

Research Article

Nearly General Septic Functional Equation

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If a mapping can be expressed by sum of a septic mapping, a sextic mapping, a quintic mapping, a quartic mapping, a cubic mapping, a quadratic mapping, an additive mapping, and a constant mapping, we say that it is a general septic mapping. A functional equation is said to be a general septic functional equation provided that each solution of that equation is a general septic mapping. In fact, there are a lot of ways to show the stability of functional equations, but by using the method of Găvruta, we examine the stability of general septic functional equation $\sum_{i=0}^8 C_i (-1)^{8-i} f(x + (i-4)y) = 0$ which considered. The method of Găvruta as just mentioned was given in the reference Gavruta (1994).

1. Introduction

The concept of stability for a functional equation arising when replacing the functional equation by an inequality which acts as a perturbation of the equation. Ulam [1] posed the question concerning the stability of group homomorphisms. Hyers [2] gave the first partial affirmative answer to the question of Ulam, which states that if $\delta > 0$ and $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a mapping with \mathcal{X} a normed space, \mathcal{Y} a Banach space such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta \text{ for all } x, y \in \mathcal{X}, \quad (1)$$

then there exists a unique additive mapping $T : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\|f(x) - T(x)\| \leq \delta \text{ for all } x \in \mathcal{X}. \quad (2)$$

Thereafter, many authors have generalized the Hyers' result (refer to [3–8]). On the other hand, a generalization of the Hyers result was also obtained by Găvruta [9]. That is, let $(G, +)$ be an abelian group and let Y be a Banach space. Suppose that a function $\varphi : G^2 \rightarrow [0, \infty)$ satisfies the assumption

$$\tilde{\varphi}(x, y) := \sum_{k=0}^{\infty} 2^{-k} \varphi(x, y) < \infty, \quad (3)$$

for all $x, y \in G$, and that $f : G \rightarrow Y$ is a mapping such that the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y), \quad (4)$$

holds for all $x, y \in G$. Then, there exists a unique additive mapping $T : G \rightarrow Y$ with

$$\|f(x) - T(x)\| \leq \frac{1}{2} \tilde{\varphi}(x, x) \text{ for all } x \in G. \quad (5)$$

We, meanwhile, call the functional equation

$$\sum_{i=0}^n C_i (-1)^{n-i} f(x + iy) - n! f(y) = 0, \quad (6)$$

the n -monomial functional equation. The 1-monomial, the 2-monomial, the 3-monomial, the 4-monomial, the 5-monomial, the 6-monomial, and the 7-monomial functional equation are called additive, quadratic, cubic, quartic, quintic, sextic, and septic functional equation, respectively. In this case,

we say that each solution of the previous equation is additive, quadratic, cubic, quartic, quintic, sextic, and septic mapping, respectively. The function $f : R \rightarrow R$ defined by $f(x) := ax^n$ is a particular solution of the n -monomial functional equation. Quite recently, Lee [6] showed the stability of the n -monomial functional equation in the sense of Găvruta.

The following functional equation is called

$$\sum_{i=0}^n {}_n C_i (-1)^{n-i} f(x + iy) = 0, \tag{7}$$

as Jensen, general quadratic, general cubic, general quartic, general quintic, general sextic, and general septic functional equation, respectively, for $n = 2, 3, 4, 5, 6, 7, 8$. The solution of the general septic functional equation is said to be a general septic mapping. The function $f : R \rightarrow R$ given by $f(x) := \sum_{i=0}^7 a_i x^i$ is a particular solution of the general septic mapping. More detailed term for the concept of a *general septic mapping* can be found in Baker's paper [10] by the term *generalized polynomial mapping of degree at most 7*.

For a number of years now, many interesting results of the stability problems to several functional equations (or involving the range from additive functional equation to sextic functional equation) have been investigated; see, e.g., [11–26].

Our principal purpose is to consider the following general septic functional equation

$$\sum_{i=0}^8 {}_8 C_i (-1)^{8-i} f(x + (i - 4)y) = 0, \tag{8}$$

and then we are going to obtain the stability theorems of the functional equation (8) in the spirit of Găvruta approach.

