Proof. Let ζ be a (p, h) -convex function; then, Hermite–Hadamard for a (p, h) -convex function is as follows:

$$\begin{aligned} \frac{1}{2h(1/2)} \zeta \left[\left(\frac{c^p + d^p}{2} \right) \right]^{1/p} &\leq \frac{p}{d^p - c^p} \int_c^d x^{p-1} \zeta(x) dx \\ &\leq (\zeta(c) + \zeta(d)) \int_0^1 h(t) dt. \end{aligned} \tag{14}$$

Since ζ is a (p, h) -convex function on $[c, d]$, we can write

$$\begin{aligned} \frac{1}{2h(1/2)} \zeta \left[\left(\frac{c^p + d^p}{2} \right) \right]^{1/p} &\leq \frac{p}{d^p - c^p} \int_c^d x^{p-1} \zeta(x) dx \\ &\quad \cdot \frac{2}{2h(1/2)} \zeta \left[\left(\frac{c^p + d^p}{2} \right) \right]^{1/p} \\ &\leq \frac{2p}{d^p - c^p} \int_c^d x^{p-1} \zeta(x) dx \\ &\quad \cdot \frac{1}{2h(1/2)} \zeta \left[\left(\frac{c^p + d^p}{2} \right) \right]^{1/p} \\ &\leq \frac{2p}{d^p - c^p} \left[\int_c^k x^{p-1} \zeta(x) dx \right. \\ &\quad \left. + \int_k^d x^{p-1} \zeta(x) dx \right]. \end{aligned} \tag{15}$$

Multiplying both sides of (15) with $\gamma(d^p - c^p)/2pB(\gamma)$ and adding $(k^{p-1}2(1-\gamma)/B(\gamma))\zeta(k)$, we have

$$\begin{aligned} &\frac{k^{p-1}2(1-\gamma)}{B(\gamma)} \zeta(k) + \frac{\gamma(d^p - c^p)}{2pB(\gamma)h(1/2)} \zeta \left[\left(\frac{c^p + d^p}{2} \right) \right]^{1/p} \\ &\leq \frac{k^{p-1}2(1-\gamma)}{B(\gamma)} \zeta(k) + \frac{\gamma}{B(\gamma)} \\ &\quad \cdot \left[\int_c^k x^{p-1} \zeta(x) dx + \int_k^d x^{p-1} \zeta(x) dx \right] \frac{2(1-\gamma)}{B(\gamma)} \eta(k) \\ &\quad + \frac{\gamma(d^p - c^p)}{2pB(\gamma)h(1/2)} \zeta \left[\left(\frac{c^p + d^p}{2} \right) \right]^{1/p} \\ &\leq \frac{2(1-\gamma)}{B(\gamma)} \eta(k) + \frac{\gamma}{B(\gamma)} \left[\int_c^k \eta(x) dx + \int_k^d \eta(x) dx \right] \\ &= \left[\frac{(1-\gamma)}{B(\gamma)} \eta(k) + \frac{\gamma}{B(\gamma)} \int_c^k \eta(x) dx \right. \\ &\quad \left. + \frac{(1-\gamma)}{B(\gamma)} \eta(k) + \frac{\gamma}{B(\gamma)} \int_k^d \eta(x) dx \right] \\ &= ({}^{CF}I^\gamma \eta)(k) + ({}^{CF}I^\gamma_d \eta)(k) \zeta \left[\left(\frac{c^p + d^p}{2} \right) \right]^{1/p} \\ &\leq \frac{2pB(\gamma)h(1/2)}{\gamma(d^p - c^p)} \left[({}^{CF}I^\gamma \eta)(k) + ({}^{CF}I^\gamma_d \eta)(k) \frac{2(1-\gamma)}{B(\gamma)} \eta(k) \right]. \end{aligned} \tag{16}$$

For the proof of the right hand side of the Hermite–Hadamard-type inequality, we have

$$\frac{2p}{d^p - c^p} \int_c^d x^{p-1} \zeta(x) dx \leq 2 \left[(\zeta(c) + \zeta(d)) \int_0^1 h(t) dt \right]. \tag{17}$$

