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Research Article

New Fractional Hermite–Hadamard–Mercer Inequalities for Harmonically Convex Function

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In 2003, Mercer presented an interesting variation of Jensen's inequality called Jensen–Mercer inequality for convex function. In the present paper, by employing harmonically convex function, we introduce analogous versions of Hermite–Hadamard inequalities of the Jensen–Mercer type via fractional integrals. As a result, we introduce several related fractional inequalities connected with the right and left differences of obtained new inequalities for differentiable harmonically convex mappings. As an application viewpoint, new estimates regarding hypergeometric functions and special means of real numbers are exemplified to determine the pertinence and validity of the suggested scheme. Our results presented here provide extensions of others given in the literature. The results proved in this paper may stimulate further research in this fascinating area.

1. Introduction

The definition of convexity has been improved, generalized, and expanded in several directions in recent years. In the literature, Jensen's inequality (J-I) and the Hermite–Hadamard's (H-H) inequality are highly familiar results. Several new classes of convex functions along with their respective new variants of (J-I) and (H-H) inequalities are established. One of the well-known and most significant inequalities in mathematical analysis is (J-I) and its related variants. These inequalities are useful in Physics since they provide upper and lower limits for natural phenomena defined by integrals, such as mechanical work. The definition of a classical convex function is as follows:

Definition 1. A function $\phi : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is called a convex function on $(\phi \in K(I))$, if

$$\phi(\zeta\theta + (1-\zeta)\Theta) \leq \zeta\phi(\theta) + (1-\zeta)\phi(\Theta), \tag{1}$$

holds provided that all $\theta, \Theta \in I$ and $\zeta \in [0, 1]$.

Jensen's inequality is the key to success in extracting applications in information theory. It is effective in finding estimates for several quantitative measures in information theory about continuous random variables, see [1–3]. The (J-I) can be stated as a generalization of convex functions as follows:

Theorem 2 (see [4]). *If $\phi \in K(I)$, then*

$$\phi\left(\sum_{j=1}^{m}\omega_j\mathfrak{a}_j\right) \le \sum_{j=1}^{m}\omega_j\phi\left(\mathfrak{a}_j\right), \tag{2}$$

for all $\mathfrak{a}_j \in I$ and $\omega_j \in [0,1]$, $(j=1,2,\cdots,m)$ with $\sum_{j=1}^{m}\omega_j = 1$.

The (H-H) inequality is another well-known inequality in the theory of convex analysis. Several notable results surrounding (H-H) inequality and its significance are compiled by Dragomir et al. in [5].

Theorem 3. *If $\phi \in K(I)$ on the interval $I=[\theta,\Theta]$ with $\theta<\Theta$, then*

$$\phi\left(\frac{\theta+\Theta}{2}\right) \le \frac{1}{\Theta-\theta}\int_{\theta}^{\Theta}\phi(x)dx \le \frac{\phi(\theta)+\phi(\Theta)}{2}. \tag{3}$$

In 2003, Mercer presents a variant of (J-I) which has a great impact on the theory of inequalities known as Jensen–Mercer (J-M) inequality.

Theorem 4 (see [6]). *If $\phi \in K(I)$ on the interval $I=[\theta,\Theta]$, then*

$$\phi\left(\theta+\Theta-\sum_{j=1}^{m}\omega_j\mathfrak{a}_j\right) \le \phi(\theta)+\phi(\Theta)-\sum_{j=1}^{m}\omega_j\phi\left(\mathfrak{a}_j\right), \tag{4}$$

for all $\mathfrak{a}_j \in [\theta,\Theta]$ and $\omega_j \in [0,1]$, $(j=1,2,\cdots,m)$ with $\sum_{j=1}^{m}\omega_j = 1$.

In 2013 [7], Kian and Moslehian introduced new variant (H-H) type inequalities utilizing Mercer concept via convex functions. Recently in 2019, Moradi and Furuichi in [8] worked on some improvements and generalization of (J-M) type inequalities. Then in 2020 [9], Adil et al. gave applications of (J-M) inequality in information theory. He computed new estimates for Csiszár and related divergences. Also, he gave new bounds for Zipf-Mandelbrot Entropy via (J-M) inequality.

Harmonic convex sets are introduced by investigating harmonic means. In 2003, the first harmonic convex set was introduced by Shi and Zhang [10]. The harmonic mean has been important in different fields of pure and applied sciences. Anderson et al. [11] and İşcan [12] introduced a significant class of convex functions known as harmonic convex.