2. Stability of the General Septic Functional Equation (8)

In this section, we let Y and V be a real Banach space and a real vector space, respectively. For a given mapping $f : V \rightarrow Y$ and all $x, y \in V$, we use the following abbreviations

$$\begin{aligned} f_o(x) &= \frac{f(x) - f(-x)}{2}, \\ f_e(x) &= \frac{f(x) + f(-x)}{2}, \\ Df(x, y) &= \sum_{i=0}^8 {}_8 C_i (-1)^{8-i} f(x + (i - 4)y), \\ \Gamma(x) &= Df_o(8x, 2x) + 8Df_o(6x, 2x) + 36Df_o(4x, 2x) \\ &\quad + 120Df_o(2x, 2x) + 160Df_o(4x, x) + 1280Df_o(3x, x) \\ &\quad + 4032Df_o(2x, x) + 5376Df_o(x, x), \\ \Delta f(x) &= Df_e(4x, x) + 8Df_e(3x, x) + 36Df_e(2x, x) \\ &\quad + 120Df_e(x, x) + 123Df_e(0, x). \end{aligned} \tag{9}$$

On the other hand, if \tilde{f} is a mapping defined by $\tilde{f}(x) = f(x) - f(0)$, we know that

$$D\tilde{f}(x, y) = Df(x, y) \text{ and } \tilde{f}(0) = 0. \tag{10}$$

In addition, through tedious computation, we then get the following expressions

$$\Delta f(x) = f_e(8x) - 84f_e(4x) + 1344f_e(2x) - 4096f_e(x), \tag{11}$$

$$\begin{aligned} \Gamma f(x) &= f_o(16x) - 170f_o(8x) + 5712f_o(4x) - 43520f_o(2x) \\ &\quad + 65536f_o(x), \end{aligned} \tag{12}$$

for all $x \in V$.

Lemma 1. *Let $f : V \rightarrow Y$ be a mapping with $f(0) = 0$. Suppose that $J_n f : V \rightarrow Y$ and $J'_n f : V \rightarrow Y$ are mappings given by*

$$\begin{aligned} J_n f(x) &:= \frac{4^n - 20 \cdot 16^n + 64 \cdot 64^n}{45} f_e\left(\frac{x}{2^n}\right) \\ &\quad - \frac{80(4^n - 17 \cdot 16^n + 16 \cdot 64^n)}{45} f_e\left(\frac{x}{2^{n+1}}\right) \\ &\quad + \frac{1024(4^n - 5 \cdot 16^n + 4 \cdot 64^n)}{45} f_e\left(\frac{x}{2^{n+2}}\right) \\ &\quad - \frac{2^n - 84 \cdot 8^n + 1344 \cdot 32^n - 4096 \cdot 128^n}{2835} f_o\left(\frac{x}{2^n}\right) \\ &\quad + \frac{168(2^n - 81 \cdot 8^n + 1104 \cdot 32^n - 1024 \cdot 128^n)}{2835} f_o\left(\frac{x}{2^{n+1}}\right) \\ &\quad - \frac{5376(2^n - 69 \cdot 8^n + 324 \cdot 32^n - 256 \cdot 128^n)}{2835} f_o\left(\frac{x}{2^{n+2}}\right) \\ &\quad + \frac{32768(2^n - 21 \cdot 8^n + 84 \cdot 32^n - 64 \cdot 128^n)}{2835} f_o\left(\frac{x}{2^{n+3}}\right), \end{aligned} \tag{13}$$

for all $x \in V$ and all integers $n \geq 0$ and

$$\begin{aligned} J'_n f(x) &:= -\left(\frac{1}{128^n} - \frac{84}{32^n} + \frac{1344}{8^n} - \frac{4096}{2^n}\right) \frac{512f_o(2^n x)}{512 \cdot 2835} \\ &\quad + \left(\frac{1}{128^n} - \frac{81}{32^n} + \frac{1104}{8^n} - \frac{1024}{2^n}\right) \frac{336f_o(2^{n+1}x)}{512 \cdot 2835} \\ &\quad - \left(\frac{1}{128^n} - \frac{69}{32^n} + \frac{324}{8^n} - \frac{256}{2^n}\right) \frac{42f_o(2^{n+2}x)}{512 \cdot 2835} \\ &\quad + \left(\frac{1}{128^n} - \frac{21}{32^n} + \frac{84}{8^n} - \frac{64}{2^n}\right) \frac{f_o(2^{n+3}x)}{512 \cdot 2835} \\ &\quad + \left(\frac{1}{64^n} - \frac{5}{16^n} + \frac{4}{4^n}\right) \frac{f_e(2^{n+2}x)}{64 \cdot 45} \\ &\quad - \left(\frac{20}{64^n} - \frac{340}{16^n} + \frac{320}{4^n}\right) \frac{f_e(2^{n+1}x)}{64 \cdot 45} \\ &\quad + \left(\frac{64}{64^n} - \frac{1280}{16^n} + \frac{4096}{4^n}\right) \frac{f_e(2^n x)}{64 \cdot 45}, \end{aligned} \tag{14}$$