Multiplying both sides of (17) with $\gamma(d^p - c^p)/2pB(\gamma)$ and adding $(k^{p-1}2(1-\gamma)/B(\gamma))\zeta(k)$, we get

$$\begin{aligned} ({}^{CF}I^\gamma \eta)(k) + ({}^{CF}I^\gamma_d \eta)(k) &\leq \frac{2(1-\gamma)}{B(\gamma)} \eta(k) + \frac{\gamma(d^p - c^p)}{\gamma B(\gamma)} \\ &\quad \cdot (\zeta(c) + \zeta(d)) \int_0^1 h(t) dt. \end{aligned} \tag{18}$$

By recognizing (18), the proof of the right hand side of (13) is completed. This completes the proof. \square

Remark 11. If we put $p = 1$ and $h(t) = t$, then we will get the Hermite–Hadamard inequality for convex function.

Theorem 12. *Let $\zeta, \theta : L \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a (p, h) -convex function. If $\zeta\theta \in L_1[c, d]$, then we have the following inequality:*

$$\begin{aligned} &\frac{2pB(\gamma)}{\gamma(d^p - c^p)} \left[({}^{CF}I^\gamma \eta\beta)(k) + ({}^{CF}I^\gamma_d \eta\beta)(k) - \frac{2(1-\gamma)}{B(\gamma)} \eta(k)\beta(k) \right] \\ &\leq 2 \left[M(c, d) \int_0^1 h_1(t)h_2(t) dt + N(c, d) \int_0^1 h_1(t)h_2(1-t) dt \right], \end{aligned} \tag{19}$$

with $\eta(x) = x^{1-p}\zeta(x)$ and $\beta(x) = x^{1-p}\theta(x)$, where $M(c, d) = \zeta(c)\theta(c) + \zeta(d)\theta(d)$, $N(c, d) = \zeta(c)\theta(d) + \zeta(d)\theta(c)$, and $k \in [c, d]$, $B(\gamma) > 0$ is a normalization function.

Proof. Since ζ and θ are (p, h) -convex functions on $[c, d]$, we have

$$\begin{aligned} \zeta(tc^p + (1-t)d^p)^{1/p} &\leq h_1(t)\zeta(c) + h_1(1-t)\zeta(d), \\ \theta(tc^p + (1-t)d^p)^{1/p} &\leq h_2(t)\theta(c) + h_2(1-t)\theta(d). \end{aligned} \tag{20}$$

Multiplying both sides of the above inequalities, we have

$$\begin{aligned} &\zeta(tc^p + (1-t)d^p)^{1/p}\theta(tc^p + (1-t)d^p)^{1/p} \\ &\leq h_1(t)h_2(t)\zeta(c)\theta(c) + h_1(1-t)h_2(1-t)\zeta(d)\theta(d) \\ &\quad + h_1(t)h_2(1-t)\zeta(c)\theta(d) + h_1(1-t)h_2(t)\zeta(d)\theta(c). \end{aligned} \tag{21}$$

Integrating with respect to t over $[0, 1]$ and making the change of variable, we obtain

$$\begin{aligned} &\frac{p}{d^p - c^p} \int_c^b x^{p-1}\zeta(x)\theta(x)dx \\ &\leq \int_0^1 h_1(t)h_2(t)dt[\zeta(c)\theta(c) + \zeta(d)\theta(d)] \\ &\quad + \int_0^1 h_1(t)h_2(1-t)dt[\zeta(c)\theta(d) + \zeta(d)\theta(c)], \end{aligned} \tag{22}$$

which implies

$$\begin{aligned} &\frac{2p}{d^p - c^p} \left[\int_c^k x^{p-1}\zeta(x)\theta(x)dx + \int_c^k x^{p-1}\zeta(x)\theta(x)dx \right] \\ &\leq 2 \left[M(c, d) \int_0^1 h_1(t)h_2(t)dt + N(c, d) \int_0^1 h_1(t)h_2(1-t)dt \right]. \end{aligned} \tag{23}$$

