Research Article

On Ćirić-Prešić Operators in Metric Spaces

Narongsuk Boonsri¹, Satit Saejung², and Kittipong Sitthikul³

¹Department of Applied Mathematics and Statistics, Faculty of Sciences and Liberal Arts, Rajamangala University of Technology Isan, Nakhon Ratchasima 30000, Thailand

²Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand

³Department of Mathematics, Faculty of Liberal Arts, Rajamangala University of Technology Rattanakosin, Wang Klai Kangwon Campus, Hua Hin Prachuap Khiri Khan 77110, Thailand

Correspondence should be addressed to Satit Saejung; saejung@kku.ac.th

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We show that the Prešić type operators of several variables can be regarded as an operator of a single variable and the fixed point problem of Prešić type can be regarded as a classical fixed point problem. We extend the recent result of Ćirić and Prešić by using the classical approach of Prešić. The key of the proof is based on the mappings introduced by Kada, Suzuki, and Takahashi. We also discuss the convergence problems of recursive real sequences and the Volterra integral equations as an application of our result.

Dedicated to the memory of Professor Wataru Takahashi

1. Introduction

Let (X, d) be a metric space and $k \geq 1$ be a fixed integer. Suppose that $T : X^k \rightarrow X$ is given. The following two problems were studied by Prešić [1]:

- (i) Find $\hat{x} \in X$ such that

$$\hat{x} = T(\hat{x}, \hat{x}, \dots, \hat{x}). \quad (1)$$

- (ii) For any given $x_1, x_2, \dots, x_k \in X$, if we define

$$x_{n+k} = T(x_n, x_{n+1}, \dots, x_{n+k-1}) \quad \text{for all } n \geq 1, \quad (2)$$

then what can we say about the convergence of the sequence $\{x_n\}$?

This problem is very interesting and was further studied by many authors, for example, see [2–4].

In 1965, Prešić [1] proved the following interesting theorem which can be regarded as a generalization of the classical fixed point theorem proposed by Banach [5]. Let k be a

positive integer and let $q_1, q_2, \dots, q_k$ be nonnegative real numbers such that $q_1 + q_2 + \dots + q_k < 1$. Let $\mathcal{P}(q_1, q_2, \dots, q_k)$ be a family of mappings $T : X^k \rightarrow X$ such that

$$\begin{aligned} d(T(u_1, u_2, \dots, u_k), T(u_2, u_3, \dots, u_{k+1})) \\ \leq \sum_{i=1}^k q_i d(u_i, u_{i+1}) \quad \text{for all } u_1, u_2, \dots, u_{k+1} \in X. \end{aligned} \quad (3)$$

Theorem P1. Let (X, d) be a complete metric space. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\liminf_{n \rightarrow \infty} (\alpha_{n+1}/\alpha_n) = 1$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of operators in $\mathcal{P}(q_1, q_2, \dots, q_k)$ such that

$$\begin{aligned} d(T_n(u_1, u_2, \dots, u_k), T_{n+1}(u_1, u_2, \dots, u_k)) \\ \leq \alpha_n \quad \text{for all } u_1, u_2, \dots, u_k \in X \text{ and for all } n \geq 1. \end{aligned} \quad (4)$$

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X such that $x_1, x_2, \dots, x_k \in X$ are arbitrary and

$$x_{n+k} = T_n(x_n, x_{n+1}, \dots, x_{n+k-1}) \quad \text{for all } n \geq 1. \quad (5)$$

Then, the following statements are true.

- (a) There exists an element $\hat{x} \in X$ such that $\hat{x} = \lim_{n \rightarrow \infty} x_n$
- (b) There exists an operator $T \in \mathcal{P}(q_1, q_2, \dots, q_k)$ such that $\{T_n\}_{n=1}^\infty$ converges to T uniformly; that is, $\lim_{n \rightarrow \infty} \sup_{x \in X} d(T_n x, T x) = 0$
- (c) The element $\hat{x}$ is the only one satisfying Equation (1)

Theorem P1, where $k = 1$, is nothing but Banach's fixed point theorem [5].

Remark 1. It was shown in [6] that the condition $\liminf_{n \rightarrow \infty} (\alpha_{n+1}/\alpha_n) = 1$ is superfluous.

As a consequence of Theorem P1, we have the following result.

Theorem P2. Let (X, d) be a complete metric space and let k be a positive integer. Let $q_1, q_2, \dots, q_k$ be nonnegative real numbers such that $q_1 + q_2 + \dots + q_k < 1$. Let $T \in \mathcal{P}(q_1, q_2, \dots, q_k)$. Let $\{x_n\}_{n=1}^\infty$ be a sequence in X such that $x_1, x_2, \dots, x_k \in X$ are arbitrary and Equation (2) holds. Then, the following statements are true.

- (a) There exists an element $\hat{x} \in X$ such that $\hat{x} = \lim_{n \rightarrow \infty} x_n$
- (b) The element $\hat{x}$ is the only one satisfying Equation (1).

Ćirić and Prešić [2] proved the following improvement of Theorem P2.

Theorem CP. Let (X, d) be a complete metric space. Let k be a positive integer and let q be a positive real number such that $q < 1$. Let $T : X^k \rightarrow X$ be such that

$$\begin{aligned} d(T(u_1, u_2, \dots, u_k), T(u_2, u_3, \dots, u_{k+1})) \\ \leq q \max \{d(u_i, u_{i+1}) : i = 1, 2, \dots, k\} \end{aligned} \quad (6)$$

for all $u_1, u_2, \dots, u_{k+1} \in X$.

Let $\{x_n\}_{n=1}^\infty$ be a sequence in X such that $x_1, x_2, \dots, x_k \in X$ are arbitrary and Equation (2) holds. Then, the following statements are true.

- (a) There exists an element $\hat{x} \in X$ such that $\hat{x} = \lim_{n \rightarrow \infty} x_n$ and Equation (1) holds.
- (b) If, in addition

$$d(T(u, u, \dots, u), T(v, v, \dots, v)) < d(u, v) \quad \text{for all } u, v \in X, \quad (7)$$

then the element $\hat{x}$ is the only one satisfying Equation (1).

Remark 2. It is clear that if $T : X^k \rightarrow X$ satisfies Expression (3), then it satisfies Expressions (6) and (7).

