Research Article
Infra Soft Semiopen Sets and Infra Soft Semicontinuity

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To contribute to the area of infra soft topology, we introduce one of the generalizations of infra soft open sets called infra soft semiopen sets. We establish some characterizations of them and study their main properties. We determine under what condition this class is closed under finite intersection and show that this class is preserved under infra soft continuous mappings and finite product of soft spaces. Then, we present the concepts of infra semi-interior, infra semiclosure, infra semilimit, and infra semiboundary soft points of a soft set and elucidate the relationships between them. Finally, we exploit infra soft semiopen and infra soft semiclosed sets to define new types of soft mappings. We characterize each one of these soft mappings and explore main features.

1. Introduction

In 1999, Molodtsov [1] presented a novel mathematical tool to address vagueness, namely, soft sets. He discussed its relationship with fuzzy sets and showed some applications in different fields. Then, many scholars and researchers have studied some applications of soft set in different scopes such as decision-making problems [2], computer science [3], and medical science [4].

In 2003, Maji et al. [5] began studying the main concepts and notions of soft set theory. They explored the intersection and union operators, difference of two soft sets, and a complement of a soft set. However, some shortcomings appeared in their definitions, which led to reformulate most of these definitions and present new kinds of them. Ali et al. [6] originated new operators and operations between to preserve some properties and results of the (crisp) set theory in the soft set theory. Attempts were still in this path to produce new operators and relations like those introduced in [7].

In 2011, Çağman et al. [8] and Shabir and Naz [9] made use of soft sets to define soft topological spaces. Whereas, Çağman et al.'s definition given over an absolute soft set and different sets of parameters, Shabir and Naz' definition given over a fixed set of universe and a fixed set of parameters. This paper follows the definition of Shabir and Naz. Later on, many studies which investigated the topological concepts in soft topologies have been done such as soft compactness [10], soft connectedness [11], soft separation axioms, soft basis [12], Caliber and chain conditions [13], soft bioperators, and generalized soft open sets [14]. Also, uniformity and Menger structures were introduced in the context of soft sets in [15, 16], respectively.

Soft topology was generalized to some structures; one of them is an infra soft topology [17]. The motivations of continuously investigating infra soft topological structure are that many topological properties are kept in the frame of infra soft topologies as well as the easy construction of examples that illustrate the relationships among the topological concepts. This matter was investigated for the concepts of infra soft compactness and infra soft connectedness in [18, 19].

Generalizations of (soft) open sets are a major topic in (soft) topology. One of the important generalizations is a soft semiopen set [20] which was studied in classical

topology by Levine [21]. In this article, we aim to explore the properties of this type of generalizations in the frame of infra soft topology. We elucidate the soundness of several properties of semiopen sets via infra soft topological spaces. This means that the infra soft topological spaces are flexible area to discuss the topological ideas and explore the relationships between them.

The arrangement of this article is as follows: Section 2 is allocated to mention some definitions and results relating to soft set theory and infra soft topology. In Section 3, we define a class of infra soft semiopen sets and establish some of characterizes. The concepts of infra semi-interior, infra semiclosure, infra semilimit, and infra semiboundary soft points of a soft set are introduced and probed in Section 4. In Section 5, we study the concepts of infra soft semicontinuous, infra soft semiopen, infra soft semiclosed, and infra soft semihomeomorphism mappings. Also, we formulate and study the concept of semifixed soft points in the frame of infra soft topologies. Finally, some conclusions and the possible upcoming works are given in Section 6.

2. Preliminaries

This section mentions the concepts and findings that we need to understand this manuscript.

2.1. Soft Set Theory

Definition 1 (see [1]). Consider Θ as a set of parameters, $\mathscr{T}$ a universal set, and $2^{\mathscr{T}}$ the power set of $\mathscr{T}$. An ordered pair (Ω, Θ) is called a soft set over $\mathscr{T}$ provided that $\Omega : \Theta \longrightarrow 2^{\mathscr{T}}$ is a crisp mapping. A soft set is expressed as $(\Omega, \Theta) = \{ (\theta, \Omega(\theta)) : \theta \in \Theta \text{ and } \Omega(\theta) \in 2^{\mathscr{T}} \}$. We call $\Omega(\theta)$ a θ-approximate of (Ω, Θ).

The class of all soft sets over $\mathscr{T}$ under a set of parameters Θ is symbolized by $C(\mathscr{T}_\Theta)$.

Definition 2 (see [6]). The complement of a soft set (Ω, Θ), denoted by (Ω^c, Θ), provided that a mapping $\Omega^c : \Theta \longrightarrow 2^{\mathscr{T}}$ is given by $\Omega^c(\theta) = \mathscr{T} \setminus \Omega(\theta)$ for each $\theta \in \Theta$.

Definition 3 (see [5]). We call (Ω, Θ) an absolute (resp., a null) soft set over $\mathscr{T}$ if the image of each parameter of Θ under a mapping $\Omega : \Theta \longrightarrow 2^{\mathscr{T}}$ is the universal set $\mathscr{T}$ (resp., empty set).

The absolute and null soft sets are symbolized by $\tilde{\mathscr{T}}$ and Φ, respectively.

Definition 4 (see [22]). We call (Ω, Θ) a stable (resp., finite, countable) soft set if every θ-approximate of (Ω, Θ) is equal (resp., finite, countable). Otherwise, it is called unstable (resp., infinite, uncountable).

Definition 5 (see [23]). Consider (Ω, Θ) as a soft set over $\mathscr{T}$ such that $\Omega(\theta) = t \in \mathscr{T}$ and $\Omega(\theta') = \varnothing$ for each $\theta' \neq \theta$. Then, we call (Ω, Θ) a soft point over $\mathscr{T}$. It will be symbolized by δ_θ^t.

Definition 6 (see [6]). The intersection of soft sets (Ω, Θ) and (Ψ, Δ) which are defined over $\mathscr{T}$, symbolized by $(\Omega, \Theta) \cap^\sim (\Psi, \Delta)$, is a soft set (Y, Σ), where $\Sigma = \Theta \cap \Delta \neq \varnothing$, and a mapping $Y : \Sigma \longrightarrow 2^{\mathscr{T}}$ is given by $Y(\theta) = \Omega(\theta) \cap \Psi(\theta)$ for each $\theta \in \Sigma$.

Definition 7 (see [5]). The union of soft sets (Ω, Θ) and (Ψ, Δ) which are defined over $\mathscr{T}$, symbolized by $(\Omega, \Theta) \cup^\sim (\Psi, \Delta)$, is a soft set (Y, Σ), where $\Sigma = \Theta \cup \Delta$ and a mapping $Y : \Sigma \longrightarrow 2^{\mathscr{T}}$ is given as follows:

$$Y(\theta) = \begin{cases} \Omega(\theta) & \theta \in \Theta \setminus \Delta, \\ \Psi(\theta) & \theta \in \Delta \setminus \Theta, \\ \Omega(\theta) \cup \Psi(\theta) & \theta \in \Theta \cap \Delta. \end{cases} \tag{1}$$

Definition 8 (see [24]). A soft set (Ω, Θ) is a subset of a soft set (Ψ, Δ), symbolized by $(\Omega, \Theta) \subseteq^\sim (\Psi, \Delta)$, if $\Theta \subseteq \Delta$ and $\Omega(\theta) \subseteq \Psi(\theta)$ for all $\theta \in \Theta$. The soft sets (Ω, Θ) and (Ψ, Δ) are called soft equal if $(\Omega, \Theta) \subseteq^\sim (\Psi, \Delta)$ and $(\Psi, \Delta) \subseteq^\sim (\Omega, \Theta)$.

Definition 9 (see [10]). The Cartesian product of (Ω, Θ) and (Ψ, Δ), symbolized by $(\Omega \times \Psi, \Theta \times \Delta)$, is defined as $(\Omega \times \Psi)(\theta, \theta') = \Omega(\theta) \times \Psi(\theta')$ for each $(\theta, \theta') \in \Theta \times \Delta$.

The definition of soft mappings given in [25] was reformulated in a way that reduces calculation burden and gives a justification (logical explanation) for some soft concepts such as why we determine that E_τ is injective, or surjective according to its two crisp maps E and τ.

Definition 10 (see [26]). Let $E : \mathscr{T} \longrightarrow \mathscr{S}$ and $\tau : \Theta \longrightarrow \Sigma$ be two crisp mappings. A soft mapping E_τ of $C(\mathscr{T}_\Theta)$ into $C(\mathscr{S}_\Sigma)$ is a relation such that each soft point in $C(\mathscr{T}_\Theta)$ is related to one and only one soft point in $C(\mathscr{S}_\Sigma)$ such that

$$E_\tau(\delta_\theta^t) = \delta_{\tau(\theta)}^{E(t)} \text{ for each } \delta_\theta^t \in C(\mathscr{T}_\Theta). \tag{2}$$

In addition, $E_\tau^{-1}(\delta_\sigma^s) = \cup^\sim_{\substack{\omega \in \tau^{-1}(\sigma) \\ v \in E^{-1}(s)}} \delta_\omega^v$ for each $\delta_\sigma^s \in C(\mathscr{S}_\Sigma)$.

Definition 11 (see [25]). A soft mapping $f_\tau : C(\mathscr{T}_\Theta) \longrightarrow C(\mathscr{S}_\Delta)$ is said to be injective (resp., surjective, bijective) if both f and τ are injective (resp., surjective, bijective).