By multiplying both sides with $(\gamma c^p - d^p)/2PB(\gamma)$ and adding $(k^{p-1}2(1-\gamma)/B(\gamma))\zeta(K)g(k)$, we have

$$\begin{aligned} &\frac{\gamma}{B(\gamma)} \left[\int_c^k \eta(x)\beta(x)dx + \int_k^d \eta(x)\beta(x)dx \right] + \frac{2(1-\gamma)}{B(\gamma)}\eta(k)\beta(k) \\ &\leq \frac{\gamma c^p - d^p}{2pB(\gamma)} 2 \left[M(c, d) \int_0^1 h_1(t)h_2(t)dt + N(c, d) \right. \\ &\quad \cdot \left. \int_0^1 h_1(t)h_2(1-t)dt \right] + \frac{2(1-\gamma)}{B(\gamma)}\eta(k)\beta(k). \end{aligned} \tag{24}$$

Thus,

$$\begin{aligned} &({}^{\text{CF}}I^\gamma \eta\beta)(k) + ({}^{\text{CF}}I_d^\gamma \eta\beta)(k) \\ &\leq \frac{\gamma c^p - d^p}{2pB(\gamma)} \left[2M(c, d) \int_0^1 h_1(t)h_2(t)dt + 2N(c, d) \right. \\ &\quad \cdot \left. \int_0^1 h_1(t)h_2(1-t)dt \right] + \frac{2(1-\gamma)}{B(\gamma)}\eta(k)\beta(k), \end{aligned} \tag{25}$$

and with suitable rearrangements, the proof is completed. \square

Remark 13. If we put $p = 1$ and $h(t) = t$ in the above theorem, we get the results for the classical convex function.

Theorem 14. Let $\zeta, \theta : L \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a (p, h) -convex function. If $\zeta\theta \in L_1[c, d]$, then we have the following inequality:

$$\begin{aligned} &\frac{1}{h_1(1/2)h_2(1/2)} \zeta\left(\left[\frac{c^p + d^p}{2}\right]^{1/p}\right) \theta\left(\left[\frac{c^p + d^p}{2}\right]^{1/p}\right) \\ &\quad - \frac{2p}{\gamma(d^p - c^p)} [({}^{\text{CF}}I^\gamma \eta\beta)(k) + ({}^{\text{CF}}I_d^\gamma \eta\beta)(k)] \\ &\quad + \frac{4p(1-\gamma)}{\gamma(d^p - c^p)} \eta(k)\beta(k) \\ &\leq 2M(c, d) \int_0^1 h_1(t)h_2(t)dt + 2N(c, d) \int_0^1 h_1(t)h_2(1-t)dt, \end{aligned} \tag{26}$$

with $\eta(x) = x^{1-p}\zeta(x)$ and $\beta(x) = x^{1-p}\theta(x)$, where $M(c, d) = \zeta(c)\theta(c) + \zeta(d)\theta(d)$, $N(c, d) = \zeta(c)\theta(d) + \zeta(d)\theta(c)$, and $k \in [c, d]$, $B(\gamma) > 0$ is a normalization function.

Proof. Since $(c^p + d^p)/2 = ((tc^p + (1-t)d^p)/2) + (((1-t)c^p + td^p)/2)$ for $t = 1/2$, we have

$$\begin{aligned} &\zeta\left(\left[\frac{(1-t)c^p + td^p + tc^p + (1-t)d^p}{2}\right]^{1/p}\right) \\ &\leq h_1\left(\frac{1}{2}\right)\zeta\left(\left((1-t)c^p + td^p\right)^{1/p}\right) \\ &\quad + h_1\left(\frac{1}{2}\right)\zeta\left(\left(tc^p + (1-t)d^p\right)^{1/p}\right), \end{aligned} \tag{27}$$

$$\begin{aligned} &\theta\left(\left[\frac{(1-t)c^p + td^p + tc^p + (1-t)d^p}{2}\right]^{1/p}\right) \\ &\leq h_2\left(\frac{1}{2}\right)\theta\left(\left((1-t)c^p + td^p\right)^{1/p}\right) \\ &\quad + h_2\left(\frac{1}{2}\right)\theta\left(\left(tc^p + (1-t)d^p\right)^{1/p}\right). \end{aligned} \tag{28}$$