Proof. Suppose that $T : X^k \rightarrow X$ satisfies Expression (3) with nonnegative constants $q_1, q_2, \dots, q_k$ such that $q_1 + q_2 + \dots + q_k < 1$. To show that T satisfies Expression (6) with $q := \sum_{i=1}^k q_i$, let $u_1, u_2, \dots, u_{k+1} \in X$. It follows that

$$\begin{aligned} d(T(u_1, u_2, \dots, u_k), T(u_2, u_3, \dots, u_{k+1})) \\ \leq \sum_{i=1}^k q_i d(u_i, u_{i+1}) \leq q \max \{d(u_i, u_{i+1}) : i = 1, 2, \dots, k\}. \end{aligned} \quad (8)$$

To see that T satisfies Equation (7), let $u, v \in X$. It follows that

$$\begin{aligned} d(T(u, u, \dots, u), T(v, v, \dots, v)) \\ \leq d(T(u, u, \dots, u), T(v, u, \dots, u)) \\ + d(T(v, u, \dots, u), T(v, v, u, \dots, u)) + \dots \\ + d(T(v, v, \dots, v, u), T(v, v, \dots, v)) \\ \leq (q_1 + q_2 + \dots + q_k) d(u, v) < d(u, v). \end{aligned} \quad (9)$$

This completes the proof. $\square$

It is natural to ask the following:

Question: Is it possible to generalize Theorem CP by using the approach of Theorem P1?

In this paper, we answer the question above by considering a wider class of mappings than those satisfying Expression (6). The class of mappings studied in this paper is motivated by the one in work of Kada et al. [7]. Some progress on these mappings can be found in [8].

Theorem KST (see [7]). Let (X, d) be a complete metric space. Let $r \in [0, 1)$ be given and let $T : X \rightarrow X$ be a mapping such that the following conditions hold:

- (1) **(KST1)** $d(Tx, T^2x) \leq rd(x, Tx)$ for all $x \in X$
- (2) **(KST2)** If $y \neq Ty$, then $\inf \{d(x, Tx) + d(x, y) : x \in X\} > 0$

Then, every sequence $\{x_n\}_{n=1}^\infty$ with arbitrary x_1 and $x_{n+1} := Tx_n$ for all $n \geq 1$ converges to a fixed point of T .

2. Main Results

2.1. On the Fixed Point Problem of Prešić Type. We first show that the fixed point problem of Prešić type is equivalent to the classical fixed point problem. Suppose that X is a nonempty set and $k \geq 1$ is an integer. The *fixed point problem of Prešić type* for a given mapping $T : X^k \rightarrow X$ is to find $u \in X$ such that

$$u = T(u, u, \dots, u). \quad (10)$$

We denote by $\text{PFix}(T)$ the set of all solutions of the problem above. We show that this problem is connected to the classical fixed point problem. To simplify the

notation of the following result, we write $\mathbf{X} := X^k$ and $\dot{x} := (x, x, \dots, x) \in \mathbf{X}$ where $x \in X$.

Theorem 3. Suppose that X is a nonempty set and $k \geq 1$ is an integer. Suppose that $T : X^k \rightarrow X$ is given. Then, there exists a mapping $\mathbf{T} : \mathbf{X} \rightarrow \mathbf{X}$ such that the following statements hold.

- (a) If $u \in \text{PFix}(T)$, then $\dot{u} \in \text{Fix}(\mathbf{T})$
- (b) If $(u_1, u_2, \dots, u_k) \in \text{Fix}(\mathbf{T})$, then $u_1 = u_2 = \dots = u_k \in \text{PFix}(T)$

Proof. Define $\mathbf{T} : \mathbf{X} \rightarrow \mathbf{X}$ by

$$\mathbf{T}\mathbf{u} := (u_2, \dots, u_k, T(u_1, \dots, u_k)), \quad (11)$$

where $\mathbf{u} := (u_1, \dots, u_k) \in \mathbf{X}$.

The statement (a) is trivial. We prove the statement (b). Suppose that $(u_1, u_2, \dots, u_k) \in \text{Fix}(\mathbf{T})$. It follows that

$$(u_1, u_2, \dots, u_k) = (u_2, \dots, u_k, T(u_1, \dots, u_k)). \quad (12)$$

This implies that

$$u_1 = u_2 = \dots = u_k = T(u_1, \dots, u_k). \quad (13)$$

This completes the proof. $\square$

2.2. On the Class of Mappings. In this subsection, we discuss the following classes of mappings. The first two classes are from Theorem P and Theorem CP and the last one from Theorem KST.

Suppose that $X := (X, d)$ is a metric space and $k \geq 1$ is an integer. Suppose that $r, q, q_1, \dots, q_k$ are nonnegative real numbers. Define

$$\begin{aligned} \mathcal{F}(X, d, q_1, \dots, q_k) &:= \left\{ T : X^k \rightarrow X \text{ satisfies (3)} \right\}, \\ \mathcal{G}(X, d, q) &:= \left\{ T : X^k \rightarrow X \text{ satisfies (6)} \right\}, \\ \mathcal{H}(X, d, r) &:= \left\{ T : X \rightarrow X \text{ satisfies Conditions (KST1) and (KST2)} \right\}. \end{aligned} \quad (14)$$

Remark 4. Suppose that $X := (X, d)$ is a metric space and $k \geq 1$ is an integer. The following statements are true.

- (a) If $q_1, \dots, q_k$ are nonnegative real numbers, then $\mathcal{F}(X, d, q_1, \dots, q_k) \subset \mathcal{G}(X, d, q)$, where $q := \sum_{i=1}^k q_i$
- (b) If q is a nonnegative real number and $T \in \mathcal{G}(X, d, q)$, then $\mathbf{T} \in \mathcal{H}(\mathbf{X}, \mathbf{d}, r)$, where $\mathbf{X} := X^k$, $r = q^{1/k}$, $\mathbf{T}$ is

defined by Equation (11) and

$$\begin{aligned} \mathbf{d}(\mathbf{x}, \mathbf{y}) &:= \max \left\{ q^{-i/k} d(x_i, y_i) : i = 1, \dots, k \right\} \\ \text{for all } \mathbf{x} &:= (x_1, \dots, x_k) \in \mathbf{X} \text{ and } \mathbf{y} := (y_1, \dots, y_k) \in \mathbf{X}. \end{aligned} \quad (15)$$