for all $x \in V$ and all integers $n \geq 0$. Then,

$$J_n f(x) - J_{n+1} f(x) = \left(\frac{4^n}{45} - \frac{20 \cdot 16^n}{45} + \frac{64 \cdot 64^n}{45} \right) \Delta f \left(\frac{x}{2^{n+3}} \right) - \frac{2^n - 84 \cdot 8^n + 1344 \cdot 32^n - 4096 \cdot 128^n}{2835} \Gamma f \left(\frac{x}{2^{n+4}} \right), \tag{15}$$

holds for all $x \in V$ and all integers $n \geq 0$ and

$$J'_n f(x) - J'_{n+1} f(x) = - \left(\frac{4}{4^{n+1}} - \frac{5}{16^{n+1}} + \frac{1}{64^{n+1}} \right) \frac{\Delta f(2^n x)}{2880} - \left(\frac{1}{128^{n+1}} - \frac{21}{32^{n+1}} + \frac{84}{8^{n+1}} - \frac{64}{2^{n+1}} \right) \frac{\Gamma f(2^n x)}{512 \cdot 2835}, \tag{16}$$

is fulfilled for all $x \in V$ and all integers $n \geq 0$.

Proof. By using (11) and the definitions of $J_n f$ and $J'_n f$, we can obtain the result after tedious calculations. Therefore, the proof will be omitted here. \square

Lemma 2. Assume that $f : V \rightarrow Y$ is a mapping with $f(0) = 0$ subject to the equation

$$Df(x, y) = 0, \tag{17}$$

for all $x, y \in V$. Then, we have

$$J_n f(x) = f(x) \text{ and } J'_n f(x) = f(x), \tag{18}$$

for all $x \in V$ and all positive integers n .

Proof. We have by the definitions of $\Delta f(x)$ and $\Gamma f(x)$ that

$$\Delta f(x) = 0 \text{ and } \Gamma f(x) = 0 \text{ for all } x \in V. \tag{19}$$

So, we figure out

$$f(x) - J_n f(x) = \sum_{i=0}^{n-1} (J_i f(x) - J_{i+1} f(x)) = \sum_{i=0}^{n-1} \left[\left(\frac{4^i}{45} - \frac{20 \cdot 16^i}{45} + \frac{64 \cdot 64^i}{45} \right) \Delta f \left(\frac{x}{2^{i+3}} \right) - \frac{2^i - 84 \cdot 8^i + 1344 \cdot 32^i - 4096 \cdot 128^i}{2835} \Gamma f \left(\frac{x}{2^{i+4}} \right) \right] = 0, \tag{20}$$

which implies $J_n f(x) = f(x)$ for all $x \in V$ and all positive integers n . Similarly, we get the equality $J'_n f(x) = f(x)$ for all $x \in V$ and all positive integers n . \square

We are now in a position to prove the following theorem.

Theorem 3. Suppose that a function $\varphi : V^2 \rightarrow [0, \infty)$ satisfies the condition

$$\sum_{n=0}^{\infty} 128^n \varphi \left(\frac{x}{2^n}, \frac{y}{2^n} \right) < \infty, \tag{21}$$

for all $x, y \in V$. Assume that $f : V \rightarrow Y$ is a mapping subject to the inequality

$$\|Df(x, y)\| \leq \varphi(x, y), \tag{22}$$

for all $x, y \in V$. Then, there exists a unique general septic mapping F with $F(0) = 0$ such that

$$\|f(x) - f(0) - F(x)\| \leq \sum_{n=0}^{\infty} \left[\left(\frac{4^n}{45} - \frac{20 \cdot 16^n}{45} + \frac{64 \cdot 64^n}{45} \right) \Phi \left(\frac{x}{2^{n+3}} \right) + \frac{-2^n + 84 \cdot 8^n - 1344 \cdot 32^n + 4096 \cdot 128^n}{2835} \Phi' \left(\frac{x}{2^{n+4}} \right) \right], \tag{23}$$