Multiplying (2) and (28), we have

$$\begin{aligned}
 & \zeta\left(\left[\frac{c^p+d^p}{2}\right]^{1/p}\right)\theta\left(\left[\frac{c^p+d^p}{2}\right]^{1/p}\right) \\
 & \leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[\zeta\left(\left((1-t)c^p+td^p\right)^{1/p}\right)\theta\right. \\
 & \quad \cdot\left.\left(\left((1-t)c^p+td^p\right)^{1/p}\right)\right]+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \\
 & \quad \cdot\left[\zeta\left(\left(tc^p+(1-t)d^p\right)^{1/p}\right)\theta\left(\left(tc^p+(1-t)d^p\right)^{1/p}\right)\right] \\
 & \quad +h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[h_1(t)\zeta(c)+h_1(1-t)\zeta(d)\right] \\
 & \quad \cdot[h_2(t)\theta(c)+h_2(1-t)\theta(d)]+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \\
 & \quad \cdot[h_1(1-t)(\zeta(c)+h_1(t)\zeta(d))][h_2(1-t)\theta(c)+h_2(t)\theta(d)] \\
 & =h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[\zeta\left(\left((1-t)c^p+td^p\right)^{1/p}\right)\theta\right. \\
 & \quad \cdot\left.\left(\left((1-t)c^p+td^p\right)^{1/p}\right)\right]+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \\
 & \quad \cdot\left[\zeta\left(\left(tc^p+(1-t)d^p\right)^{1/p}\right)\theta\left(\left(tc^p+(1-t)d^p\right)^{1/p}\right)\right] \\
 & \quad +h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[(h_1(t)+h_2(1-t)+h_1(1-t)h_2(t))M(c,d)\right] \\
 & \quad +h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[(h_1(t)h_2(t)+h_1(1-t)h_2(1-t))N(c,d)\right].
 \end{aligned}
 \tag{29}$$

Integrating the above inequality with respect to t over $[0, 1]$ and making the change of variable, one obtains

$$\begin{aligned}
 & \frac{1}{h_1(1/2)h_2(1/2)}\zeta\left(\left[\frac{c^p+d^p}{2}\right]^{1/p}\right)\theta\left(\left[\frac{c^p+d^p}{2}\right]^{1/p}\right) \\
 & \quad -\frac{p}{d^p-c^p}\int_c^d x^{p-1}\zeta(x)\theta(x)dx \\
 & \leq 2M(c,d)\int_0^1 h_1(t)h_2(t)dt+2N(c,d)\int_0^1 h_1(t)h_2(1-t)dt.
 \end{aligned}
 \tag{30}$$

By multiplying both sides with $\gamma(d^p-c^p)/2pB(\gamma)$ and subtracting $(K^{p-1}2(1-\gamma)/B(\gamma))\zeta(k)\theta(k)$, we have

$$\begin{aligned}
 & \frac{2\gamma(d^p-c^p)}{pB(\gamma)}\zeta\left(\left[\frac{c^p+d^p}{2}\right]^{1/p}\right)\theta\left(\left[\frac{c^p+d^p}{2}\right]^{1/p}\right) \\
 & \quad -\frac{\gamma}{B(\gamma)}\left[\int_c^k F(x)G(x)dx+\int_k^d F(x)G(x)dx\right] \\
 & \quad -\frac{2(1-\gamma)}{B(\gamma)}\zeta(k)g(k) \\
 & \leq \frac{\gamma(d^p-c^p)}{2pB(\gamma)}2M(c,d)\int_0^1 h_1(t)h_2(t)dt+2N(c,d) \\
 & \quad \cdot\int_0^1 h_1(t)h_2(1-t)dt-\frac{2(1-\gamma)}{B(\gamma)}F(k)G(k),
 \end{aligned}$$