2.2. Infra Soft Topological Spaces

Definition 12 (see [17]). A family ξ of soft sets over $\mathscr{T}$ with Θ as a parameter set is said to be an infra soft topology on $\mathscr{T}$ if it is closed under finite intersection and Φ is a member of ξ.

The triple $(\mathscr{T}, \xi, \Theta)$ is called an infra soft topological space (briefly, ISTS). We call a member of ξ an infra soft open set and call its complement an infra soft closed set. We call $(\mathscr{T}, \xi, \Theta)$ stable if all its infra soft open sets are stable.

Definition 13 (see [17]). Let (Ω, Θ) be a subset of $(\mathscr{T}, \xi, \Theta)$.

(i) The intersection of all infra soft closed subsets of $(\mathscr{T}, \xi, \Theta)$ which contains a soft set (Ω, Θ) is called the infra soft closure of (Ω, Θ). It is denoted by $Cl(\Omega, \Theta)$

(ii) The union of all infra soft open subsets of $(\mathscr{T}, \xi, \Theta)$ which are contained in a soft set (Ω, Θ) is called the infra soft interior of (Ω, Θ). It is denoted by $Int(\Omega, \Theta)$

It was shown in [17] that $Cl(\Omega, \Theta)$ and $Int(\Omega, \Theta)$ need not be infra soft closed and infra soft open, respectively. Through this paper, (Ω, Θ) is called ξ-infra soft open (resp., ξ-infra soft closed) if $Int(\Omega, \Theta) = (\Omega, \Theta)$ $(resp., Cl(\Omega, \Theta)) = (\Omega, \Theta)$.

Proposition 14 (see [17]). *Let* (Ω, Θ) *and* (Ψ, Θ) *subsets of an ISTS* $(\mathscr{T}, \xi, \Theta)$. *Then*

(i) $Cl[(\Omega, \Theta) \cup \tilde{} (\Psi, \Theta)] = Cl(\Omega, \Theta) \cup \tilde{} Cl(\Psi, \Theta)$

(ii) $Int[(\Omega, \Theta) \cap \tilde{} (\Psi, \Theta)] = Int(\Omega, \Theta) \cap \tilde{} Int(\Psi, \Theta)$

Proposition 15 (see [17]). *Let* (Ω, Θ) *be an infra soft open set. Then*

$$(\Omega, \Theta) \cap \tilde{} Cl(\Psi, \Theta) \subseteq \tilde{} Cl[(\Omega, \Theta) \cup \tilde{} (\Psi, \Theta)] \quad \text{for any subset} \quad (\Psi, \Theta) \text{of} (\mathscr{T}, \xi, \Theta).$$
(3)

Proposition 16 (see [17]). *Let* (Ω, Θ) *be an infra soft closed set. Then*

$$Int[(\Omega, \Theta) \cup \tilde{} (\Psi, \Theta)] \subseteq \tilde{} (\Omega, \Theta) \cap \tilde{} Int(\Psi, \Theta) \quad \text{for any subset} \quad (\Omega, \Theta) \text{of} (\mathscr{T}, \xi, \Theta).$$
(4)

Definition 17. A soft mapping $f_\tau : (\mathscr{T}, \xi, \Theta) \longrightarrow (\mathscr{S}, \pi, \Delta)$ is said to be an infra soft homeomorphism if it is bijective, infra soft continuous (i.e., the image of every infra soft open set is infra soft open), and infra soft open (i.e., the image of every infra soft open set is infra soft open).

A property is called an infra soft topological property (briefly, IST property) if it is preserved by any infra soft homeomorphism.

Definition 18 (see [17]). Let $E_\tau : (\mathscr{T}, \xi, \Theta) \longrightarrow (\mathscr{S}, \pi, \Delta)$ be a soft mapping and $\mathscr{M} \neq \varnothing$ be a subset of $\mathscr{T}$. A soft mapping $E_{\tau_{|\mathscr{M}}} : (\mathscr{M}, \xi_{\mathscr{M}}, \Theta) \longrightarrow (\mathscr{S}, \pi, \Delta)$ which given by $E_{\tau_{|\mathscr{M}}}(\delta_\theta^m) = E_\tau(\delta_\theta^m)$ for every $\delta_\theta^m \in \tilde{\mathscr{M}}$ is called a restriction soft mapping of E_τ on $\mathscr{M}$.

Proposition 19. *Let* $\{(\mathscr{T}_k, \xi_k, \Theta_k): k \in K\}$ *be a family of ISTSs. Then,* $\xi = \{\prod_{k \in K}(\theta_k, \Theta_k): (\theta_k, \Theta_k) \in \tau_k\}$ *is an infra soft topology on* $\mathscr{T} = \prod_{k \in K} \mathscr{T}_k$ *under a set of parameters* $\mathscr{B} = \prod_{k \in K} \Theta_k$.

We call ξ given in proposition above, a product of infra soft topologies, and $(\mathscr{T}, \xi, \mathscr{B})$ a product of infra soft spaces.

3. Infra Soft Semiopen Sets and Basic Properties

In this section, we introduce the concept of infra soft semiopen sets which represents a class of generalizations of infra soft open sets. We give some characterizations of infra soft semiopen and infra soft semiclosed sets and establish main properties. Also, we prove that this class is closed under arbitrary unions and determine under what condition this class is closed under finite intersection. Finally, we show that an infra soft semiopen set and its complement are preserved under infra soft continuous mappings and finite product of soft spaces.

Definition 20. A subset (Ω, Θ) of an ISTS $(\mathscr{T}, \xi, \Theta)$ is said to be infra soft semiopen if $(\Omega, \Theta) \subseteq \tilde{} Cl(Int(\Omega, \Theta))$. Its complement is said to be an infra soft semiclosed set.

The following two propositions give some descriptions for infra soft semiopen and infra soft semiclosed sets.

Proposition 21. *Let* (Ω, Θ) *be a subset of an ISTS* $(\mathscr{T}, \xi, \Theta)$. *Then, we have the following equivalent properties:*

(i) (Ω, Θ) *is an infra soft semiopen set*

(ii) $Cl(Int(\Omega, \Theta)) = Cl(\Omega, \Theta)$

(iii) *There exists an* ξ-*infra soft open set* (Ψ, Θ) *such that* $(\Psi, \Theta) \subseteq \tilde{} (\Omega, \Theta) \subseteq \tilde{} Cl(\Psi, \Theta)$

Proof. $(i) \Leftrightarrow (ii)$: Let (Ω, Θ) be an infra soft semiopen set. Then, $(\Omega, \Theta) \subseteq \tilde{} Cl(Int(\Omega, \Theta))$. Therefore, $Cl(\Omega, \Theta) \subseteq \tilde{} Cl(Int(\Omega, \Theta))$. It is well known that $Cl(Int(\Omega, \Theta)) \subseteq \tilde{} Cl(\Omega, \Theta)$. Thus, $Cl(Int(\Omega, \Theta)) = Cl(\Omega, \Theta)$. The direction $(ii) \Rightarrow (i)$ is obvious.

$(ii) \Leftrightarrow (iii)$: Since (Ω, Θ) is an infra soft semiopen set, $(\Omega, \Theta) \subseteq \tilde{} Cl(Int(\Omega, \Theta))$. Taking $(\Psi, \Theta) = Int(\Omega, \Theta)$, we obtain $(\Psi, \Theta) \subseteq \tilde{} (\Omega, \Theta) \subseteq \tilde{} Cl(\Psi, \Theta)$. Since $Int(\Psi, \Theta) = Int(Int(\Omega, \Theta)) = Int(\Omega, \Theta) = (\Psi, \Theta)$, (Ψ, Θ) is an ξ-infra soft open set. Conversely, let (Ψ, Θ) is an ξ-infra soft open set such that $(\Psi, \Theta) \subseteq \tilde{} (\Omega, \Theta) \subseteq \tilde{} Cl(\Psi, \Theta)$. Then, $Cl(\Psi, \Theta) \subseteq \tilde{} Cl(Int(\Omega, \Theta))$. By assumption, $(\Omega, \Theta) \subseteq \tilde{} Cl(Int(\Omega, \Theta))$ which means that (Ω, Θ) is an infra soft semiopen set.□ □

Proposition 22. *Let* (Ω, Θ) *be a subset of an ISTS* $(\mathscr{T}, \xi, \Theta)$. *Then, we have the following equivalent properties:*

(i) (Ω, Θ) *is an infra soft semiclosed set*

(ii) $Int(Cl(\Omega, \Theta)) \subseteq \tilde{} (\Omega, \Theta)$

(iii) $Int(Cl(\Omega, \Theta)) = Int(\Omega, \Theta)$

(iv) *There exists an* ξ-*infra soft closed set* (Ψ, Θ) *such that* $Int(\Psi, \Theta) \subseteq \tilde{} (\Omega, \Theta) \subseteq \tilde{} (\Psi, \Theta)$

Proof. One can prove it following similar arguments given in the proof of Proposition 21.□ □

Proposition 23. *The class of infra soft semiopen sets is closed under arbitrary unions.*

Proof. Consider $\{(\Omega_j, \Theta): j \in J\}$ as a family of infra soft semiopen sets. If the index J is an empty set, then $\tilde{\cup}_{j \in \varnothing}(\Omega_j, \Theta) = \Phi$ which is an infra soft semiopen set. Suppose that $J \neq \varnothing$. Then $(\Omega_j, \Theta) \tilde{\subseteq} Cl(Int(\Omega_j, \Theta))$ for each $j \in J$. Consequentially, $\tilde{\cup}_{j \in J}(\Omega_j, \Theta) \tilde{\subseteq} \tilde{\cup}_{j \in J} Cl(Int(\Omega_j, \Theta)) \tilde{\subseteq} Cl(Int(\tilde{\cup}_{j \in J}(\Omega_j, \Theta)))$. Hence, $\tilde{\cup}_{j \in J}(\Omega_j, \Theta)$ is infra soft semiopen.□ □

Corollary 24. *The class of infra soft semiclosed sets is closed under arbitrary intersections.*

Corollary 25. *The class of infra soft semiopen subsets of an ISTS $(\mathcal{T}, \xi, \Theta)$ forms a supra soft topology over $\mathcal{T}$.*

To illustrate that the class of infra soft semiopen sets does not form an infra soft topology, we present the following example.