Definition 5 (see [12]). A function $\phi : I \subset \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ is said to be harmonically convex on ($\phi \in H_K(I)$), if

$$\phi\left(\frac{\theta\Theta}{\zeta\theta+(1-\zeta)\Theta}\right) \le \zeta\phi(\Theta)+(1-\zeta)\phi(\theta), \tag{5}$$

for all $\theta,\Theta \in I$ and $\zeta \in [0,1]$ holds.

The harmonic mean is useful in electrical circuit theory and different fields of research. It is well known that the total resistance of a set of parallel resistors can be calculated by adding the reciprocals of each individual resistance value and then taking the reciprocal of the total resistance. For example, if t_1 and t_2 are the resistance of two parallel resistors, the total resistance is

$$T=\frac{t_1t_2}{t_1+t_2}=\frac{1}{2}H(t_1,t_2), \tag{6}$$

which is the half of the harmonic mean [13]. The harmonic mean is also important in the development of parallel algorithms for solving nonlinear problems [14]. The harmonic mean of the effective masses, as well as the three crystallographic directions, is often used to describe a semiconductor's "conductivity effective mass" [15]; see also [16].

Dragomir is the first to introduce (J-I) for $\phi \in H_K(I)$ as:

Theorem 6 (see [17]). *If $\phi \in H_K(I)$ on the interval $I \subseteq (0,\infty)$, then*

$$\phi\left(\frac{1}{\sum_{j=1}^{m}\omega_j/\mathfrak{a}_j}\right) \le \sum_{j=1}^{m}\omega_j\phi\left(\mathfrak{a}_j\right), \tag{7}$$

for all $\mathfrak{a}_j \in I$ and $\omega_j \in [0,1]$, $(j=1,2,\cdots,m)$ with $\sum_{j=1}^{m}\omega_j = 1$

.

In [12], İşcan proved the (H-H) inequality for $\phi \in H_K(I)$ as:

Theorem 7 (see [12]). *Let $I \subseteq (0,\infty)$ be an interval. If $\phi \in H_K(I)$ and $\phi \in L[\theta,\Theta]$ and for all θ, $\Theta \in I$ with $\theta<\Theta$ then*

$$\phi\left(\frac{2\theta\Theta}{\theta+\Theta}\right) \le \frac{\theta\Theta}{\Theta-\theta}\int_{0}^{1}\frac{\phi(x)}{x^2}dx \le \frac{\phi(\theta)+\phi(\Theta)}{2}. \tag{8}$$

Very recently, Baloch et al. [18] present a variant of (J-I) which has a great impact on the theory of inequalities known as (J-M) inequality for $\phi \in H_K(I)$:

Theorem 8 (see [18]). *Let $I=[\theta,\Theta]\subseteq(0,\infty)$ be an interval. If $\phi\in H_K(I)$, then inequality*

$$\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-\sum_{j=1}^{m}\omega_j/\mathfrak{a}_j}\right)\leq\phi(\theta)+\phi(\Theta)-\sum_{j=1}^{m}\omega_j\phi(\mathfrak{a}_j),\tag{9}$$

for all $\mathfrak{a}_j\in[\theta,\Theta]$ and $\omega_j\in[0,1]$, $(j=1,2,\cdots,m)$ with $\sum_{j=1}^{m}\omega_j=1$.

For some recent results connected with (J-M) inequality for $\phi\in H_K(I)$, see [18, 19].

Let us recall some important functions and inequality.

(i) Beta function

$$\beta(\xi_1,\xi_2)=\frac{\Gamma(\xi_1)\Gamma(\xi_2)}{\Gamma(\xi_1+\xi_2)}=\int_0^1\zeta^{\xi_1-1}(1-\zeta)^{\xi_2-1}\,d\zeta,\xi_1,\xi_2>0.\tag{10}$$

(ii) Hypergeometric function: [20]

$${}_2F_1(x,y;z;k)=\frac{1}{\beta(y,z-y)}\int_0^1\zeta^{y-1}(1-\zeta)^{z-y-1}(1-k\zeta)^{-x}\,d\zeta,z>y>0,|k|<1.\tag{11}$$

Lemma 9 (see [21, 22]). *For $0<\alpha\leq1$ and $0\leq x<y$, we have*

$$|x^\alpha-y^\alpha|\leq(y-x)^\alpha.\tag{12}$$

One of the concepts that have played a significant role in the growth of inequality theory in recent years is fractional analysis. Fractional integrals are the most commonly used concept in calculus analysis to obtain new generalizations, extensions, and versions of classical integral inequalities. Since fractional calculus was presented toward the end of the nineteenth century, the subject has become a quickly developing area and has discovered numerous applications in different research fields. Fractional calculus is now concerned with fractional-order integral and derivative operators in real and complex analysis and their applications. Fractional calculus is used in several fields of engineering and science worldwide, including fluid dynamics, electrochemistry, electromagnetics, viscoelasticity, biological population models, optics, and signal processing. It has been used to model physical and engineering processes that are best represented by fractional differential equations.