$$\begin{aligned}
 & \frac{2\gamma(d^p-c^p)}{pB(\gamma)}\zeta\left(\left[\frac{c^p+d^p}{2}\right]^{1/p}\right)\theta\left(\left[\frac{c^p+d^p}{2}\right]^{1/p}\right) \\
 & \quad -\frac{\gamma}{B(\gamma)}\left[({}^C I^\gamma \eta\beta)(k)+({}^C I^\gamma_a \eta\beta)(k)\right] \\
 & \leq \frac{\gamma(d^p-c^p)}{2pB(\gamma)}2\left[M(c,d)\int_0^1 h_1(t)h_2(t)dt\right. \\
 & \quad \left.+N(c,d)\int_0^1 h_1(t)h_2(1-t)dt\right]-\frac{2(1-\gamma)}{B(\gamma)}\eta(k)\beta(k).
 \end{aligned}
 \tag{31}$$

By multiplying both sides of the above inequality by $2pB(\gamma)/\gamma(d^p-c^p)$, we get the required inequality (28). \square

Remark 15. If we put $p = 1$ and $h(t) = t$ in the above theorem, then we will get the result for the convex function.

4. Some New Results Related to the Caputo-Fabrizio Fractional Operator

In this section, firstly we generalize a lemma; then, we prove our main theorem with the help of this lemma.

Lemma 16. *Let $\zeta : L \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $L, c, d \in L$ with $c < d$. If $\zeta' \in L_1[c, d]$ and $\gamma \in [0, 1]$, the following equality holds:*

$$\begin{aligned}
 & \frac{d^p-c^p}{2}\int_0^1 M^{-1}(c,d;t)(1-2t)\zeta'(M_p(c,d;t))dt-\frac{2(1-\gamma)}{\gamma(d^p-c^p)}\eta(k) \\
 & =\frac{\zeta(c)+\zeta(d)}{2}-\frac{pB(\gamma)}{\gamma(d^p-c^p)}\left[({}^C I^\gamma \eta)(k)+({}^C I^\gamma_a \eta)(k)\right],
 \end{aligned}
 \tag{32}$$

where $k \in [c, d]$ and $B(\gamma) > 0$ is a normalization function with $\eta(x) = \zeta(x)/x^{1-p}$.

Proof. From Lemma 9, we can observe that

$$\begin{aligned}
 & \int_0^1 M^{-1}(c,d;t)(1-2t)\zeta'(M_p(c,d;t))dt \\
 & =\frac{\zeta(c)+\zeta(d)}{d^p-c^p}-\frac{2p}{(d^p-c^p)^2}\left(\int_c^k \frac{\zeta(x)}{x^{1-p}}dx+\int_k^d \frac{\zeta(x)}{x^{1-p}}dx\right).
 \end{aligned}
 \tag{33}$$

By multiplying $\gamma(d^p-c^p)^2/2pB(\gamma)$ and subtracting $(2(1-\gamma)/x^{1-p}B(\gamma))\zeta(k)$, we have