for all $x \in V$, where $\varphi_e : V^2 \rightarrow [0, \infty)$ and $\Phi, \Phi' : V \rightarrow [0, \infty)$ are functions defined by

$$\begin{aligned} \varphi_e(x, y) &:= \frac{\varphi(x, y) + \varphi(-x, -y)}{2}, \\ \Phi(x) &:= \varphi_e(4x, x) + 8\varphi_e(3x, x) + 36\varphi_e(2x, x) + 120\varphi_e(x, x) + 123\varphi_e(0, x), \\ \Phi'(x) &:= \varphi_e(8x, 2x) + 8\varphi_e(6x, 2x) + 36\varphi_e(4x, 2x) + 120\varphi_e(2x, 2x) + 160\varphi_e(4x, x) + 1280\varphi_e(3x, x) + 4032\varphi_e(2x, x) + 5376\varphi_e(x, x). \end{aligned} \tag{24}$$

Proof. Considering a mapping \tilde{f} defined by $\tilde{f}(x) = f(x) - f(0)$, we see that \tilde{f} satisfies the properties

$$D\tilde{f}(x, y) = Df(x, y) \text{ and } \tilde{f}(0) = 0. \tag{25}$$

Then, by (11) and the definitions of Γf and Δf , we obtain that

$$\begin{aligned} \|\Gamma \tilde{f}(x)\| &= \|Df_o(8x, 2x) + 8Df_o(6x, 2x) + 36Df_o(4x, 2x) + 120Df_o(2x, 2x) + 160Df_o(4x, x) + 1280Df_o(3x, x) + 4032Df_o(2x, x) + 5376Df_o(x, x)\| \leq \Phi'(x), \end{aligned} \tag{26}$$

$$\begin{aligned} \|\Delta \tilde{f}(x)\| &= \|Df_e(4x, x) + 8Df_e(3x, x) + 36Df_e(2x, x) + 120Df_e(x, x) + 123Df_e(0, x)\| \leq \Phi(x), \end{aligned} \tag{27}$$

hold for all $x \in V$. It follows from (15) and (26) that

$$\begin{aligned} \|J_n \tilde{f}(x) - J_{n+1} \tilde{f}(x)\| &= \left\| \left(\frac{4^n}{45} - \frac{20 \cdot 16^n}{45} + \frac{64 \cdot 64^n}{45} \right) \Delta \tilde{f} \left(\frac{x}{2^{n+3}} \right) \right. \\ &\quad \left. - \frac{2^n - 84 \cdot 8^n + 1344 \cdot 32^n - 4096 \cdot 128^n}{2835} \Gamma \tilde{f} \left(\frac{x}{2^{n+4}} \right) \right\| \\ &\leq \left(\frac{4^n}{45} - \frac{20 \cdot 16^n}{45} + \frac{64 \cdot 64^n}{45} \right) \Phi \left(\frac{x}{2^{i+3}} \right) \\ &\quad + \frac{-2^n + 84 \cdot 8^n - 1344 \cdot 32^n + 4096 \cdot 128^n}{2835} \Phi' \left(\frac{x}{2^{i+4}} \right), \end{aligned} \quad (28)$$

for all $x \in V$. This gives that

$$\begin{aligned} \|J_n \tilde{f}(x) - J_{n+m} \tilde{f}(x)\| &\leq \sum_{i=n}^{n+m-1} \left[\left(\frac{4^i}{45} - \frac{20 \cdot 16^i}{45} + \frac{64 \cdot 64^i}{45} \right) \Phi \left(\frac{x}{2^{i+3}} \right) \right. \\ &\quad \left. + \frac{-2^i + 84 \cdot 8^i - 1344 \cdot 32^i + 4096 \cdot 128^i}{2835} \Phi' \left(\frac{x}{2^{i+4}} \right) \right], \end{aligned} \quad (29)$$

for all $x \in V$ and all nonnegative integers n and m . By the definition of Φ and Φ' together with (21) and (29), the sequence $\{J_n \tilde{f}(x)\}$ is Cauchy in Y . And since Y is complete, the sequence $\{J_n \tilde{f}(x)\}$ converges. Therefore, we can define a mapping $F : V \rightarrow Y$ by

$$F(x) := \lim_{n \rightarrow \infty} J_n \tilde{f}(x) \text{ for all } x \in V. \quad (30)$$