Example 1. Let $\mathcal{T} = \{t_1, t_2, t_3, t_4\}$ and $\Theta = \{\theta_1, \theta_2\}$. Then, $\xi = \{\Phi, \tilde{\mathcal{T}}, (\Omega_1, \Theta), (\Omega_2, \Theta)\}$ is an infra soft topology on $\mathcal{T}$ with Θ as a set of parameters, where

$$(\Omega_1, \Theta) = \{(\theta_1, \{t_1\}), (\theta_2, \{t_1\})\}, \tag{5}$$

and

$$(\Omega_2, \Theta) = \{(\theta_1, \{t_2\}), (\theta_2, \{t_2\})\}. \tag{6}$$

Let $(\Omega_3, \Theta) = \{(\theta_1, \{t_1, t_3\}), (\theta_2, \{t_1, t_3\})\}$ and $(\Omega_4, \Theta) = \{(\theta_1, \{t_2, t_3\}), (\theta_2, \{t_2, t_3\})\}$. Then, (Ω_3, Θ) and (Ω_4, Θ) are infra soft semiopen sets because $Cl(Int(\Omega_3, \Theta)) = \{(\theta_1, \{t_1, t_3, t_4\}), (\theta_2, \{t_1, t_3, t_4\})\}$ and $Cl(Int(\Omega_4, \Theta)) = \{(\theta_1, \{t_2, t_3, t_4\}), (\theta_2, \{t_2, t_3, t_4\})\}$. But $(\Omega_3, \Theta) \tilde{\cap} (\Omega_4, \Theta)$ is not infra soft semiopen because $Cl(Int[(\Omega_3, \Theta) \tilde{\cap} (\Omega_4, \Theta)]) = \Phi$.

Proposition 26. *The intersection of infra soft open and infra soft semiopen sets is an infra soft semiopen set.*

Proof. Let (Ω_1, Θ) be an infra soft open set and (Ω_2, Θ) be an infra soft semiopen set. Then, $(\Omega_1, \Theta) \tilde{\cap} (\Omega_2, \Theta) \tilde{\subseteq} (\Omega_1, \Theta) \tilde{\cap} Cl(Int(\Omega_2, \Theta))$; by Proposition 15, we obtain $(\Omega_1, \Theta) \tilde{\cap} (\Omega_2, \Theta) \tilde{\subseteq} Cl[(\Omega_1, \Theta) \tilde{\cap} Int(\Omega_2, \Theta)] = Cl[Int(\Omega_1, \Theta) \tilde{\cap} Int(\Omega_2, \Theta)] = Cl(Int[(\Omega_1, \Theta) \tilde{\cap} (\Omega_2, \Theta)])$. Hence, we obtain the desired result.□ □

Corollary 27. *The union of infra soft closed and infra soft semiclosed sets is an infra soft semiclosed set.*

Definition 28. An ISTS $(\mathcal{T}, \xi, \Theta)$ is said to be infra soft hyperconnected if the intersection of any two nonnull ξ-infra soft open sets is nonnull. Otherwise, $(\mathcal{T}, \xi, \Theta)$ is said to be infra soft dishyperconnected.

Proposition 29. *The intersection of two infra soft semiopen subsets of an infra soft hyperconnected space is an infra soft semiopen set.*

Proof. Let (Ω_1, Θ) and (Ω_2, Θ) be infra soft semiopen sets. If one of them is the null soft set, then, we obtain the desired result. Suppose that (Ω_1, Θ) and (Ω_2, Θ) are nonnull soft sets. According to Proposition 21, there are two ξ-infra soft open sets $(\Psi_1, \Theta) \neq \Phi$ and $(\Psi_2, \Theta) \neq \Phi$ such that $(\Psi_1, \Theta) \tilde{\subseteq} (\Omega_1, \Theta) \tilde{\subseteq} Cl(\Psi_1, \Theta)$ and $(\Psi_2, \Theta) \tilde{\subseteq} (\Omega_2, \Theta) \tilde{\subseteq} Cl(\Psi_2, \Theta)$. By hypothesis of infra soft hyperconnectedness, $(\Psi_1, \Theta) \tilde{\cap} (\Psi_2, \Theta)$ is a nonnull ξ-infra soft open set. Now, $(\Psi_1, \Theta) \tilde{\cap} (\Psi_2, \Theta) \tilde{\subseteq} (\Omega_1, \Theta) \tilde{\cap} (\Omega_2, \Theta) \tilde{\subseteq} Cl[(\Psi_1, \Theta) \tilde{\cap} (\Psi_2, \Theta)]$. Hence, $(\Omega_1, \Theta) \tilde{\cap} (\Omega_2, \Theta)$ is an infra soft semiopen set.□ □

Lemma 30. *Let $E_\tau: (\mathcal{T}_1, \xi_1, \Theta_1) \longrightarrow (\mathcal{T}_2, \xi_2, \Theta_2)$ be an infra soft homeomorphism map. Then, for any subset (Ω, Θ_1), we have the next two results.*

(i) $E_\tau(Int(\Omega, \Theta_1)) = Int(E_\tau(\Omega, \Theta_1))$

(ii) $E_\tau(Cl(\Omega, \Theta_1)) = Cl(E_\tau(\Omega, \Theta_1))$

Proof. To prove (i), let $\delta_{\theta'}^s \in E_\tau(Int(\Omega, \Theta_1))$. Then, there is $\delta_\theta^t \in Int(\Omega, \Theta_1)$ such that $E_\tau(\delta_\theta^t) = \delta_{\theta'}^s$. This means there exists an infra soft open set (Ψ, Θ_1) such that $\delta_\theta^t \in (\Psi, \Theta_1) \tilde{\subseteq} (\Omega, \Theta_1)$. Therefore, $\delta_{\theta'}^s = E_\tau(\delta_\theta^t) \in E_\tau(\Psi, \Theta_1) \tilde{\subseteq} E_\tau(\Omega, \Theta_1)$. This implies that $\delta_{\theta'}^s \in Int(E_\tau(\Omega, \Theta_1))$. Thus, $E_\tau(Int(\Omega, \Theta_1)) \tilde{\subseteq} Int(E_\tau(\Omega, \Theta_1))$. Conversely, let $\delta_{\theta'}^s \in Int(E_\tau(\Omega, \Theta_1))$. Then, there exists an infra soft open set (Ψ, Θ_2) such that $\delta_{\theta'}^s \in (\Psi, \Theta_2) \tilde{\subseteq} E_\tau(\Omega, \Theta_1)$. Therefore, $E_\tau^{-1}(\delta_{\theta'}^s) \in E_\tau^{-1}(\Psi, \Theta_2) \tilde{\subseteq} (\Omega, \Theta_1)$. Automatically, we obtain $E_\tau^{-1}(\delta_{\theta'}^s) \in Int(\Omega, \Theta_1)$. So that, $\delta_{\theta'}^s \in E_\tau(Int(\Omega, \Theta_1))$. Thus, $Int(E_\tau(\Omega, \Theta_1)) \tilde{\subseteq} E_\tau(Int(\Omega, \Theta_1))$. Hence, the proof is complete.

Following similar arguments, one can prove (ii).□ □

Proposition 31. *The infra soft homeomorphism image of an infra soft semiopen set is an infra soft semiopen set.*

Proof. Consider $E_\tau: (\mathcal{T}_1, \xi_1, \Theta_1) \longrightarrow (\mathcal{T}_2, \xi_2, \Theta_2)$ as an infra soft continuous mapping and let (Ω, Θ_1) be an infra soft semiopen subset of $(\mathcal{T}_1, \xi_1, \Theta_1)$. Then, $E_\tau(\Omega, \Theta_1) \tilde{\subseteq} E_\tau(Cl(Int(\Omega, \Theta_1)))$. It follows from the above lemma that $E_\tau(\Omega, \Theta_1) \tilde{\subseteq} Cl(Int(E_\tau(\Omega, \Theta_1)))$. Hence, $E_\tau(\Omega, \Theta_1)$ is an infra soft semiopen subset of $(\mathcal{T}_2, \xi_2, \Theta_2)$, as required.□ □

Lemma 32. *Consider (Ω_1, Θ_1) and (Ω_2, Θ_2) as subsets of $(\mathcal{T}_1, \xi_1, \Theta_1)$ and $(\mathcal{T}_2, \xi_2, \Theta_2)$, respectively. Then*

(i) $Cl[(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2)] = Cl(\Omega_1, \Theta_1) \times Cl(\Omega_2, \Theta_2)$

(ii) $Int[(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2)] = Int(\Omega_1, \Theta_1) \times Int(\Omega_2, \Theta_2)$