$$\begin{aligned}
& \frac{\gamma(d^p - c^p)^2}{2pB(\gamma)} \int_0^1 M^{-1}(c, d; t)(1-2t)\zeta'(M_p(c, d; t))dt - \frac{2(1-\gamma)}{x^{1-p}B(\gamma)}\zeta(k) \\
&= \frac{\gamma(d^p - c^p)\zeta(c) + \zeta(d)}{2pB(\gamma)} + \frac{2(1-\gamma)}{x^{1-p}B(\gamma)}\zeta(k) - \frac{\gamma}{B(\gamma)} \left(\int_c^k \frac{\zeta(x)}{x^{1-p}} dx + \int_k^d \frac{\zeta(x)}{x^{1-p}} dx \right) \\
&= \frac{\gamma(d^p - c^p)\zeta(c) + \zeta(d)}{2pB(\gamma)} + \frac{2(1-\gamma)}{B(\gamma)}\eta(k) - \frac{\gamma}{B(\gamma)} \left(\int_c^k \eta(x) dx + \int_k^d \eta(x) dx \right) \\
&= \frac{\gamma(d^p - c^p)\zeta(c) + \zeta(d)}{2pB(\gamma)} - \left(\frac{(1-\gamma)}{B(\gamma)}\eta(k) + \frac{\gamma}{B(\gamma)} \int_c^k \eta(x) dx \right) \\
&\quad - \left(\frac{(1-\gamma)}{B(\gamma)}\eta(k) + \frac{\gamma}{B(\gamma)} \int_c^k \eta(x) dx \right) \\
&= \frac{\gamma(d^p - c^p)\zeta(c) + \zeta(d)}{2pB(\gamma)} - \left[({}^{\text{CF}}I_{d^-}^{\gamma}\eta)(k) + ({}^{\text{CF}}I_{d^-}^{\gamma}\eta)(k) \right].
\end{aligned} \tag{34}$$

This completes the proof. \square

Theorem 17. Let $\zeta : L \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differential mapping on L and $|\zeta'|$ be (p, h) -convex on $[c, d]$ where $c, d \in L$ with $c < d$, $p > 0$. If $\zeta' \in L_1[c, d]$ and $\gamma \in [0, 1]$, the following inequality holds:

$$\begin{aligned}
& \left| \frac{\zeta(c) + \zeta(d)}{2} + \frac{2(1-\gamma)}{\gamma(d^p - c^p)}\zeta(k) - \frac{pB(\gamma)}{\gamma(d^p - c^p)} \left[({}^{\text{CF}}I_{d^-}^{\gamma}\zeta)(k) + ({}^{\text{CF}}I_{d^-}^{\gamma}\zeta)(k) \right] \right| \\
&\leq \frac{d^p - c^p}{2p} \left[d_1(c, d, p)|\zeta'(c)| + d_2(c, d, p)|\zeta'(d)| \right],
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
d_1(c, d, p) &= \int_0^{1/2} \frac{(1-2t)h(t)}{(tc^p + (1-t)d^p)^{(1/p)-1}} dt + \int_{1/2}^1 \frac{(2t-1)h(t)}{(tc^p + (1-t)d^p)^{(1/p)-1}} dt, \\
d_2(c, d, p) &= \int_0^{1/2} \frac{(1-2t)h(1-t)}{(tc^p + (1-t)d^p)^{(1/p)-1}} dt + \int_{1/2}^1 \frac{(2t-1)h(1-t)}{(tc^p + (1-t)d^p)^{(1/p)-1}} dt,
\end{aligned} \tag{36}$$

where $k \in [c, d]$ and $B(\gamma) > 0$ is a normalization function.

Proof.

$$\begin{aligned}
& \left| \frac{\zeta(c) + \zeta(d)}{2} + \frac{2(1-\gamma)}{\gamma(d^p - c^p)}\zeta(k) - \frac{pB(\gamma)}{\gamma(d^p - c^p)} \left[({}^{\text{CF}}I_{d^-}^{\gamma}\zeta)(k) + ({}^{\text{CF}}I_{d^-}^{\gamma}\zeta)(k) \right] \right| \\
&\leq \frac{d^p - c^p}{2p} \int_0^1 M^{-1}(c, d; t)(1-2t)|\zeta'(M_p(c, d; t))| dt \\
&\leq \frac{d^p - c^p}{2p} \int_0^1 \left(\frac{|1-2t| |\zeta'(tc^p(1-t)d^p)^{1/p}|}{|(tc^p(1-t)d^p)^{(1/p)-1}|} \right) dt \\
&\leq \frac{d^p - c^p}{2p} \int_0^1 \left(\frac{|1-2t| (h(t)|\zeta'(c)| + h(1-t)|\zeta'(d)|)}{|(tc^p(1-t)d^p)^{(1/p)-1}|} \right) dt \\
&= \frac{d^p - c^p}{2p} \left[\int_0^{1/2} \frac{(1-2t)h(t)|\zeta'(c)|}{(tc^p + (1-t)d^p)^{(1/p)-1}} dt + \int_{1/2}^1 \frac{(2t-1)h(t)|\zeta'(c)|}{(tc^p + (1-t)d^p)^{(1/p)-1}} dt \right] \\
&\quad + \frac{d^p - c^p}{2p} \left[\int_0^{1/2} \frac{(1-2t)h(1-t)|\zeta'(d)|}{(tc^p + (1-t)d^p)^{(1/p)-1}} dt + \int_{1/2}^1 \frac{(2t-1)h(1-t)|\zeta'(d)|}{(tc^p + (1-t)d^p)^{(1/p)-1}} dt \right] \\
&\leq \frac{d^p - c^p}{2p} \left[d_1(c, d, p)|\zeta'(c)| + d_2(c, d, p)|\zeta'(d)| \right].
\end{aligned} \tag{37}$$