Note that $J_0 \tilde{f}(x) = f(x) - f(0)$ and $F(0) = 0$ follow from $\tilde{f}(0) = 0$. Furthermore, by letting $n = 0$ and passing the limit as $m \rightarrow \infty$ in (29), we arrive at the inequality (23). On the other hand, from (21) and the definition of F , we yield the following inequality

$$\begin{aligned} \|DF(x, y)\| &= \lim_{n \rightarrow \infty} \|DJ_n f(x, y)\| \leq \lim_{n \rightarrow \infty} \left(\frac{64^{n+1}}{45} \varphi_e \left(\frac{x}{2^n}, \frac{y}{2^n} \right) \right. \\ &\quad + \frac{20 \cdot 64^{n+1}}{45} \varphi_e \left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}} \right) + \frac{64^{n+2}}{45} \varphi_e \left(\frac{x}{2^{n+2}}, \frac{y}{2^{n+2}} \right) \\ &\quad + \frac{32 \cdot 128^{n+1}}{2835} \varphi_e \left(\frac{x}{2^n}, \frac{y}{2^n} \right) + \frac{1344 \cdot 128^{n+1}}{2835} \varphi_e \left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}} \right) \\ &\quad \left. + \frac{42 \cdot 128^{n+2}}{2835} \varphi_e \left(\frac{x}{2^{n+2}}, \frac{y}{2^{n+2}} \right) + \frac{128^{n+3}}{2835} \varphi_e \left(\frac{x}{2^{n+3}}, \frac{y}{2^{n+3}} \right) \right) \\ &= 0, \end{aligned} \quad (31)$$

for all $x, y \in V$.

In order to prove the uniqueness of F , we suppose that $F' : V \rightarrow Y$ is another general septic mapping satisfying (23) and $F'(0) = 0$. However, it is also possible to show uniqueness by replacing condition (23) with weaker condition. So, we want to prove that there is a unique mapping satisfying the weaker condition

$$\|\tilde{f}(x) - F(x)\| \leq \sum_{i=0}^{\infty} 128^{i+1} \left(\Phi \left(\frac{x}{2^{i+3}} \right) + \Phi' \left(\frac{x}{2^{i+4}} \right) \right), \quad (32)$$

for all $x \in V$. According to Lemma 2, we get $F'(x) = J_n F'(x)$ for all positive integers n . Thus, by using the condition (32) and the definition of J_n , after tedious calculations, we have

$$\begin{aligned} \|J_n \tilde{f}(x) - F'(x)\| &= \|J_n \tilde{f}(x) - J_n F'(x)\| \\ &\leq 2 \sum_{i=0}^{\infty} 128^{i+n+2} \left(\Phi \left(\frac{x}{2^{i+n+3}} \right) + \Phi' \left(\frac{x}{2^{i+n+4}} \right) \right) \\ &\quad + 2 \sum_{i=0}^{\infty} 128^{i+n+2} \left(\Phi \left(\frac{x}{2^{i+n+4}} \right) + \Phi' \left(\frac{x}{2^{i+n+5}} \right) \right) \\ &\quad + 2 \sum_{i=0}^{\infty} 128^{i+n+3} \left(\Phi \left(\frac{x}{2^{i+n+5}} \right) + \Phi' \left(\frac{x}{2^{i+n+6}} \right) \right) \\ &\quad + \sum_{i=0}^{\infty} 128^{i+n+3} \left(\Phi \left(\frac{x}{2^{i+n+6}} \right) + \Phi' \left(\frac{x}{2^{i+n+7}} \right) \right) \\ &\leq \sum_{i=n+3}^{\infty} 128^i \left(\Phi \left(\frac{x}{2^i} \right) + \Phi' \left(\frac{x}{2^i} \right) \right), \end{aligned} \quad (33)$$

for all $x \in V$ and all positive integer n . Taking the limit as $n \rightarrow \infty$ in the last inequality, we then have

$$F'(x) = \lim_{n \rightarrow \infty} J_n \tilde{f}(x) \text{ for all } x \in V. \quad (34)$$