Proof. The statement (a) is trivial. We prove the statement (b). Let $T \in \mathcal{G}(X, d, q)$. It is clear that $\mathbf{d}$ is a metric on $\mathbf{X}$. We now prove that $\mathbf{T} \in \mathcal{H}(\mathbf{X}, \mathbf{d}, r)$. To see that $\mathbf{T}$ satisfies Condition (KST1), let $\mathbf{u} := (u_1, \dots, u_k) \in \mathbf{X}$. We write $u_{k+1} := T(u_1, \dots, u_k)$ and $u_{k+2} := T(u_2, \dots, u_{k+1})$. It follows from $T \in \mathcal{G}(X, d, q)$ that

$$d(u_{k+1}, u_{k+2}) \leq q \max \{ d(u_i, u_{i+1}) : i = 1, \dots, k \}. \quad (16)$$

Note that

$$\begin{aligned} \mathbf{T}\mathbf{u} &= (u_2, \dots, u_{k+1}), \\ \mathbf{T}^2\mathbf{u} &= (u_3, \dots, u_{k+2}). \end{aligned} \quad (17)$$

Put $r := 1/s \in (0, 1)$ where $s = q^{-1/k}$. It follows from Equation (6) that

$$\begin{aligned} \mathbf{d}(\mathbf{T}\mathbf{u}, \mathbf{T}^2\mathbf{u}) &= \max \{ s^i d(u_{i+1}, u_{i+2}) : i = 1, \dots, k \} \\ &= \max \left\{ \max \{ s^i d(u_{i+1}, u_{i+2}) : i = 1, \dots, k-1 \}, \right. \\ &\quad \left. s^k d(u_{k+1}, u_{k+2}) \right\} \\ &= \max \left\{ \max \{ s^i d(u_{i+1}, u_{i+2}) : i = 1, \dots, k-1 \}, \right. \\ &\quad \left. s^k d(T(u_1, u_2, \dots, u_k), T(u_2, u_3, \dots, u_{k+1})) \right\} \\ &\leq \max \left\{ \max \{ s^i d(u_{i+1}, u_{i+2}) : i = 1, \dots, k-1 \}, \right. \\ &\quad \left. s^k q \max \{ d(u_i, u_{i+1}) : i = 1, \dots, k \} \right\} \\ &= \max \left\{ \max \{ s^i d(u_{i+1}, u_{i+2}) : i = 1, \dots, k-1 \}, \right. \\ &\quad \left. \max \{ d(u_i, u_{i+1}) : i = 1, \dots, k \} \right\} \\ &= \max \left\{ \max \{ s^i d(u_{i+1}, u_{i+2}) : i = 1, \dots, k-1 \}, \right. \\ &\quad \left. d(u_1, u_2), \max \{ d(u_i, u_{i+1}) : i = 2, \dots, k \} \right\} \\ &= \max \{ d(u_1, u_2), \max \{ s^i d(u_{i+1}, u_{i+2}) : i = 1, \dots, k-1 \} \} \\ &= \frac{1}{s} \max \{ s d(u_1, u_2), s \max \{ s^i d(u_{i+1}, u_{i+2}) : i = 1, \dots, k-1 \} \} \\ &= r \max \{ s d(u_1, u_2), \max \{ s^i d(u_i, u_{i+1}) : i = 2, \dots, k \} \} \\ &= r \max \{ s^i d(u_i, u_{i+1}) : i = 1, \dots, k \} \\ &= r \max \{ q^{-i/k} d(u_i, u_{i+1}) : i = 1, \dots, k \} = r \mathbf{d}(\mathbf{u}, \mathbf{T}\mathbf{u}). \end{aligned} \quad (18)$$

Finally, we show that $\mathbf{T}$ satisfies Condition (KST2). To see this, let $\{\mathbf{u}_n\}$ be a sequence in $\mathbf{X}$ and let $\mathbf{u} \in \mathbf{X}$ be such that $\lim_{n \rightarrow \infty} \mathbf{d}(\mathbf{u}_n, \mathbf{T}\mathbf{u}_n) = \lim_{n \rightarrow \infty} \mathbf{d}(\mathbf{u}_n, \mathbf{u}) = 0$. For each $n \in \mathbb{N}$,

we write $\mathbf{u}_n := (u_1^{(n)}, u_2^{(n)}, \dots, u_k^{(n)})$ and $\mathbf{u} := (u_1, u_2, \dots, u_k)$. It follows from the definition of $\mathbf{d}$ that

$$\lim_{n \rightarrow \infty} d(u_i^{(n)}, u_{i+1}^{(n)}) = \lim_{n \rightarrow \infty} d(u_i^{(n)}, u_i) = 0, \quad \text{for all } i = 1, 2, \dots, k, \text{ where } u_{k+1}^{(n)} := T(u_1^{(n)}, u_2^{(n)}, \dots, u_k^{(n)}) \text{ for all } n \geq 1. \quad (19)$$

In particular, we have $u_1 = u_2 = \dots = u_k$. This implies that $\mathbf{u} = (u_1, u_1, \dots, u_1)$ and $\lim_{n \rightarrow \infty} d(u_1, T(u_1^{(n)}, u_2^{(n)}, \dots, u_k^{(n)})) = 0$. We consider

$$\begin{aligned} & d(T(u_1^{(n)}, u_2^{(n)}, \dots, u_k^{(n)}), T(u_1, u_1, \dots, u_1)) \\ & \leq d(T(u_1^{(n)}, u_2^{(n)}, \dots, u_k^{(n)}), T(u_2^{(n)}, u_3^{(n)}, \dots, u_k^{(n)}, u_1)) \\ & \quad + d(T(u_2^{(n)}, u_3^{(n)}, \dots, u_k^{(n)}, u_1), T(u_3^{(n)}, \dots, u_k^{(n)}, u_1, u_1)) \\ & \quad + \dots + d(T(u_k^{(n)}, u_1, \dots, u_1), T(u_1, u_1, \dots, u_1)) \\ & \leq q \max \{d(u_k^{(n)}, u_1), d(u_i^{(n)}, u_{i+1}^{(n)}): i = 1, \dots, k-1\} \\ & \quad + q \max \{d(u_k^{(n)}, u_1), d(u_i^{(n)}, u_{i+1}^{(n)}): i = 2, \dots, k-1\} \\ & \quad + \dots + qd(u_k^{(n)}, u_1). \end{aligned} \quad (20)$$