Proof. (i): Let $\delta_{(\theta,\vartheta)}^{(t,s)} \notin Cl[(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2)]$. Then, there is an infra soft open subset $(\Psi_1, \Theta_1) \times (\Psi_2, \Theta_2)$ of $\widetilde{\mathcal{T}}_1 \times \widetilde{\mathcal{T}}_2$ containing $\delta_{(\theta,\vartheta)}^{(t,s)}$ such that $[(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2)] \cap \tilde{} [(\Psi_1, \Theta_1) \times (\Psi_2, \Theta_2)] = \Phi_{\Theta_1 \times \Theta_2}$. This implies that $(\Omega_1, \Theta_1) \cap \tilde{}(\Psi_1, \Theta_1) = \Phi_{\Theta_1}$ or $(\Omega_2, \Theta_2) \cap \tilde{}(\Psi_2, \Theta_2) = \Phi_{\Theta_2}$. Therefore, $\delta_\theta^t \notin Cl(\Omega_1, \Theta_1)$ or $\delta_\vartheta^s \notin Cl(\Omega_2, \Theta_2)$. Thus, $\delta_{(\theta,\vartheta)}^{(t,s)} \notin [Cl(\Omega_1, \Theta_1) \times Cl(\Omega_2, \Theta_2)]$. Hence, $Cl(\Omega_1, \Theta_1) \times Cl(\Omega_2, \Theta_2) \subseteq \tilde{} Cl[(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2)]$. Conversely, let $\delta_{(\theta,\vartheta)}^{(t,s)} \notin Cl(\Omega_1, \Theta_1) \times Cl(\Omega_2, \Theta_2)$. Then, $\delta_\theta^t \notin Cl(\Omega_1, \Theta_1)$ or $\delta_\vartheta^s \notin Cl(\Omega_2, \Theta_2)$. Suppose, without loss of generality, that $\delta_\theta^t \notin Cl(\Omega_1, \Theta_1)$. Then, there is an infra soft open subset (Ψ_1, Θ_1) of $(\mathcal{T}_1, \xi_1, \Theta_1)$ containing δ_θ^t such that $(\Omega_1, \Theta_1) \cap \tilde{}(\Psi_1, \Theta_1) = \Phi_{\Theta_1}$. Obviously, $(\Psi_1, \Theta_1) \times \widetilde{\mathcal{T}}_2$ is an infra soft open subset of $\widetilde{\mathcal{T}}_1 \times \widetilde{\mathcal{T}}_2$ containing $\delta_{(\theta,\vartheta)}^{(t,s)}$ such that $[(\Psi_1, \Theta_1) \times \mathcal{T}_2 \tilde{}] \cap \tilde{}[(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2)] = \Phi_{\Theta_1 \times \Theta_2}$. Therefore, $\delta_{(\theta,\vartheta)}^{(t,s)} \notin Cl[(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2)]$. Thus, $Cl[(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2)] \subseteq \tilde{} Cl(\Omega_1, \Theta_1) \times Cl(\Omega_2, \Theta_2)$. Hence, the proof is complete. Following similar arguments, one can prove (ii).□ ∎

Proposition 33. *The product of infra soft semiopen sets is an infra soft semiopen set.*

Proof. Let (Ω_1, Θ_1) and (Ω_2, Θ_2) be infra soft semiopen subsets of $(\mathcal{T}_1, \xi_1, \Theta_1)$ and $(\mathcal{T}_2, \xi_2, \Theta_2)$, respectively. Then, $(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2) \subseteq \tilde{} Cl(Int(\Omega_1, \Theta_1)) \times Cl(Int(\Omega_2, \Theta_2))$. According to the above lemma, we obtain $(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2) \subseteq \tilde{} Cl(Int[(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2)])$ which means that $(\Omega_1, \Theta_1) \times (\Omega_2, \Theta_2)$ is an infra soft semiopen subset of $\widetilde{\mathcal{T}}_1 \times \widetilde{\mathcal{T}}_2$.□ ∎

4. Infra Semi-Interior, Infra Semiclosure, Infra Semilimit, and Infra Semiboundary Soft Points of a Soft Set

In this section, we first present the infra soft semi-interior and infra soft semiclosure operators and scrutinize their essential properties. Then, we define infra soft semilimit and infra soft semiboundary soft points of a soft set. We discuss their main features and revel the relationships between them with the help of examples.

Definition 34. Let (Ω, Θ) be a subset of $(\mathcal{T}, \xi, \Theta)$. Then

(i) The infra soft semi-interior of (Ω, Θ), denoted by $sInt(\Omega, \Theta)$, is the union of all infra soft semiopen sets that are contained in (Ω, Θ)

(ii) The infra soft semiclosure of (Ω, Θ), denoted by $sCl(\Omega, \Theta)$, is the intersection of all infra soft semiclosed sets containing (Ω, Θ)

Proposition 35. *We have the following properties.*

(i) (Ω, Θ) *is an infra soft semiopen subset of* $(\mathcal{T}, \xi, \Theta)$ *iff* $sInt(\Omega, \Theta) = (\Omega, \Theta)$

(ii) (Ω, Θ) *is an infra soft semiclosed subset of* $(\mathcal{T}, \xi, \Theta)$ *iff* $sCl(\Omega, \Theta) = (\Omega, \Theta)$

Proof. It follows from Proposition 23 and Corollary 24.□ ∎

It should be noted that the infra soft open and infra soft closed sets do not satisfy the above two properties.

Proposition 36. *Let* (Ω, Θ) *be a subset of* $(\mathcal{T}, \xi, \Theta)$.

(i) $\delta_\theta^t \in sInt(\Omega, \Theta)$ *iff there is an infra soft semiopen set* (Ψ, Θ) *such that* $\delta_\theta^t \in (\Psi, \Theta) \subseteq \tilde{} (\Omega, \Theta)$

(ii) $\delta_\theta^t \in sCl(\Omega, \Theta)$ *iff the intersection of any infra soft semiopen set* (Ψ, Θ) *containing* δ_θ^t *and* (Ω, Θ) *is nonnull*

Proof. The proof of (i) is obvious, so we prove (ii).

Let $\delta_\theta^t \in sCl(\Omega, \Theta)$. Then, every infra soft semiclosed set contains (Ω, Θ) contains δ_θ^t as well. Suppose that there exists an infra soft semiopen set (Ψ, Θ) containing δ_θ^t such that $(\Omega, \Theta) \cap \tilde{}(\Psi, \Theta) = \Phi$. Therefore, $(\Omega, \Theta) \subseteq \tilde{}(\Psi^c, \Theta)$ which means that $\delta_\theta^t \notin sCl(\Omega, \Theta)$. This is a contradiction. Conversely, suppose that there exists an infra soft semiopen set (Ψ, Θ) containing δ_θ^t such that $(\Omega, \Theta) \cap \tilde{}(\Psi, \Theta) = \Phi$. Therefore, $sCl(\Omega, \Theta) \subseteq \tilde{}(\Psi^c, \Theta)$ which means that $\delta_\theta^t \notin sCl(\Omega, \Theta)$. Hence, we obtain the desired result.□ ∎

Proposition 37. *Let* (Ω, Θ) *be a subset of* $(\mathcal{T}, \xi, \Theta)$. *Then,*

(i) $(sInt(\Omega, \Theta))^c = sCl(\Omega^c, \Theta)$

(ii) $(sCl(\Omega, \Theta))^c = sInt(\Omega^c, \Theta)$

Proof. (i): $(sInt(\Omega, \Theta))^c = \{\cup^{\tilde{}}_{j \in J}(\Psi_j, \Theta): (\Psi_j, \Theta)$ is an infra soft semiopen set contained in $(\Omega, \Theta)\}^c = \cap^{\tilde{}}_{j \in J}\{(\Psi_j^c, \Theta): (\Psi_j^c, \Theta)$ is an infra soft semi-closed set containing $(\Omega^c, \Theta)\} = sCl(\Omega^c, \Theta)$.

The proof of (ii) is similar to (i).□ ∎

Proposition 38. *Let* (Ψ, Θ) *be an infra soft open set and* (Λ, Θ) *be an infra soft closed set in* $(\mathcal{T}, \xi, \Theta)$. *Then,*

(i) $(\Psi, \Theta) \cap \tilde{} sCl(\Omega, \Theta) \subseteq \tilde{} sCl((\Psi, \Theta) \cap \tilde{}(\Omega, \Theta))$

(ii) $sInt((\Lambda, \Theta) \cup \tilde{}(\Omega, \Theta)) \subseteq \tilde{}(\Lambda, \Theta) \cup \tilde{} sInt(\Omega, \Theta)$

Proof. (i): Let $\delta_\theta^t \in (\Psi, \Theta) \cap \tilde{} sCl(\Omega, \Theta)$. Then, $\delta_\theta^t \in (\Psi, \Theta)$ and $\delta_\theta^t \in sCl(\Omega, \Theta)$. This implies that $(\Gamma, \Theta) \cap \tilde{}(\Omega, \Theta) \neq \Phi$ for every infra soft semiopen set (Γ, Θ) containing δ_θ^t. It

follows from Proposition 26 that $(\Psi,\Theta) \cap^{\sim} (\Gamma,\Theta)$ is an infra soft semiopen set containing δ_θ^t. Therefore, $[(\Psi,\Theta) \cap^{\sim} (\Gamma,\Theta)] \cap^{\sim} (\Omega,\Theta) \neq \Phi$. Now, $(\Gamma,\Theta) \cap^{\sim} [(\Psi,\Theta) \cap^{\sim} (\Omega,\Theta)] \neq \Phi$ which means that $\delta_\theta^t \in sCl((\Psi,\Theta) \cap^{\sim} (\Omega,\Theta))$. Hence, $(\Psi,\Theta) \cap^{\sim} sCl(\Omega,\Theta) \subseteq^{\sim} sCl((\Psi,\Theta) \cap^{\sim} (\Omega,\Theta))$.