This implies that $F = F'$. \square

Theorem 4. Suppose that a function $\varphi : V^2 \rightarrow [0, \infty)$ satisfies the condition

$$\sum_{n=0}^{\infty} 2^{-n} \varphi(2^n x, 2^n y) < \infty, \quad (35)$$

for all $x, y \in V$. Assume that $f : V \rightarrow Y$ is a mapping subject to the inequality

$$\|Df(x, y)\| \leq \varphi(x, y), \quad (36)$$

for all $x, y \in V$. Then, there exists a unique general septic mapping F with $F(0) = 0$ such that

$$\begin{aligned} \|f(x) - f(0) - F(x)\| &\leq \sum_{n=0}^{\infty} \left[\left(\frac{4}{4^{n+1}} - \frac{5}{16^{n+1}} + \frac{1}{64^{n+1}} \right) \frac{\Phi(2^n x)}{2880} \right. \\ &\quad \left. + \left(\frac{64}{2^{n+1}} - \frac{84}{8^{n+1}} + \frac{21}{32^{n+1}} - \frac{1}{128^{n+1}} \right) \frac{\Phi'(2^n x)}{512 \cdot 2835} \right], \end{aligned} \quad (37)$$

for all $x \in V$, where φ_e, Φ and Φ' are the functions in Theorem 3.

Proof. We first define a mapping \tilde{f} by $\tilde{f}(x) = f(x) - f(0)$. Then, we find that

$$D\tilde{f}(x, y) = Df(x, y) \text{ and } \tilde{f}(0) = 0. \quad (38)$$

With help of the definitions of Γf and Δf , the relations (16), (26), and (36) guarantee that

$$\begin{aligned} \|J'_n \tilde{f}(x) - J'_{n+1} \tilde{f}(x)\| &\leq \left(\frac{4}{4^{n+1}} - \frac{5}{16^{n+1}} + \frac{1}{64^{n+1}}\right) \frac{\Phi(2^n x)}{2880} \\ &\quad + \left(\frac{64}{2^{n+1}} - \frac{84}{8^{n+1}} + \frac{21}{32^{n+1}} - \frac{1}{128^{n+1}}\right) \frac{\Phi'(2^n x)}{512 \cdot 2835}, \end{aligned} \tag{39}$$

for all $x \in V$, which leads to the inequality

$$\begin{aligned} \|J'_n \tilde{f}(x) - J'_{n+m} \tilde{f}(x)\| &\leq \sum_{i=n}^{n+m-1} \left[\left(\frac{4}{4^{i+1}} - \frac{5}{16^{i+1}} + \frac{1}{64^{i+1}}\right) \frac{\Phi(2^i x)}{2880} \right. \\ &\quad \left. + \left(\frac{64}{2^{i+1}} - \frac{84}{8^{i+1}} + \frac{21}{32^{i+1}} - \frac{1}{128^{i+1}}\right) \frac{\Phi'(2^i x)}{512 \cdot 2835} \right], \end{aligned} \tag{40}$$

$$\begin{aligned} \|DF(x, y)\| &= \lim_{n \rightarrow \infty} \|DJ'_n f(x, y)\| \\ &\leq \lim_{n \rightarrow \infty} \left(\frac{2\varphi_e(2^n x, 2^n y) + \varphi_e(2^{n+1} x, 2^{n+1} y) + \varphi_e(2^{n+2} x, 2^{n+2} y)}{4^n} + \frac{2\varphi_e(2^n x, 2^n y) + \varphi_e(2^{n+1} x, 2^{n+1} y) + \varphi_e(2^{n+2} x, 2^{n+2} y) + \varphi_e(2^{n+3} x, 2^{n+3} y)}{2^n} \right) \\ &= 0, \text{ for all } x, y \in V. \end{aligned} \tag{42}$$

To show the aforementioned uniqueness, let us assume that $F' : V \rightarrow Y$ is another general septic mapping with (37) and $F'(0) = 0$. Then, this proof is analogue to that of Theorem 3. That is, instead of the condition (37), it suffices to prove that there is a unique mapping satisfying the weaker condition

$$\|\tilde{f}(x) - F(x)\| \leq \sum_{i=0}^{\infty} \frac{\Phi(2^i x) + \Phi'(2^i x)}{2^i}, \tag{43}$$

for all $x \in V$. From Lemma 2, the equality $F'(x) = J'_n F'(x)$ holds for all positive integer n . So, we have by (43) that