Hence, $\lim_{n \rightarrow \infty} d(T(u_1^{(n)}, u_2^{(n)}, \dots, u_k^{(n)}), T(u_1, u_1, \dots, u_1)) = 0$. Since

$$\lim_{n \rightarrow \infty} d(T(u_1^{(n)}, u_2^{(n)}, \dots, u_k^{(n)}), u_1) = \lim_{n \rightarrow \infty} d(u_{k+1}^{(n)}, u_1) = 0, \quad (21)$$

we have $u_1 = T(u_1, u_1, \dots, u_1)$. Hence, $\mathbf{u} = \mathbf{T}\mathbf{u}$. This completes the proof. $\square$

Remark 5. The classes $\mathcal{F}(X, d, q_1, \dots, q_k)$ and $\mathcal{G}(X, d, q)$ are closed under the pointwise convergence, that is, if $\{T_n\}$ is a sequence in $\mathcal{F}(X, d, q_1, \dots, q_k)$ ($\mathcal{G}(X, d, q)$), respectively and there exists a mapping $T : X^k \rightarrow X$ such that

$$\begin{aligned} & \lim_{n \rightarrow \infty} d(T_n(u_1, u_2, \dots, u_k), T(u_1, u_2, \dots, u_k)) \\ & = 0 \quad \text{for all } (u_1, u_2, \dots, u_k) \in X^k, \end{aligned} \quad (22)$$

then $T \in \mathcal{F}(X, d, q_1, \dots, q_k)$ ($T \in \mathcal{G}(X, d, q)$), respectively).

Proof. Suppose that $\{T_n\}$ is a sequence in $\mathcal{G}(X, d, q)$ and there exists a mapping $T : X \rightarrow X$ such that

$$\begin{aligned} & \lim_{n \rightarrow \infty} d(T_n(u_1, u_2, \dots, u_k), T(u_1, u_2, \dots, u_k)) \\ & = 0 \quad \text{for all } (u_1, u_2, \dots, u_k) \in X^k. \end{aligned} \quad (23)$$

We prove that $T \in \mathcal{G}(X, d, q)$. To see this, let $u_1, u_2, \dots, u_k, u_{k+1} \in X$. Note that

$$\begin{aligned} & \lim_{n \rightarrow \infty} d(T(u_1, u_2, \dots, u_k), T_n(u_1, u_2, \dots, u_k)) \\ & = \lim_{n \rightarrow \infty} d(T_n(u_2, u_3, \dots, u_{k+1}), T(u_2, u_3, \dots, u_{k+1})) = 0. \end{aligned} \quad (24)$$

It follows that

$$\begin{aligned} & d(T(u_1, u_2, \dots, u_k), T(u_2, u_3, \dots, u_{k+1})) \\ & \leq d(T(u_1, u_2, \dots, u_k), T_n(u_1, u_2, \dots, u_k)) \\ & \quad + d(T_n(u_1, u_2, \dots, u_k), T_n(u_2, u_3, \dots, u_{k+1})) \\ & \quad + d(T_n(u_2, u_3, \dots, u_{k+1}), T(u_2, u_3, \dots, u_{k+1})) \\ & \leq d(T(u_1, u_2, \dots, u_k), T_n(u_1, u_2, \dots, u_k)) \\ & \quad + q \max \{d(u_i, u_{i+1}): i = 1, 2, \dots, k\} \\ & \quad + d(T_n(u_2, u_3, \dots, u_{k+1}), T(u_2, u_3, \dots, u_{k+1})). \end{aligned} \quad (25)$$

In particular,

$$\begin{aligned} & d(T(u_1, u_2, \dots, u_k), T(u_2, u_3, \dots, u_{k+1})) \\ & \leq q \max \{d(u_i, u_{i+1}): i = 1, 2, \dots, k\}. \end{aligned} \quad (26)$$

This implies that $T \in \mathcal{G}(X, d, q)$. The case that $\{T_n\}$ is a sequence in $\mathcal{F}(X, d, q_1, q_2, \dots, q_k)$ can be done similarly. $\square$

2.3. A Generalization of Theorem KST and Its Consequences. The following result is analogous to Theorem P1 with a wider class of mappings.

Theorem 6. Let (X, d) be a complete metric space and $r \in [0, 1)$ be given. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \alpha_n < \infty$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of operators in $\mathcal{H}(X, d, r)$ such that

$$d(T_n x, T_{n+1} x) \leq \alpha_n \quad \text{for all } x \in X. \quad (27)$$

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X such that $x_1 \in X$ is arbitrary and $x_{n+1} := T_n x_n$ for all $n \geq 1$. Then, the following statements are true.

- (a) There exists an element $\hat{x} \in X$ such that $\hat{x} = \lim_{n \rightarrow \infty} x_n$
- (b) There exists an operator $T : X \rightarrow X$ such that $\{T_n\}$ converges to T uniformly
- (c) If T satisfies Condition (KST2), then $\hat{x} \in \text{Fix}(T)$

Lemma 7. Let $\{s_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ be two sequences of non-negative real numbers such that $\sum_{n=1}^{\infty} \alpha_n < \infty$. Let $r \in (0, 1)$ be given. If

$$s_{n+1} \leq rs_n + \alpha_n \quad \text{for all } n \geq 1, \quad (28)$$

then $\sum_{n=1}^{\infty} s_n < \infty$.

Proof. Note that, for each $m \geq 1$, we have $\sum_{n=2}^{m+1} s_n \leq r \sum_{n=1}^m s_n + \sum_{n=1}^m \alpha_n$. In particular, $(1-r) \sum_{n=2}^m s_n \leq (1-r) \sum_{n=2}^m s_n + \sum_{n=1}^m \alpha_n$ for all $m \geq 2$. Hence, the conclusion follows. $\square$

Proof of Theorem 6. First, we note that

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(T_n x_n, T_{n+1} x_{n+1}) \leq d(T_n x_n, T_n x_{n+1}) \\ &\quad + d(T_n x_{n+1}, T_{n+1} x_{n+1}) \\ &= d(T_n x_n, T_n^2 x_n) + d(T_n x_{n+1}, T_{n+1} x_{n+1}) \quad (29) \\ &\leq rd(x_n, T_n x_n) + \alpha_n \\ &= rd(x_n, x_{n+1}) + \alpha_n. \end{aligned}$$

It follows from Lemma 7 that $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$. In particular, $\{x_n\}$ is a Cauchy sequence and hence $\lim_{n \rightarrow \infty} x_n = \hat{x}$ for some $\hat{x} \in X$ by the completeness of X . Hence, (a) is proved.