One can prove (ii) following similar arguments.□   □

Theorem 39. *Let (Ω,Θ) and (Ψ,Θ) be subsets of $(\mathscr{T},\xi,\Theta)$. Then, we have the following properties.*

  (i) $sInt(\tilde{\mathscr{T}}) = \tilde{\mathscr{T}}$

  (ii) $sInt(\Omega,\Theta) \subseteq^{\sim} (\Omega,\Theta)$

  (iii) If $(\Psi,\Theta) \subseteq^{\sim} (\Omega,\Theta)$, then $sInt(\Psi,\Theta) \subseteq^{\sim} sInt(\Omega,\Theta)$

  (iv) $sInt(sInt(\Omega,\Theta)) = sInt(\Omega,\Theta)$

  (v) $sInt(\Psi,\Theta) \cap^{\sim} sInt(\Omega,\Theta) \subseteq^{\sim} sInt((\Psi,\Theta) \cap^{\sim} (\Omega,\Theta))$

Proof. (i): Since $\tilde{\mathscr{T}}$ is infra soft semiopen, $sInt(\tilde{\mathscr{T}}) = \tilde{\mathscr{T}}$.

(ii) and (iii) are obvious.

(iv): It is clear that $sInt(sInt(\Omega,\Theta))$ is the largest infra soft semiopen set contained in $sInt(\Omega,\Theta)$; however, $sInt(\Omega,\Theta)$ is an infra soft semiopen set; hence, $sInt(sInt(\Omega,\Theta)) = sInt(\Omega,\Theta)$.

(v): It comes from (iii).□   □

Theorem 40. *Let (Ω,Θ) and (Ψ,Θ) be subsets of $(\mathscr{T},\xi,\Theta)$. Then, we have the following properties.*

  (i) $sCl(\Phi) = \Phi$

  (ii) $(\Omega,\Theta) \subseteq^{\sim} sCl(\Omega,\Theta)$

  (iii) If $(\Psi,\Theta) \subseteq^{\sim} (\Omega,\Theta)$, then $sCl(\Psi,\Theta) \subseteq^{\sim} sCl(\Omega,\Theta)$

  (iv) $sCl(sCl(\Omega,\Theta)) \subseteq^{\sim} sCl(\Omega,\Theta)$

  (v) $sCl((\Psi,\Theta) \cup^{\sim} (\Omega,\Theta)) = sCl(\Psi,\Theta) \cup^{\sim} sCl(\Omega,\Theta)$

Proof. It can be proved following similar arguments given in the proof of Theorem 39.□   □

The next example shows that the inclusion relations given in the above two theorems are proper.

Example 2. Let $\mathscr{T} = \{t_1, t_2\}$ and $\Theta = \{\theta_1, \theta_2\}$. Then, $\xi = \{\Phi, \tilde{\mathscr{T}}, (\Omega_j,\Theta): j = 1, 2, 3\}$ is an infra soft topology on $\mathscr{T}$ over $\mathscr{T}$ with Θ as a set of parameters, where

$$(\Omega_1,\Theta) = \{(\theta_1, \{t_1\}), (\theta_2, \varnothing)\}; \tag{7}$$

$$(\Omega_2,\Theta) = \{(\theta_1, \varnothing), (\theta_2, \{t_1\})\}, \tag{8}$$

and

$$(\Omega_3,\Theta) = \{(\theta_1, \mathscr{T}), (\theta_2, \{t_2\})\}. \tag{9}$$

Let $(\Psi_1,\Theta) = \{(\theta_1, \{t_2\}), (\theta_2, \{t_1\})\}$. Then, $sInt(\Psi_1,\Theta)$ $= \{(\theta_1, \varnothing), (\theta_2, \{t_1\})\} \subset^{\sim} (\Psi_1,\Theta)$ and $sCl(\Psi_1,\Theta) = \{(\theta_1, \{t_2\}), (\theta_2, \mathscr{T})\} \supset^{\sim} (\Psi_1,\Theta)$. Also, consider $(\Psi_2,\Theta) = \{(\theta_1, \{t_2\}), (\theta_2, \varnothing)\}$. Then, $sCl((\Psi_1,\Theta) \cup^{\sim} (\Psi_2,\Theta)) = \{(\theta_1, \{t_2\}), (\theta_2, \mathscr{T})\} \supseteq^{\sim} sCl(\Psi_1,\Theta) \cap^{\sim} sCl(\Psi_2,\Theta) = \{(\theta_1, \{t_2\}), (\theta_2, \{t_2\})\}$.

Definition 41. A soft point δ_θ^t is said to be an infra soft semi-limit point of a subset (Ω,Θ) of $(\mathscr{T},\xi,\Theta)$ provided that $[(\Psi,\Theta) \setminus \delta_\theta^t] \cap^{\sim} (\Omega,\Theta) \neq \Phi$ for every infra soft semiopen set (Ψ,Θ) containing δ_θ^t.

The soft set of all infra soft semilimit points of (Ω,Θ) is said to be an infra semiderived soft set. It is denoted by $(\Omega,\Theta)^{is'}$.

Proposition 42. *Consider (Ψ,Θ) and (Ω,Θ) as subsets of $(\mathscr{T},\xi,\Theta)$. Then,*

  (i) $\Phi^{is'} = \Phi$ and $\mathscr{T}^{\sim is'} \subseteq^{\sim} \tilde{\mathscr{T}}$

  (ii) If $(\Psi,\Theta) \subseteq^{\sim} (\Omega,\Theta)$, then $(\Psi,\Theta)^{is'} \subseteq^{\sim} (\Omega,\Theta)^{is'}$

  (iii) If $\delta_\theta^t \in (\Omega,\Theta)^{is'}$, then $\delta_\theta^t \in ((\Omega,\Theta) \setminus \delta_\theta^t)^{is'}$

  (iv) $(\Psi,\Theta)^{is'} \cup^{\sim} (\Omega,\Theta)^{is'} \subseteq^{\sim} ((\Psi,\Theta) \cup^{\sim} (\Omega,\Theta))^{is'}$.

Proof. Straightforward.□   □

Theorem 43. *Let (Ω,Θ) be a subset of $(\mathscr{T},\xi,\Theta)$. Then,*

  (i) If (Ω,Θ) is an infra soft semiclosed set, then, $(\Omega,\Theta)^{is'} \subseteq (\Omega,\Theta)$

  (ii) $((\Omega,\Theta) \cup^{\sim} (\Omega,\Theta)^{is'})^{is'} \subseteq^{\sim} (\Omega,\Theta) \cup^{\sim} (\Omega,\Theta)^{is'}$

  (iii) $sCl(\Omega,\Theta) = (\Omega,\Theta) \cup^{\sim} (\Omega,\Theta)^{is'}$

Proof.

  (i) Consider (Ω,Θ) as an infra soft semiclosed set such that $\delta_\theta^t \notin (\Omega,\Theta)$. Then, $\delta_\theta^t \in (\Omega^c,\Theta)$. Now, (Ω^c,Θ) is an infra soft semiopen set such that $(\Omega^c,\Theta) \cup^{\sim} (\Omega,\Theta) = \Phi$ which means that $\delta_\theta^t \notin (\Omega,\Theta)^{is'}$. Thus, $(\Omega,\Theta)^{is'} \subseteq^{\sim} (\Omega,\Theta)$

  (ii) Consider $\delta_\theta^t \notin (\Omega,\Theta) \cup^{\sim} (\Omega,\Theta)^{is'}$. Then, $\delta_\theta^t \notin (\Omega,\Theta)$ and $\delta_\theta^t \notin (\Omega,\Theta)^{is'}$. Therefore, there exists an infra soft semiopen set (Ψ,Θ) such that

$$(\Psi,\Theta) \widetilde{\bigcap} (\Omega,\Theta) = \Phi, \tag{10}$$

This implies that

$$(\Psi,\Theta) \widetilde{\bigcap} (\Omega,\Theta)^{is'} = \Phi. \tag{11}$$

It follows from (10) and (11) that (Ψ, Θ) $\cap \tilde{} ((\Omega, \Theta) \cup \tilde{} (\Omega, \Theta)^{is'}) = \Phi$. Thus, $\delta_\theta^t \notin ((\Omega, \Theta)$ $\cup \tilde{} (\Omega, \Theta)^{is'})^{is'}$. Hence, $((\Omega, \Theta) \cup \tilde{} (\Omega, \Theta)^{is'})^{is'} \subseteq \tilde{} ((\Omega, \Theta)$ $\cup \tilde{} (\Omega, \Theta)^{is'})$, as required.

(iii) It is clear that $(\Omega, \Theta) \cup \tilde{} (\Omega, \Theta)^{is'} \subseteq \tilde{} sCl(\Omega, \Theta)$. Conversely, let $\delta_\theta^t \in sCl(\Omega, \Theta)$. Then, for every infra soft semiopen set containing δ_θ^t, we have (Ω, Θ) $\cap \tilde{} (\Psi, \Theta) \neq \Phi$. Without loss of generality, let $\delta_\theta^t \notin (\Omega, \Theta)$. Then, $[(\Omega, \Theta) \setminus \delta_\theta^t] \cap \tilde{} (\Psi, \Theta) \neq \Phi$. Consequentially, $\delta_\theta^t \in (\Omega, \Theta)^{is'}$. Hence, the proof is complete.