$$\begin{aligned} \|J'_n \tilde{f}(x) - F'(x)\| &= \|J'_n \tilde{f}(x) - J'_n F'(x)\| \\ &\leq 4 \sum_{i=0}^{\infty} \frac{\Phi(2^{n+i} x) + \Phi'(2^{n+i} x)}{2^{n+i}} \\ &\quad + 2 \sum_{i=0}^{\infty} \frac{\Phi(2^{n+i+1} x) + \Phi'(2^{n+i+1} x)}{2^{n+i+1}} \\ &\quad + 2 \sum_{i=0}^{\infty} \frac{\Phi(2^{n+i+2} x) + \Phi'(2^{n+i+2} x)}{2^{n+i+2}} \\ &\quad + 2 \sum_{i=0}^{\infty} \frac{\Phi(2^{n+i+3} x) + \Phi'(2^{n+i+3} x)}{2^{n+i+3}} \\ &\leq 10 \sum_{i=n}^{\infty} \frac{\Phi(2^i x) + \Phi'(2^i x)}{2^i}, \end{aligned} \tag{44}$$

for all $x \in V$ and all nonnegative integers n and m . In view of (35) and (40), the sequence $\{J'_n \tilde{f}(x)\}$ is Cauchy in $x \in Y$. By completeness of Y , the sequence $\{J'_n \tilde{f}(x)\}$ converges. Thereby, we can define a mapping $F : V \rightarrow Y$ by

$$F(x) := \lim_{n \rightarrow \infty} J'_n \tilde{f}(x) \text{ for all } x \in V. \tag{41}$$

Since $\tilde{f}(0) = 0$, we have $F(0) = 0$. Note that $J'_0 \tilde{f}(x) = f(x) - f(0)$ for all $x \in V$. Letting $n = 0$ and sending the limit in (40) as $m \rightarrow \infty$, we obtain the desired result (37).

But, we intend to claim that $DF(x, y) = 0$. The expression (35) and the definition of F provide that

for all $x \in V$ and all positive integer n . Taking the limit as $n \rightarrow \infty$ in the previous inequality, we get the relation

$$F'(x) = \lim_{n \rightarrow \infty} J'_n \tilde{f}(x) \text{ for all } x \in V, \tag{45}$$

which means that $F = F'$. □

3. Discussion

In this paper, we investigated the stability of general septic functional equation by using the method of Găvruta. In Theorem 3, we proved that if the function $f : V \rightarrow Y$ satisfies the inequality

$$\|Df(x, y)\| = \left\| \sum_{i=0}^8 {}_8C_i (-1)^{8-i} f(x + (i-4)y) \right\| \leq \varphi(x, y), \tag{46}$$

where a function $\varphi : V^2 \rightarrow [0, \infty)$ satisfies the condition

$$\sum_{n=0}^{\infty} 128^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) < \infty, \tag{47}$$

for all $x, y \in V$, then there exists a unique general septic mapping F near the function f .

And in Theorem 4, we proved similar result when the function $\varphi : V^2 \rightarrow [0, \infty)$ satisfies the condition

$$\sum_{n=0}^{\infty} 2^{-n} \varphi(2^n x, 2^n y) < \infty, \text{ for all } x, y \in V. \quad (48)$$

Lee [27] proved the stability of the general sextic functional equation for the mapping f such that

$$Df(x, y) = \sum_{i=0}^7 {}_7C_i (-1)^{7-i} f(x + iy) \leq \theta (\|x\|^p + \|y\|^p). \quad (49)$$

For the future work, we want to see if we can get similar results for the general septic functional equation. And by using our results, we want to know what we can say for the stability of the general octic functional equation.

Also, by fixed point theorem, Roh et al. [24] showed the stability of the general sextic functional equation for the mapping f such that

$$Df(x, y) = \sum_{i=0}^7 {}_7C_i (-1)^{7-i} f(x + iy) \leq \varphi(x, y), \quad (50)$$

where $\varphi : V^2 \rightarrow [0, \infty)$ is a function for which there exists a constant $0 < L < 1$ such that

$$\varphi(2x, 2y) \leq (4\sqrt{21} \cos \theta - 14) L \varphi(x, y), \quad (51)$$

for all $x, y \in V$, and θ is a real constant such that $0 < \theta < \pi/4$ and $\cos(3\theta) = -17/21\sqrt{21}$. In fact, for the stability of the septic functional equation, we first tried this method, but we failed to get the good result. The calculation became complicated because it resulted in a quadratic equation problem. So, we leave this problem as an open problem.

Data Availability

For the manuscript, there is no data we need.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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