We now prove (b). For each $x \in X$ and for each $n, k \geq 1$, we note that

$$d(T_n x, T_{n+k} x) \leq \sum_{j=n}^{n+k-1} d(T_j x, T_{j+1} x) \leq \sum_{j=n}^{n+k-1} \alpha_j \leq \sum_{j=n}^{\infty} \alpha_j. \quad (30)$$

It follows that $\{T_n x\}_{n=1}^{\infty}$ is a Cauchy sequence and hence $\lim_{n \rightarrow \infty} T_n x$ exists. We then define $T : X \rightarrow X$ by

$$Tx := \lim_{n \rightarrow \infty} T_n x. \quad (31)$$

It follows then that

$$\begin{aligned} d(T_n x, Tx) &\leq \lim_{k \rightarrow \infty} d(T_n x, T_{n+k} x) \leq \lim_{k \rightarrow \infty} \sum_{j=n}^{n+k-1} d(T_j x, T_{j+1} x) \\ &\leq \lim_{k \rightarrow \infty} \sum_{j=n}^{n+k-1} \alpha_j = \sum_{j=n}^{\infty} \alpha_j \quad \text{for all } x \in X \text{ and for all } n \geq 1. \end{aligned} \quad (32)$$

To see that $\lim_{n \rightarrow \infty} \sup_{x \in X} d(T_n x, Tx) = 0$, let $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} \alpha_n < \infty$, there exists an integer M such that $\sum_{j=n}^{\infty} \alpha_j < \varepsilon$ for all $n \geq M$. For each $x \in X$ and $n \geq M$, we have

$$d(T_n x, Tx) \leq \sum_{j=n}^{\infty} \alpha_j < \varepsilon. \quad (33)$$

That is, $\sup_{x \in X} d(T_n x, Tx) \leq \varepsilon$ for all $n \geq M$. Hence,

$$\lim_{n \rightarrow \infty} \sup_{x \in X} d(T_n x, Tx) = 0. \quad (34)$$

Finally, we assume that T satisfies (KST2). Note that $\lim_{n \rightarrow \infty} d(x_n, \hat{x}) = 0$. Moreover,

$$\begin{aligned} \lim_{n \rightarrow \infty} d(x_n, Tx_n) &\leq \lim_{n \rightarrow \infty} (d(x_n, T_n x_n) + d(T_n x_n, Tx_n)) \\ &\leq \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) + \lim_{n \rightarrow \infty} \sum_{j=n}^{\infty} \alpha_j = 0. \end{aligned} \quad (35)$$

It follows from Condition (KST2) that $\hat{x} = T\hat{x}$.

We are now ready to give an affirmative answer of the problem in the introduction. In fact, we can generalize Theorem CP by using the approach of Theorem P1.

Theorem 8. Let (X, d) be a complete metric space and $k \geq 1$ be a fixed integer. Let q be a positive real number such that $q < 1$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \alpha_n < \infty$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of operators in $\mathcal{G}(X, d, q)$ such that

$$\begin{aligned} d(T_n(u_1, u_2, \dots, u_k), T_{n+1}(u_1, u_2, \dots, u_k)) \\ \leq \alpha_n \quad \text{for all } u_1, u_2, \dots, u_k \in X \text{ and for all } n \geq 1. \end{aligned} \quad (36)$$

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X such that $x_1, x_2, \dots, x_k \in X$ are arbitrary and

$$x_{n+k} := T_n(x_n, x_{n+1}, \dots, x_{n+k-1}) \quad \text{for all } n \geq 1. \quad (37)$$

Then, the following statements are true.

- (a) There exists an element $\hat{x} \in X$ such that $\hat{x} = \lim_{n \rightarrow \infty} x_n$
- (b) There exists an operator $T \in \mathcal{G}(X, d, q)$ such that $\{T_n\}_{n=1}^{\infty}$ converges uniformly to T and $\hat{x} \in \text{PFix}(T)$
- (c) $\text{PFix}(T) = \{\hat{x}\}$, that is, $\hat{x}$ is the only solution of the fixed point problem of Prešić type for T , provided that

$$d(T(u, u, \dots, u), T(v, v, \dots, v)) < d(u, v) \quad \text{for all } u, v \in X. \quad (38)$$

Proof. Suppose that $\{\alpha_n\}$, $\{T_n\}$, and $\{x_n\}$ are given as in the statement of the theorem. Let $\mathbf{X}$ and $\mathbf{d}$ be defined as in Theorem 3 and Remark 4. For each $n \geq 1$, we define $\mathbf{T}_n : \mathbf{X} \rightarrow \mathbf{X}$ by

$$\mathbf{T}_n \mathbf{u} := (u_2, \dots, u_k, T_n(u_1, \dots, u_k)), \quad (39)$$

where $\mathbf{u} := (u_1, \dots, u_k) \in \mathbf{X}$. By Remark 4, we have $\mathbf{T}_n \in \mathcal{H}(\mathbf{X}, \mathbf{d}, r)$ for all $n \geq 1$. Define $\{\mathbf{x}_n\}_{n=1}^{\infty}$ be a sequence in $\mathbf{X}$ by

$$\mathbf{x}_n := (x_n, x_{n+1}, \dots, x_{n+k-1}) \quad \text{for all } n \geq 1. \quad (40)$$

We prove that $\mathbf{x}_{n+1} := \mathbf{T}_n \mathbf{x}_n$ for each $n \geq 1$. To see this, we have

$$\begin{aligned} \mathbf{x}_{n+1} &= (x_{n+1}, x_{n+2}, \dots, x_{n+k}) \\ &= (x_{n+1}, x_{n+2}, \dots, T_n(x_n, x_{n+1}, \dots, x_{n+k-1})) \\ &= \mathbf{T}_n \mathbf{x}_n. \end{aligned} \quad (41)$$

Next, we prove that

$$\mathbf{d}(\mathbf{T}_n \mathbf{u}, \mathbf{T}_{n+1} \mathbf{u}) \leq \frac{\alpha_n}{q} \quad \text{for all } n \geq 1 \text{ and } \mathbf{u} \in \mathbf{X}. \quad (42)$$

In fact, for each $n \geq 1$ and $\mathbf{u} := (u_1, \dots, u_k) \in \mathbf{X}$, we have

$$\begin{aligned} \mathbf{d}(\mathbf{T}_n \mathbf{u}, \mathbf{T}_{n+1} \mathbf{u}) &= \mathbf{d}((u_2, u_3, \dots, u_k, T_n(u_1, \dots, u_k)), \\ &\quad \cdot (u_2, u_3, \dots, u_k, T_{n+1}(u_1, \dots, u_k))) \\ &= q^{-1} d(T_n(u_1, \dots, u_k), T_{n+1}(u_1, \dots, u_k)) \\ &\leq \frac{\alpha_n}{q}. \end{aligned} \quad (43)$$