□

Definition 44. The infra soft semiboundary points of a subset (Ω, Θ) of $(\mathcal{T}, \xi, \Theta)$, denoted by $sB(\Omega, \Theta)$, are all the soft points which belong to the complement of $sInt(\Omega, \Theta)$ $\cup \tilde{} sInt(\Omega^c, \Theta)$.

Proposition 45. *Let (Ω, Θ) be a subset of $(\mathcal{T}, \xi, \Theta)$. Then,*

(i) $sB(\Omega, \Theta) = sCl(\Omega, \Theta) \widetilde{\cap} sCl((\Omega^c, \Theta))$

(ii) $sB(\Omega, \Theta) = sCl(\Omega, \Theta) \setminus sInt(\Omega, \Theta)$

Proof.

(i) $sB(\Omega, \Theta) = \{\delta_\theta^t \in \tilde{\mathcal{T}} : \delta_\theta^t \notin sInt(\Omega, \Theta) \text{ and } \delta_\theta^t \notin sInt((\Omega^c, \Theta))\} = \{\delta_\theta^t \in \tilde{\mathcal{T}} : \delta_\theta^t \notin (sCl(\Omega^c, \Theta))^c \text{ and } \delta_\theta^t \notin (sCl(\Omega, \Theta))^c\} = \{\delta_\theta^t \in \tilde{\mathcal{T}} : \delta_\theta^t \in sCl(\Omega^c, \Theta) \text{ and } \delta_\theta^t \in sCl(\Omega, \Theta)\}1.9cm = sCl(\Omega, \Theta) \cap \tilde{} sCl(\Omega^c, \Theta)$

(ii) $sB(\Omega, \Theta) = sCl(\Omega, \Theta) \cap \tilde{} sCl(\Omega^c, \Theta)1.7cm = sCl(\Omega, \Theta) \cap \tilde{} (sInt(\Omega, \Theta))^c 1.7cm = sCl(\Omega, \Theta) \setminus sInt(\Omega, \Theta)$

□ □

Corollary 46. *Let (Ω, Θ) be a subset of $(\mathcal{T}, \xi, \Theta)$. Then*

(i) $sB(\Omega, \Theta) = sB(\Omega^c, \Theta)$

(ii) $sCl(\Omega, \Theta) = sInt(\Omega, \Theta) \cup \tilde{} sB(\Omega, \Theta)$

Proposition 47. *Let (Ω, Θ) be a subset of $(\mathcal{T}, \xi, \Theta)$. Then,*

(i) (Ω, Θ) is infra soft semiopen iff $sB(\Omega, \Theta)$ $\cap \tilde{} (\Omega, \Theta) = \Phi$

(ii) (Ω, Θ) is infra soft semiclosed iff $sB(\Omega, \Theta) \subseteq \tilde{} (\Omega, \Theta)$

Proof.

(i) $sB(\Omega, \Theta) \cap (\Omega, \Theta) = sB(\Omega, \Theta) \cap sInt(\Omega, \Theta) = \Phi$. Conversely, let $\delta_\theta^t \in (\Omega, \Theta)$. Then $\delta_\theta^t \in sInt(\Omega, \Theta)$ or $\delta_\theta^t \in sB(\Omega, \Theta)$. Since $sB(\Omega, \Theta) \cap (\Omega, \Theta) = \Phi$, $\delta_\theta^t \in sI$

$nt(\Omega, \Theta)$. Thus, $(\Omega, \Theta) \subseteq sInt(\Omega, \Theta)$ which means that $(\Omega, \Theta) = sInt(\Omega, \Theta)$. Hence, (Ω, Θ) is infra soft semiopen

(ii) (Ω, Θ) is infra soft semiclosed $\Leftrightarrow (\Omega^c, \Theta)$ is infra soft semiopen $\Leftrightarrow sB(\Omega^c, \Theta) \cap (\Omega^c, \Theta) = \Phi \Leftrightarrow sB(\Omega^c, \Theta) \cap (\Omega^c, \Theta) = \Phi \Leftrightarrow sB(\Omega, \Theta) \subseteq (\Omega, \Theta)$

□

Corollary 48. *A subset (Ω, Θ) of $(\mathcal{T}, \xi, \Theta)$ is infra soft semiopen and infra soft semiclosed iff $sB(\Omega, \Theta) = \Phi$.*

5. Infra Soft Semihomeomorphism Maps

This section introduces the concepts of infra soft semicontinuous, infra soft semiopen, infra soft semiclosed, and infra soft semihomeomorphism maps. We give some characterizations of each one of these concepts and demonstrate some interrelations between them. Finally, we study the concept of fixed soft points with respect to infra soft semiopen sets.

Definition 49. A soft mapping $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is said to be infra soft semicontinuous at $\delta_\theta^t \in \tilde{\mathcal{T}}$ if for any infra soft semiopen set (Ψ, Δ) containing $E_\tau(\delta_\theta^t)$, there is an infra soft semiopen set (Ω, Θ) containing δ_θ^t such that $E_\tau(\Omega, \Theta)$ $\subseteq \tilde{} (\Psi, \Delta)$.

If E_τ is infra soft semicontinuous at all soft points of the domain, then, it is called infra soft semicontinuous.

Theorem 50. *$E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is an infra soft semicontinuous mapping iff the preimage of each infra soft semiopen set is infra soft semiopen.*

Proof. Necessity: let (Ω, Δ) be an infra soft semiopen subset of $(\mathcal{S}, \pi, \Delta)$. If $E_\tau^{-1}(\Omega, \Delta) = \Phi$, then, the proof is trivial. So, consider $E_\tau^{-1}(\Omega, \Delta) \neq \Phi$. Now, for any $\delta_\theta^t \in E_\tau^{-1}(\Omega, \Delta)$, there, is an infra soft semiopen subset (Ψ_i, Θ) of $(\mathcal{T}, \xi, \Theta)$ containing δ_θ^t such that $E_\tau(\Psi_i, \Theta) \subseteq \tilde{} (\Omega, \Delta)$. Thus, $\delta_\theta^t \in (\Psi_i, \Theta)$ $\subseteq \tilde{} E_\tau^{-1}(\Omega, \Delta)$ and $\widetilde{\bigcup}\{(\Psi_i, \Theta)\} = E_\tau^{-1}(\Omega, \Delta)$. This means that $E_\tau^{-1}(\Omega, \Delta)$ is infra soft semiopen.

Sufficiency: let $\delta_\theta^t \in \tilde{\mathcal{T}}$ and (Ψ, Θ) be an infra soft semiopen set containing $E_\tau(\delta_\theta^t)$. Then, $E_\tau^{-1}(\Psi, \Theta)$ is an infra soft semiopen set containing δ_θ^t such that $E_\tau(E_\tau^{-1}(\Psi, \Theta))$ $\subseteq \tilde{} (\Psi, \Theta)$. This means that E_τ is infra soft semicontinuous at δ_θ^t. This ends the proof that E_τ is infra soft semicontinuous. □ □

Example 3. Consider $\mathcal{T}$ is the set of real numbers, $\mathcal{S}$ is the set of natural numbers and $\Theta = \{\theta_1, \theta_2\}$. Let $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{T}, \pi, \Theta)$ and $F_\nu : (\mathcal{S}, \mu, \Theta) \longrightarrow (\mathcal{S}, \pi, \Theta)$ be two soft mapping such that E, F, τ, and ν are identity mappings, and π is the discrete soft topology (it is also infra soft topology). Let $\xi = \{\mathcal{T}, (\Omega, \Theta) \subseteq \tilde{} \mathcal{T} : (\Omega, \Theta) \text{ is finite}\}$ and $\mu = \{\Phi, \mathcal{S}, \{(\theta_1, \{2\}), (\theta_2, \{3\})\}, \{(\theta_1, \{3\}), (\theta_2, \{2\})\}\}$ are two infra soft topologies on $\mathcal{T}$ and $\mathcal{S}$, respectively. It is clear that every subset of $(\mathcal{T}, \xi, \Theta)$ is infra soft semiopen. So that, E_τ is

infra soft semicontinuous. On the other hand, F_ν is not infra soft semicontinuous because $(\Psi, \Theta) = \{(\theta_1, \{1\}), (\theta_2, \{1\})\}$ is an infra soft semiopen subset of $(\mathcal{S}, \pi, \Theta)$, whereas its pre-image under a soft mapping F_ν is not an infra soft semiopen subset of $(\mathcal{S}, \mu, \Theta)$.

Theorem 51. *Let $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ be an infra soft semicontinuous mapping. Then, we have the following five equivalent statements:*

    (i) E_τ is an infra soft semicontinuous mapping

    (ii) The preimage of each infra soft semiclosed set is infra soft semiclosed

    (iii) $sCl(E_\tau^{-1}(\Omega, \Delta)) \subseteq \tilde{} E_\tau^{-1}(sCl(\Omega, \Delta))$ for each $(\Omega, \Delta) \subseteq \tilde{} \tilde{\mathcal{S}}$

    (iv) $E_\tau(sCl(\Psi, \Theta)) \subseteq \tilde{} sCl(E_\tau(\Psi, \Theta))$ for each $(\Psi, \Theta) \subseteq \tilde{} \tilde{\mathcal{T}}$

    (v) $E_\tau^{-1}(sInt(\Omega, \Delta)) \subseteq \tilde{} sInt(E_\tau^{-1}(\Omega, \Delta))$ for each $(\Omega, \Delta) \subseteq \tilde{} \tilde{\mathcal{S}}$

Proof. $(i) \Rightarrow (ii)$: Let (Ω, Δ) be an infra soft semiclosed set in $(\mathcal{S}, \pi, \Delta)$.