This completes the proof. \square

Theorem 18. Let $\zeta : L \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differential mapping on L and $|\zeta'|$ be (p, h) -convex on $[c, d]$ where $c, d \in L$ with $c < d$, $p > 1$, and $(1/r) + (1/q) = 1$. If $\zeta' \in L_1[c, d]$ and $\gamma \in [0, 1]$, the following inequality holds:

$$\begin{aligned}
& \left| \frac{\zeta(c) + \zeta(d)}{2} + \frac{2(1-\gamma)}{\gamma(d^p - c^p)}\zeta(k) - \frac{pB(\gamma)}{\gamma(d^p - c^p)} \left[({}^{\text{CF}}I_{d^-}^{\gamma}\zeta)(k) + ({}^{\text{CF}}I_{d^-}^{\gamma}\zeta)(k) \right] \right| \\
&\leq \frac{d^p - c^p}{2p} (d_1(c, d, p))^{1/r} (d_2(c, d, p)|\zeta'(c)|^q + d_3(c, d, p)|\zeta'(d)|^q)^{1/q},
\end{aligned} \tag{38}$$

where

$$\begin{aligned}
d_1(c, d, p) &= \int_0^1 \left(\frac{|1-2t|^r}{(tc^p(1-t)d^p)^{(1/p)-1}} \right) dt, \\
d_2(c, d, p) &= \int_0^1 \left(\frac{h(t)}{(tc^p(1-t)d^p)^{(1/p)-1}} \right) dt, \\
d_3(c, d, p) &= \int_0^1 \left(\frac{h(1-t)}{(tc^p(1-t)d^p)^{(1/p)-1}} \right) dt,
\end{aligned} \tag{39}$$

where $k \in [c, d]$ and $B(\gamma) > 0$ is a normalization function.

Proof. By a similar argument to the proof of the previous theorem, by using lemma, the Hölder inequality, and convexity of $|\zeta'|$, we get