It follows from Theorem 6 that there exists an element $\hat{\mathbf{x}} := (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_k) \in \mathbf{X}$ such that $\hat{\mathbf{x}} = \lim_{n \rightarrow \infty} \mathbf{x}_n$. In particular, we have $\lim_{n \rightarrow \infty} x_n = \hat{x}_1 = \hat{x}_2 = \dots = \hat{x}_k$. It also follows from Theorem 6 that there exists an operator $\mathbf{S} : \mathbf{X} \rightarrow \mathbf{X}$ such that $\{\mathbf{T}_n\}$ converges uniformly to $\mathbf{S}$. Note that, for each $(u_1, u_2, \dots, u_k) \in \mathbf{X}$, we have

$$\begin{aligned} \mathbf{S}(u_1, u_2, \dots, u_k) &= \lim_{n \rightarrow \infty} \mathbf{T}_n(u_1, u_2, \dots, u_k) \\ &= \lim_{n \rightarrow \infty} (u_2, u_3, \dots, u_k, T_n(u_1, u_2, \dots, u_k)) \\ &= (u_2, u_3, \dots, u_k, \lim_{n \rightarrow \infty} T_n(u_1, u_2, \dots, u_k)). \end{aligned} \quad (44)$$

We now define $T : \mathbf{X} \rightarrow X$ (from $\mathbf{S}$) by

$$T(u_1, u_2, \dots, u_k) := v_k \stackrel{\text{def}}{\iff} \mathbf{S}(u_1, u_2, \dots, u_k) := (v_1, v_2, \dots, v_k). \quad (45)$$

In particular, we have

- (i) $T(u_1, u_2, \dots, u_k) = \lim_{n \rightarrow \infty} T_n(u_1, u_2, \dots, u_k)$,
- (ii) $\mathbf{S}(u_1, u_2, \dots, u_k) = (u_2, u_3, \dots, u_k, T(u_1, u_2, \dots, u_k))$
for all $(u_1, u_2, \dots, u_k) \in \mathbf{X}$.

It follows from each $T_n \in \mathcal{G}(X, d, q)$ and Remark 5 that $T \in \mathcal{G}(X, d, q)$. Hence $\mathbf{S} \in \mathcal{H}(\mathbf{X}, \mathbf{d}, r)$. In particular, $\mathbf{S}$ satisfies Condition KST2 and hence $\hat{\mathbf{x}} = \mathbf{S}\hat{\mathbf{x}}$ by Theorem 6. This implies that

$$(\hat{x}_1, \hat{x}_1, \dots, \hat{x}_1) = \mathbf{S}(\hat{x}_1, \hat{x}_1, \dots, \hat{x}_1) = (\hat{x}_1, \hat{x}_1, \dots, \hat{x}_1, T(\hat{x}_1, \hat{x}_1, \dots, \hat{x}_1)). \quad (46)$$

Hence,

$$T(\hat{x}_1, \hat{x}_1, \dots, \hat{x}_1) = \hat{x}_1. \quad (47)$$

The uniqueness of the fixed point of Prešić type is obvious if the additional hypothesis is assumed. $\square$

3. Applications

We finally discuss some applications of our result.

3.1. Some Convergence Problem of Recursive Real Sequences. We reconsider the following example studied by Ćirić and Prešić [2, Example 1] and give some remark.

Example 9. Let $X := [0, 1] \cup [2, 3]$ be a metric space endowed with the usual metric d and $T : X^2 \rightarrow X$ be defined by

$$T(x, y) := \begin{cases} \frac{x+y}{4} & \text{if } (x, y) \in [0, 1]^2, \\ 1 + \frac{x+y}{4} & \text{if } (x, y) \in [2, 3]^2, \\ \frac{x+y}{4} - \frac{1}{2} & \text{if } (x, y) \in ([0, 1] \times [2, 3]) \cup ([2, 3] \times [0, 1]). \end{cases} \quad (48)$$

It was proved in [2] that

$$d(T(x, y), T(y, z)) \leq \frac{1}{2} \max \{d(x, y), d(y, z)\} \quad (49)$$

for all $x, y \in X$ and $z := T(x, y)$.

The author of [2] claimed that $T \in \mathcal{G}(X, d, 1/2)$. We note that $T \notin \mathcal{G}(X, d, q)$ for all $0 < q < 1$. In fact, let $x := 3$, $y := 2$, and $z := 1$; it follows that

$$\begin{aligned} d(T(x, y), T(y, z)) &= d\left(\frac{9}{4}, \frac{1}{4}\right) \\ &= 2 \text{ while } \max \{d(x, y), d(y, z)\} \\ &= 1. \end{aligned} \quad (50)$$

Using our approach, let $\mathbf{X} := X^2$ and

$$\mathbf{d}(\mathbf{x}, \mathbf{y}) := \max \left\{ \sqrt{2} d(x_1, x_2), 2d(y_1, y_2) \right\}, \quad (51)$$

where $\mathbf{x} := (x_1, x_2) \in \mathbf{X}$ and $\mathbf{y} := (y_1, y_2) \in \mathbf{X}$. Moreover, let $\mathbf{T} : \mathbf{X} \rightarrow \mathbf{X}$ be defined by

$$\mathbf{T}\mathbf{x} := (x_2, T(x_1, x_2)), \quad (52)$$

where $\mathbf{x} := (x_1, x_2) \in \mathbf{X}$. We can follow the proof in Remark 4 to show that

$$\mathbf{d}(\mathbf{T}\mathbf{x}, \mathbf{T}^2\mathbf{x}) \leq \frac{1}{\sqrt{2}} \mathbf{d}(\mathbf{x}, \mathbf{T}\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathbf{X}. \quad (53)$$

Next, we show that $\mathbf{T}$ satisfies Condition (KST2). Note

TABLE 1: A numerical experiment for Example 10.