Then, $E_\tau^{-1}(\Omega^c, \Delta)$ is an infrasoft semiopen subset of $\tilde{\mathcal{T}}$. Obviously, $E_\tau^{-1}(\Omega^c, \Delta) = \tilde{\mathcal{T}} - E_\tau^{-1}(\Omega, \Delta)$; hence, $E_\tau^{-1}(\Omega, \Delta)$ is an infra soft semiclosed subset of $\tilde{\mathcal{T}}$.

$(ii) \Rightarrow (iii)$: According to (ii), $E_\tau^{-1}(sCl(\Omega, \Delta))$ is an infra soft semiclosed subset of $\tilde{\mathcal{T}}$. Then, $sCl(E_\tau^{-1}(\Omega, \Delta)) \subseteq \tilde{} sCl(E_\tau^{-1}(sCl(\Omega, \Delta))) = E_\tau^{-1}(sCl(\Omega, \Delta))$.

$(iii) \Rightarrow (vi)$: According to (iii), $sCl(E_\tau^{-1}(E_\tau(\Psi, \Theta))) \subseteq \tilde{} E_\tau^{-1}(sCl(E_\tau(\Psi, \Theta)))$. Then, $E_\tau(sCl(\Psi, \Theta)) \subseteq \tilde{} E_\tau(E_\tau^{-1}(sCl(E_\tau(\Psi, \Theta)))) \subseteq \tilde{} sCl(E_\tau(\Psi, \Theta))$.

$(iv) \Rightarrow (v)$: According to (iv), $E_\tau(sCl(\tilde{\mathcal{T}} - E_\tau^{-1}(\Omega, \Delta))) \subseteq \tilde{} sCl(E_\tau(\tilde{\mathcal{T}} - E_\tau^{-1}(\Omega, \Delta)))$. Therefore, $E_\tau(\tilde{\mathcal{T}} - sInt(E_\tau^{-1}(\Omega, \Delta))) = E_\tau(sCl(\tilde{\mathcal{T}} - E_\tau^{-1}(\Omega, \Delta))) \subseteq sCl(\tilde{\mathcal{S}} - (\Omega, \Delta)) = \tilde{\mathcal{S}} - sInt(\Omega, \Delta)$. Thus, $\tilde{\mathcal{T}} - sInt(E_\tau^{-1}(\Omega, \Delta)) \subseteq \tilde{} E_\tau^{-1}(\tilde{\mathcal{S}} - sInt(\Omega, \Delta)) = E_\tau^{-1}(\tilde{\mathcal{S}}) - E_\tau^{-1}(sInt(\Omega, \Delta))$. Hence, $E_\tau^{-1}(sInt(\Omega, \Delta)) \subseteq \tilde{} sInt(E_\tau^{-1}(\Omega, \Delta))$.

$(v) \Rightarrow (i)$: Let (Ω, Δ) be an infra soft open subset of $\tilde{\mathcal{S}}$. According to (v), $E_\tau^{-1}(\Omega, \Delta) \subseteq \tilde{} sInt(E_\tau^{-1}(\Omega, \Delta))$. This implies that $E_\tau^{-1}(\Omega, \Delta) = sInt(E_\tau^{-1}(\Omega, \Delta))$. Hence, E_τ is infra soft semicontinuous.

Theorem 52. *If $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is infra soft semicontinuous, then, the restriction soft mapping $E_{\tau_{|\mathcal{M}}} : (\mathcal{M}, \xi_{\mathcal{M}}, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is infra soft semicontinuous provided that $\tilde{\mathcal{M}}$ is an infra soft open set.*

Proof. Consider (Ω, Δ) is an infra soft semiopen set in $(\mathcal{S}, \pi, \Delta)$. By hypothesis, $E_\tau^{-1}(\Omega, \Delta)$ is infra soft semiopen. Now, $E_{\tau_{|\mathcal{M}}}^{-1}(\Omega, \Delta) = E_\tau^{-1}(\Omega, \Delta) \cap \tilde{} \tilde{\mathcal{M}}$. Since $\tilde{\mathcal{M}}$ is an infra soft open set, it follows from Proposition 26 that $E_{\tau_{|\mathcal{M}}}^{-1}(\Omega, \Delta)$ is

infra soft semiopen. Hence, $E_{\tau_{|\mathcal{M}}}$ is an infra soft semicontinuous map.

Proposition 53. *Let $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ and $F_\nu : (\mathcal{S}, \pi, \Delta) \longrightarrow (\mathcal{V}, \sigma, \Gamma)$ be infra soft semicontinuous. Then, $F_\nu \circ E_\tau$ is infra soft semicontinuous.*

Proof. Straightforward.

Definition 54. A soft mapping $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is said to be infra soft semiopen (resp., infra soft semiclosed) if the image of each infra soft semiopen (resp., infra soft semiclosed) set is infra soft semiopen (resp., infra soft semiclosed).

Proposition 55. *$E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is an infra soft semiopen mapping iff $E_\tau(sInt(\Omega, \Theta)) \subseteq \tilde{} sInt(E_\tau(\Omega, \Theta))$ for each subset of (Ω, Θ) of $\tilde{\mathcal{T}}$.*

Proof. $\Rightarrow$: Let (Ω, Θ) be a subset of $\tilde{\mathcal{T}}$. Now, $E_\tau(sInt(\Omega, \Theta)) \subseteq \tilde{} E_\tau(\Omega, \Theta)$ and $sInt(\Omega, \Theta)$ are an infra soft semiopen set. By hypothesis, $E_\tau(sInt(\Omega, \Theta))$ is infra soft semiopen. Therefore, $E_\tau(sInt(\Omega, \Theta)) \subseteq \tilde{} sInt(E_\tau(\Omega, \Theta))$.

$\Leftarrow$: Let (Λ, Θ) be an infra soft open subset of $\tilde{\mathcal{T}}$. Then, $E_\tau(\Omega, \Theta) \subseteq \tilde{} sInt(E_\tau(\Omega, \Theta))$. Therefore, $E_\tau(\Omega, \Theta) = sInt(E_\tau(\Omega, \Theta))$ which means that E_τ is an infra soft semiopen map.

Proposition 56. *$E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is an infra soft semiclosed mapping iff $sCl(E_\tau(\Omega, \Theta)) \subseteq \tilde{} E_\tau(sCl(\Omega, \Theta))$ for each subset (Ω, Θ) of $\tilde{\mathcal{T}}$.*

Proof. $\Rightarrow$: Let E_τ be an infra soft semiclosed mapping and (Ω, Θ) be a subset of $\tilde{\mathcal{T}}$. By hypothesis, $E_\tau(sCl(\Omega, \Theta))$ is infra soft semiclosed. Since $E_\tau(\Omega, \Theta) \subseteq \tilde{} E_\tau(sCl(\Omega, \Theta))$, $sCl(E_\tau(\Omega, \Theta)) \subseteq \tilde{} E_\tau(sCl(\Omega, \Theta))$.

$\Leftarrow$: Suppose that (Ω, Θ) is an infra soft semiclosed subset of $\tilde{\mathcal{T}}$. By hypothesis, $E_\tau(\Omega, \Theta) \subseteq \tilde{} sCl(E_\tau(\Omega, \Theta)) \subseteq \tilde{} E_\tau(sCl(\Omega, \Theta)) = E_\tau(\Omega, \Theta)$. Therefore, $E_\tau(\Omega, \Theta)$ is infra soft semiclosed. Hence, E_τ is an infra soft semiclosed map.

Proposition 57. *The concepts of infra soft semiopen and infra soft semiclosed mappings are equivalent under bijectiveness.*

Proof. It comes from the fact that a bijective soft mapping $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ implies that $E_\tau(\Omega^c, \Theta) = (E_\tau(\Omega, \Theta))^c$.

Proposition 58. *Let $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ and $F_\nu : (\mathcal{S}, \pi, \Delta) \longrightarrow (\mathcal{V}, \sigma, \Gamma)$ be two soft maps. Then,*

    (i) If E_τ and F_ν are infra soft semiopen maps, then, $F_\nu \circ E_\tau$ is an infra soft semiopen map

(ii) If $F_\nu \circ E_\tau$ is an infra soft semiopen mapping and E_τ is a surjective infra soft semicontinuous map, then, F_ν is an infra soft semiopen map

(iii) If $F_\nu \circ E_\tau$ is an infra soft semiopen mapping and F_ν is an injective infra soft semicontinuous map, then, E_τ is an infra soft semiopen map

Proof.