$$\begin{aligned}
& \left| \frac{\zeta(c) + \zeta(d)}{2} + \frac{2(1-\gamma)}{\gamma(d^p - c^p)}f(k) - \frac{pB(\gamma)}{\gamma(d^p - c^p)} \left[({}^{\text{CF}}I_{d^-}^{\gamma}\zeta)(k) + ({}^{\text{CF}}I_{d^-}^{\gamma}\zeta)(k) \right] \right| \\
&\leq \frac{d^p - c^p}{2p} \int_0^1 M^{-1}(c, d; t)(1-2t)|\zeta'(M_p(c, d; t))| dt \\
&\leq \frac{d^p - c^p}{2p} \left(\int_0^1 \left(\frac{|1-2t|^r}{(tc^p(1-t)d^p)^{(1/p)-1}} \right) dt \right)^{1/r} \left(\int_0^1 \left(\frac{|\zeta'(tc^p(1-t)d^p)^{q/p}|}{(tc^p(1-t)d^p)^{(1/p)-1}} \right) dt \right)^{1/q} \\
&\leq \frac{d^p - c^p}{2p} \left(\int_0^1 \left(\frac{|1-2t|^r}{(tc^p(1-t)d^p)^{(1/p)-1}} \right) dt \right)^{1/r} \left(\int_0^1 \left(\frac{(h(t)|\zeta'(c)|^q + h(1-t)|\zeta'(d)|^q)}{(tc^p(1-t)d^p)^{(1/p)-1}} \right) dt \right)^{1/q} \\
&\leq \frac{d^p - c^p}{2p} \left(\int_0^1 \left(\frac{|1-2t|^r}{(tc^p(1-t)d^p)^{(1/p)-1}} \right) dt \right)^{1/r} \left(\int_0^1 \left(\frac{h(t)|\zeta'(c)|^q}{(tc^p(1-t)d^p)^{(1/p)-1}} \right) dt \right)^{1/q} \\
&\quad + \int_0^1 \left(\frac{h(1-t)|\zeta'(d)|^q}{(tc^p(1-t)d^p)^{(1/p)-1}} \right) dt \right)^{1/q} \\
&\leq \frac{d^p - c^p}{2p} (d_1(c, d, p))^{1/r} (d_2(c, d, p)|\zeta'(c)|^q + d_3(c, d, p)|\zeta'(d)|^q)^{1/q}.
\end{aligned} \tag{40}$$

This completes the proof. \square

5. Application to Special Means

The applications to special means are also used to confirm the accuracy of the findings for real numbers c, d such that $c \neq d$.

The arithmetic mean of two numbers c and d is defined as

$$A = A(c, d) = \frac{c + d}{2}, c, d \in \mathbb{R}. \tag{41}$$

The generalized logarithmic mean is defined as

$$L = L_r^r = \frac{c^{t+1} - d^{t+1}}{(r+1)(d-c)} r \in \mathbb{R} - [-1, 0], t \in \mathbb{R}, c, d \in \mathbb{R}, c \neq d. \quad (42)$$

Now, using the results in Section 4, we have some applications to the special means of real numbers.

Proposition 19. *Let $c, d \in \mathbb{R}, c < d$; then,*

$$|A(c^2, d^2) - pL_2^2(c, d)| \leq \frac{d^p - c^p}{2p} [d_1(2|c|) + d_2(2|d|)]. \quad (43)$$

Proof. In the inequality proven in Theorem 17, if we set $\zeta(z) = z^2$ with $\gamma = 1$ and $B(\gamma) = B(1) = 1$, then we obtain the result immediately. \square

Remark 20. If we put $h(t) = t$ and $p = 1$ in this proposition, we will obtain this result for the convex function.

Proposition 21. *Let $c, d \in \mathbb{R}, c < d$; then,*

$$|A(c^n, d^n) - pL_n^n(c, d)| \leq \frac{n(d^p - c^p)}{2p} [d_1(2|c^{n-1}|) + d_2(2|d^{n-1}|)]. \quad (44)$$

Proof. In the inequality proven in Theorem 17, if we set $\zeta(z) = z^n$ where n is an even number with $\gamma = 1$ and $B(\gamma) = B(1) = 1$, then we obtain the result immediately. \square

Remark 22. If we put $h(t) = t$ and $p = 1$ in this proposition, we will obtain this result for the convex function.

6. Conclusion

Convexity is very important for solving optimization problems. Fractional calculus together with convexity plays an important role in solving real-life problems. In this paper, we established several Hermite–Hadamard-type inequalities in the setting of a fractional integral operator for (p, h) -convex functions. We also presented some applications in means. Our results generalized several existing results.

Conflicts of Interest

The authors do not have any competing interests.

Authors' Contributions

Dong Zhang proved the main results, Muhammad Shoab Saleem proposed the problem and supervised this work, Thongchai Botmart analyzed the results and arranged the funding for this paper, M. S. Zahoor proved the main results, and R. Bano wrote the first version of this paper.

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