x_1	0.5	1	1	2	3
x_2	1	1	2	3	3
x_3	0.375	0.5	0.25	2.25	2.5
x_4	0.34375	0.375	0.0625	2.3125	2.375
x_5	0.1796875	0.21875	0.078125	2.140625	2.21875
x_6	0.130859375	0.1484375	0.03515625	2.11328125	2.1484375
x_7	0.077636719	0.091796875	0.028320313	2.063476563	2.091796875
x_8	0.052124023	0.060058594	0.015869141	2.044189453	2.060058594
x_9	0.032440186	0.037963867	0.011047363	2.026916504	2.037963867
x_{10}	0.021141052	0.024505615	0.006729126	2.017776489	2.024505615
x_{11}	0.013395309	0.015617371	0.004444122	2.011173248	2.015617371
x_{12}	0.00863409	0.010030746	0.002793312	2.007237434	2.010030746
x_{13}	0.00550735	0.006412029	0.001809359	2.004602671	2.006412029
x_{14}	0.00353536	0.004110694	0.001150668	2.002960026	2.004110694
x_{15}	0.002260678	0.002630681	0.000740007	2.001890674	2.002630681
x_{16}	0.001449009	0.001685344	0.000472669	2.001212675	2.001685344
x_{17}	0.000927422	0.001079006	0.000303169	2.000775837	2.001079006
x_{18}	0.000594108	0.000691087	0.000193959	2.000497128	2.000691087
x_{19}	0.000380382	0.000442523	0.000124282	2.000318241	2.000442523
x_{20}	0.000243623	0.000283403	7.95603E - 05	2.000203842	2.000283403

that $\text{PFix}(T) = \{0, 2\}$ so $\text{Fix}(\mathbf{T}) = \{(0, 0), (2, 2)\}$. Let $\{\mathbf{u}_n\}$ be a sequence in $\mathbf{X}$ and let $\mathbf{u} \in \mathbf{X}$ be such that $\lim_{n \rightarrow \infty} \mathbf{d}(\mathbf{u}_n, \mathbf{T}\mathbf{u}_n) = \lim_{n \rightarrow \infty} \mathbf{d}(\mathbf{u}_n, \mathbf{u}) = 0$. For each $n \in \mathbb{N}$, we write $\mathbf{u}_n := (u_1^{(n)}, u_2^{(n)})$ and $\mathbf{u} := (u_1, u_2)$. It follows from the definition of $\mathbf{d}$ that

$$\begin{aligned}
 \lim_{n \rightarrow \infty} d(u_1^{(n)}, u_2^{(n)}) &= \lim_{n \rightarrow \infty} d(u_2^{(n)}, T(u_1^{(n)}, u_2^{(n)})) \\
 &= \lim_{n \rightarrow \infty} d(u_1^{(n)}, u_1) \\
 &= \lim_{n \rightarrow \infty} d(u_2^{(n)}, u_2) = 0.
 \end{aligned} \tag{54}$$

In particular, we have $u_1 = u_2$. This implies that $\mathbf{u} = (u_1, u_1)$ and $\lim_{n \rightarrow \infty} d(u_1, T(u_1^{(n)}, u_2^{(n)})) = 0$. We consider the following two cases.

Case 1. $u_1 \in [0, 1]$. In this case, we may assume that $\{u_1^{(n)}\}$ and $\{u_2^{(n)}\}$ are sequences in $[0, 1]$ and hence $T(u_1^{(n)}, u_2^{(n)}) = (u_1^{(n)} + u_2^{(n)})/4$. This implies that $u_1 = (u_1 + u_2)/4 = u_1/2$ and hence $u_1 = 0$. Then, $\mathbf{u} := (u_1, u_2) = (u_1, u_1) = (0, 0) \in \text{Fix}(\mathbf{T})$.

Case 2. $u_1 \in [2, 3]$. In this case, we may assume that $\{u_1^{(n)}\}$ and $\{u_2^{(n)}\}$ are sequences in $[2, 3]$ and hence $T(u_1^{(n)}, u_2^{(n)}) = 1 + (u_1^{(n)} + u_2^{(n)})/4$. This implies that $u_1 = 1 + (u_1 + u_2)/4 = 1 + u_1/2$ and hence $u_1 = 2$. Then $\mathbf{u} := (u_1, u_2) = (u_1, u_1) = (2, 2) \in \text{Fix}(\mathbf{T})$.

Hence, $\mathbf{T} \in \mathcal{H}(\mathbf{X}, \mathbf{d}, 1/\sqrt{2})$. In particular, we can apply this example to our Theorem 8. Note that, since $\text{PFix}(T) = \{0, 2\}$, the condition (7) cannot be omitted for the uniqueness of the solution as claimed in [2].

We now discuss the following convergence problem of real sequences inspired by [6].

Example 10. Suppose that $\{x_n\}$ is a real sequence satisfying the following recursive relation: $x_1, x_2 \in [0, 1] \cup [2, 3]$ and for each $n \geq 1$

$$x_{n+2} := \begin{cases} \frac{x_n + x_{n+1}}{4} & \text{if } (x_n, x_{n+1}) \in [0, 1]^2, \\ 1 + \frac{x_n + x_{n+1}}{4} & \text{if } (x_n, x_{n+1}) \in [2, 3]^2, \\ \frac{x_n + x_{n+1}}{4} - \frac{1}{2} & \text{if } (x_n, x_{n+1}) \in ([0, 1] \times [2, 3]) \cup ([2, 3] \times [0, 1]). \end{cases} \tag{55}$$

It is clear that this example is related to the preceding one. Table 1 shows the numerical experiment regarding to the problem with respect to the initial inputs x_1 and x_2 . For example, if $x_1 := 0.5$ and $x_2 := 1$, then $\lim_{n \rightarrow \infty} x_n = 0$; if $x_1 := 2$ and $x_2 := 3$, then $\lim_{n \rightarrow \infty} x_n = 2$.

3.2. Volterra Type Integral Equations. We discuss a further application of our Theorem 8 in the context of Volterra type integral equations.

Theorem 12. Suppose that $X := C([0, T], \mathbb{R})$ is the space of continuous real-valued functions defined on an interval $[0, T]$, where $T > 0$, equipped with the supremum metric d defined by

$$d(u, v) := \sup \{|u(t) - v(t)| : t \in [0, T]\} \quad \text{for all } u, v \in X. \quad (56)$$

Suppose that $\lambda, \mu : [0, T] \times \mathbb{R} \rightarrow [0, \infty)$ and $f, g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions such that

(i) f and g satisfy the Lipschitz condition for the second argument with constants α and β , respectively, that is,

$$\begin{aligned} |f(t, s) - f(t, s')| &\leq \alpha |s - s'| \text{ and } |g(t, s) - g(t, s')| \\ &\leq \beta |s - s'| \quad \text{for all } t \in [0, T] \text{ and for all } s, s' \in \mathbb{R}, \end{aligned} \quad (57)$$

(ii) $q := \alpha \max_{t \in [0, T]} \int_0^T \lambda(t, s) ds + \beta \max_{t \in [0, T]} \int_0^T \mu(t, s) ds < 1$.