(i) Straightforward

(ii) Consider (Ω, Δ) as an infra soft semiopen subset of $(\mathcal{S}, \pi, \Delta)$. By hypothesis, $E_\tau^{-1}(\Omega, \Delta)$ is an infra soft semiopen subset of $(\mathcal{T}, \xi, \Theta)$. Again, by hypothesis, $(F_\nu \circ E_\tau)(E_\tau^{-1}(\Omega, \Delta))$ is an infra soft semiopen subset of $(\mathcal{V}, \sigma, \Gamma)$. Since E_τ is surjective, then, $(F_\nu \circ E_\tau)(E_\tau^{-1}(\Omega, \Delta)) = F_\nu(E_\tau(E_\tau^{-1}(\Omega, \Delta))) = F_\nu(\Omega, \Delta)$. Hence, F_ν is an infra soft semiopen map

(iii) Consider (Ω, Θ) as an infra soft semiopen subset of $(\mathcal{T}, \xi, \Theta)$. By hypothesis, $(F_\nu \circ E_\tau)(\Omega, \Theta)$ is an infra soft semiopen subset of $(\mathcal{V}, \sigma, \Gamma)$. Again, by hypothesis, $F_\nu^{-1}(F_\nu \circ E_\tau(\Omega, \Theta))$ is an infra soft semiopen subset of $(\mathcal{S}, \pi, \Delta)$. Since F_ν is injective, then, $F_\nu^{-1}(F_\nu \circ E_\tau(\Omega, \Theta)) = (F_\nu^{-1}F_\nu)(E_\tau(\Omega, \Theta)) = E_\tau(\Omega, \Theta)$. Hence, E_τ is an infra soft semiopen map. □

□

The following result can be proved following similar arguments given in proposition' proof above.

Proposition 59. *Let $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ and $F_\nu : (\mathcal{S}, \pi, \Delta) \longrightarrow (\mathcal{V}, \sigma, \Gamma)$ be two infra soft maps. Then, the following statements hold.*

(i) *If E_τ and F_ν are infra soft semiclosed maps, then, $F_\nu \circ E_\tau$ is an infra soft semiclosed map*

(ii) *If $F_\nu \circ E_\tau$ is an infra soft semiclosed mapping and E_τ is a surjective infra soft semicontinuous map, then, F_ν is an infra soft semiclosed map*

(iii) *If $F_\nu \circ E_\tau$ is an infra soft semiclosed mapping and F_ν is an injective infra soft semicontinuous map, then, E_τ is an infra soft semiclosed map*

Definition 60. A bijective soft mapping $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is said to be an infra soft semihomeomorphism if it is infra soft semicontinuous and infra soft semiopen.

We cancel the proofs of the next two results because they are easy.

Proposition 61. *Let $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ and $F_\nu : (\mathcal{S}, \pi, \Delta) \longrightarrow (\mathcal{V}, \sigma, \Gamma)$ be infra soft semihomeomorphism maps. Then, $F_\nu \circ E_\tau$ is an infra soft semihomeomorphism map.*

Proposition 62. *If $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is a bijective soft map, then, the following statements are equivalent.*

(i) *E_τ is an infra soft semihomeomorphism*

(ii) *E_τ and E_τ^{-1} is infra soft semicontinuous*

(iii) *E_τ is infra soft semiclosed and infra soft semicontinuous*

Proposition 63. *If $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ is an infra soft semihomeomorphism map, then, the following statements hold for each $(\Omega, \Theta) \in S(X)_A$.*

(i) *$E_\tau(sInt(\Omega, \Theta)) = sInt(E_\tau(\Omega, \Theta))$*

(ii) *$E_\tau(sCl(\Omega, \Theta)) = sCl(E_\tau(\Omega, \Theta))$*

Proof. (i): According to Proposition 55 (i), we obtain $E_\tau(sInt(\Omega, \Theta)) \subseteq \widetilde{\ } sInt(E_\tau(\Omega, \Theta))$.

Conversely, let $\delta_\kappa^s \in sInt(E_\tau(\Omega, \Theta))$. Then, there is an infra soft semiopen set (Ψ, Δ) such that $\delta_\kappa^s \in (\Psi, \Delta) \subseteq \widetilde{\ } E_\tau(\Omega, \Theta)$. By hypothesis, $\delta_\theta^t = E_\tau^{-1}(\delta_\kappa^s) \in E_\tau^{-1}(\Psi, \Delta) \subseteq \widetilde{\ } (\Omega, \Theta)$ such that $E_\tau^{-1}(\Psi, \Delta)$ is an infra soft semiopen set. So that, $\delta_\theta^t \in sInt(\Omega, \Theta)$ which means that $\delta_\kappa^s \in E_\tau(sInt(\Omega, \Theta))$.

One can achieve item (ii) following similar arguments. □

□

Theorem 64. *The property of an infra soft semidense set is an infra soft topological invariant.*

Proof. Let $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{S}, \pi, \Delta)$ be an infra soft semihomeomorphism mapping and consider (Ω, Θ) as an infra soft semidense subset of $(\mathcal{T}, \xi, \Theta)$, i.e., $sCl(\Omega, \Theta) = \tilde{\mathcal{T}}$. It comes from Proposition 63 (ii) that $sCl(E_\tau(\Omega, \Theta)) = E_\tau(sCl(\Omega, \Theta)) = E_\tau(\tilde{\mathcal{T}}) = sCl(\tilde{\mathcal{S}}) = \tilde{\mathcal{S}}$. Thus, $E_\tau(\Omega, \Theta)$ is an infra soft semidense set in $(\mathcal{S}, \pi, \Delta)$, as required. □

□

We complete this section by studying the concept of fixed soft points with respect to infra soft semiopen sets. For more details in fixed soft points in the crisp setting, see [27–29].

Definition 65. We say that $(\mathcal{T}, \xi, \Theta)$ has a semifixed soft point property provided that for every infra soft semicontinuous mapping $E_\tau : (\mathcal{T}, \xi, \Theta) \longrightarrow (\mathcal{T}, \xi, \Theta)$, there exists $\delta_\theta^s \in \mathcal{T}$ such that $E_\tau(\delta_\theta^s) = \delta_\theta^s$.

Proposition 66. *The property of being a semifixed soft point is preserved under an infra soft semihomeomorphism.*

Proof. Consider $(\mathcal{T}_1, \xi_1, \Theta_1)$ and $(\mathcal{T}_2, \xi_2, \Theta_2)$ as two infra soft semihomeomorphism. This means that there exists a bijective soft mapping $E_\tau : (\mathcal{T}_1, \xi_1, \Theta_1) \longrightarrow (\mathcal{T}_2, \xi_2, \Theta_2)$ such that E_τ and E_τ^{-1} are infra soft semicontinuous. Suppose that $(\mathcal{T}_1, \xi_1, \Theta_1)$ has the property of semifixed soft point.

That is any infra soft semicontinuous mapping $E_\tau : (\mathscr{T}_1, \xi_1, \Theta_1) \longrightarrow (\mathscr{T}_1, \xi_1, \Theta_1)$ has a semifixed soft point. Now, consider $C_\tau : (\mathscr{T}_2, \xi_2, \Theta_2) \longrightarrow (\mathscr{T}_2, \xi_2, \Theta_2)$ is infra soft semicontinuous. It is clear that $C_\tau \circ E_\tau : (\mathscr{T}_1, \xi_1, \Theta_1) \longrightarrow (\mathscr{T}_2, \xi_2, \Theta_2)$ is infra soft semicontinuous. Therefore, $E_\tau^{-1} \circ C_\tau \circ E_\tau : (\mathscr{T}_1, \xi_1, \Theta_1) \longrightarrow (\mathscr{T}_1, \xi_1, \Theta_1)$ is infra soft semicontinuous. Since $(\mathscr{T}_1, \xi_1, \Theta_1)$ has a semifixed soft point property, $E_\tau^{-1}(h_\tau(E_\tau(\delta_\theta^s))) = \delta_\theta^s$ for some $\delta_\theta^s \in \tilde{\mathscr{T}}$. Thus, $E_\tau(E_\tau^{-1}(h_\tau(E_\tau(\delta_\theta^s)))) = E_\tau(\delta_\theta^s)$. This implies that $h_\tau(E_\tau(\delta_\theta^s)) = E_\tau(\delta_\theta^s)$. Hence, $E_\tau(\delta_\theta^s)$ is a semifixed soft point of C_τ which means that $(\mathscr{T}_2, \xi_2, \Theta_2)$ has a semifixed soft point property.☐ □

6. Concluding Remark and Further Work

This article contributes to the expanding literature on soft topological spaces. The obtained results demonstrate that most soft topological properties of the presented concepts are preserved in structure of infra soft topologies which means we can dispense of some topological stipulations. This gives an advantage of discussing soft topological ideas via infra soft topologies because it relaxes the restrictions imposed in the study. The obtained results in this manuscript and those given in [17–19] validate this viewpoint.

On the other hand, there are a few properties of some topological concepts that are partially losing via infra soft topology such as the equivalence between an infra soft semiopen set (Ω, Θ) and the existence of an infra soft open set (Ψ, Θ) such that $(\Psi, \Theta) \subseteq^\sim (\Omega, \Theta) \subseteq^\sim Cl(\Psi, \Theta)$. However, we have addressed this matter by defining an ξ-infra soft open set and proving the counterpart equivalence as given in Proposition 21. As we have shown in Corollary 25 that the class of infra soft semiopen subsets on ISTSs forms a new generalization of soft topology called a supra soft topology.

This work considers a promising line for future work; for example, we will complete introducing the main topological concepts using infra soft semiopen sets such as soft separation axioms, soft compact, and soft connected spaces. Our roadmap for research also comprises the examination of the concepts and results initiated herein using another generalization of infra soft open sets such as infra soft α-open and infra soft b-open sets. Moreover, we will introduce new types of rough approximations using these generalizations of infra soft open sets and apply them to improve the accuracy measures of sets.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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