Suppose that $\{h_n\}$ is a sequence in X such that $\sum_{n=1}^{\infty} \alpha_n < \infty$ where $\alpha_n := d(h_n, h_{n+1})$ for all $n \geq 1$. If $u_1, u_2 \in X$ and for each $n \geq 1$

$$\begin{aligned} u_{n+2}(t) &:= h_n(t) + \int_0^T \lambda(t, s) f(s, u_n(s)) ds \\ &\quad + \int_0^T \mu(t, s) g(s, u_{n+1}(s)) ds \quad (t \in [0, T]), \end{aligned} \quad (58)$$

then $u_n \in X$ for all $n \geq 3$ and there exists an element $\hat{u} \in X$ such that $\lim_{n \rightarrow \infty} d(u_n, \hat{u}) = 0$ and $\hat{u}$ is a unique solution of the following Volterra type integral equation

$$\begin{aligned} u(t) &= h(t) + \int_0^T \lambda(t, s) f(s, u(s)) ds \\ &\quad + \int_0^T \mu(t, s) g(s, u(s)) ds \quad (t \in [0, T]), \end{aligned} \quad (59)$$

where $h(t) := \lim_{n \rightarrow \infty} h_n(t)$.

Proof. Now, for each $n \geq 1$, we define $F_n : X \times X \rightarrow X$, by for each $u, v \in X$,

$$\begin{aligned} F_n(u, v)(t) &:= h_n(t) + \int_0^T \lambda(t, s) f(s, u(s)) ds \\ &\quad + \int_0^T \mu(t, s) g(s, v(s)) ds \quad (t \in [0, T]). \end{aligned} \quad (60)$$

To apply our Theorem 8, it is sufficient to prove that

- (a) $F_n \in \mathcal{G}(X, d, q)$ for all $n \geq 1$
- (b) $d(F_n(u_1, u_2), F_{n+1}(u_1, u_2)) = \alpha_n$ for all $n \geq 1$ and for all $u_1, u_2 \in X$

We now prove the statements (a) and (b).

(a) Let $n \geq 1$ and let $u_1, u_2, u_3 \in X$. Then,

$$\begin{aligned} d(F_n(u_1, u_2), F_n(u_2, u_3)) &= \sup_{t \in [0, T]} |F_n(u_1, u_2)(t) - F_n(u_2, u_3)(t)| \\ &\leq \sup_{t \in [0, T]} \int_0^T \lambda(t, s) |f(s, u_1(s)) - f(s, u_2(s))| ds \\ &\quad + \sup_{t \in [0, T]} \int_0^T \mu(t, s) |g(s, u_2(s)) - g(s, u_3(s))| ds \\ &\leq \left(\alpha \max_{t \in [0, T]} \int_0^T \lambda(t, s) ds \right) d(u_1, u_2) \\ &\quad + \left(\beta \max_{t \in [0, T]} \int_0^T \mu(t, s) ds \right) d(u_2, u_3) \\ &\leq q \max \{d(u_1, u_2), d(u_2, u_3)\}. \end{aligned} \quad (61)$$

(b) Let $n \geq 1$ and let $u_1, u_2 \in X$. Then,

$$\begin{aligned} d(F_n(u_1, u_2), F_{n+1}(u_1, u_2)) &= \sup_{t \in [0, T]} |F_n(u_1, u_2)(t) - F_{n+1}(u_1, u_2)(t)| \\ &= \sup_{t \in [0, T]} |h_n(t) - h_{n+1}(t)| \\ &= d(h_n, h_{n+1}) \\ &= \alpha_n. \end{aligned} \quad (62)$$

Note that

$$u_{n+2}(t) = F_n(u_n(t), u_{n+1}(t)) \quad (t \in [0, T]). \quad (63)$$

It follows from our Theorem 8, where $k = 2$, there exists an element $\hat{u} \in X$ such that

$$\lim_{n \rightarrow \infty} d(u_n, \hat{u}) = 0. \quad (64)$$

Moreover,

$$\hat{u} = F(\hat{u}, \hat{u}) \quad (\iff \hat{u} \text{ is a solution of (11)}), \quad (65)$$

where $F : X \times X \rightarrow X$ is defined by

$$\begin{aligned} F(u, v)(t) &:= h(t) + \int_0^T \lambda(t, s) f(s, u(s)) ds \\ &\quad + \int_0^T \mu(t, s) g(s, v(s)) ds \quad (t \in [0, T]), \end{aligned} \quad (66)$$

and $h(t) = \lim_{n \rightarrow \infty} h_n(t)$. Finally, to show that $\hat{u}$ is a unique solution of the Volterra type integral Equation (59), we show that $d(F(u, u), F(v, v)) < d(u, v)$ for all $u, v \in X$. Let $u, v \in X$. Then,

$$\begin{aligned}
d(F(u, u), F(v, v)) &= \sup_{t \in [0, T]} |F(u, u)(t) - F(v, v)(t)| \\
&\leq \sup_{t \in [0, T]} \int_0^T \lambda(t, s) |f(s, u(s)) - f(s, v(s))| ds \\
&\quad + \sup_{t \in [0, T]} \int_0^T \mu(t, s) |g(s, u(s)) - g(s, v(s))| ds \\
&\leq \left(\alpha \max_{t \in [0, T]} \int_0^T \lambda(t, s) ds \right) d(u, v) \\
&\quad + \left(\beta \max_{t \in [0, T]} \int_0^T \mu(t, s) ds \right) d(u, v) \\
&= qd(u, v) < d(u, v).
\end{aligned} \tag{67}$$

This completes the proof. $\square$

4. Conclusion

We show that the fixed point problem of Prešić type (with respect to several variables) can be regarded a fixed point problem (of a single variable) by using a product space approach. With an appropriate metric on the product space, the Ćirić-Prešić operator can be regarded a mapping studied by Kada et al. in the product space. In particular, we deduce the fixed point result under a weaker assumption. We apply our result for the convergence problems of recursive real sequences and the Volterra type integral equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally and significantly in writing this paper. They read and approved the final manuscript.

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