Now, we give the definition of Riemann-Liouville (RL) integrals which we will use in this paper.

Definition 10. Let $\phi\in L[\theta,\Theta]$. The left and right sided (RL) fractional integrals of order $\alpha>0$ with $\theta\geq0$ are stated as:

$$\begin{aligned}J_{\theta^+}^{\alpha}\phi(r)&=\frac{1}{\Gamma(\alpha)}\int_\theta^r(r-u)^{\alpha-1}\phi(u)\,du,r>\theta,\\J_{\Theta^-}^{\alpha}\phi(r)&=\frac{1}{\Gamma(\alpha)}\int_r^\Theta(u-r)^{\alpha-1}\phi(u)\,du,r<\Theta,\end{aligned}\tag{13}$$

respectively, with $\Gamma(\alpha)=\int_0^\infty e^{-u}u^{\alpha-1}du$ and $J_{\theta^+}^{0}\phi(r)=J_{\Theta^-}^{0}\phi(r)=\phi(r)$.

In recent times, the topic of investigating fractional (H-H) inequalities by employing the Mercer concept along with its applications is worth study, as evident from several publications in this direction (see [23–26]). This study is done by utilizing convex functions. But in this paper, we first time introduce and analyze this concept for harmonic convex functions. In this paper, by using (J-M) inequality, we derive Hermite-Hadamard–Mercer's (H-H-M) inequalities for $\phi\in H_K(I)$ via (RL) fractional integral, and we established several new fractional inequalities pertaining (H-H-M) type inequalities for differentiable harmonically convex mappings. Some applications to special means of positive real numbers will also be provided in Section 4. We hope that the new idea and techniques formulated in the present paper are more invigorating than the accessible ones.

2. (H-H-M) Inequalities for $\phi\in H_K(I)$ via (RL) Fractional Integrals

By using (J-M) inequality, we give the following (H-H-M) inequalities for $\phi\in H_K(I)$.

Theorem 11. *Let $\phi:I=[\theta,\Theta]\subseteq(0,\infty)\longrightarrow\mathbb{R}$ be a function such that $\phi\in L[\theta,\Theta]$ with $0<\theta<\Theta$. If $\phi\in H_K(I)$ on the interval $I=[\theta,\Theta]$, then*

$$\begin{aligned}\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-((x+y)/2xy)}\right)&\leq\phi(\theta)+\phi(\Theta)-\frac{\Gamma(\alpha+1)}{2}\\&\quad\times\left(\frac{xy}{y-x}\right)^\alpha\left\{J_{1/y^+}^{\alpha}(\phi\circ\hbar)\left(\frac{1}{x}\right)+J_{1/x^-}^{\alpha}(\phi\circ\hbar)\left(\frac{1}{y}\right)\right\}\\&\leq\phi(\theta)+\phi(\Theta)-\phi\left(\frac{2xy}{x+y}\right),\end{aligned}\tag{14}$$

$$\begin{aligned}&\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-((x+y)/2xy)}\right)\\&\leq\frac{\Gamma(\alpha+1)}{2}\left(\frac{xy}{y-x}\right)^\alpha\left\{J^{\alpha}_{((1/\theta)+(1/\Theta)-(1/x))^+}(\phi\circ\hbar)\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right)\right.\\&\quad\left.+J^{\alpha}_{((1/\theta)+(1/\Theta)-(1/y))^-}(\phi\circ\hbar)\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{x}\right)\right\}\\&\leq\frac{1}{2}\left[\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-(1/x)}\right)+\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right]\\&\leq\phi(\theta)+\phi(\Theta)-\frac{\phi(x)+\phi(y)}{2},\end{aligned}\tag{15}$$

for all $x,y\in[\theta,\Theta]$, $\alpha>0$, and $\hbar(r)=1/r$, $r\in[1/\Theta,1/\theta]$.

Proof. By employing (J-M) inequality for $\phi \in H_K(I)$, we have

$$\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-((x_1+y_1)/2x_1y_1)}\right) \leq \phi(\theta)+\phi(\Theta)-\frac{\phi(x_1)+\phi(y_1)}{2}, \tag{16}$$

for all $x_1, y_1 \in [\theta, \Theta]$. By changing of the variables $x_1 = xy/(\zeta x+(1-\zeta)y)$, $y_1 = xy/(\zeta y+(1-\zeta)a)$ for all $x, y \in [\theta, \Theta]$, and $\zeta \in [0,1]$ in (16), we obtain

$$\begin{aligned}\phi&\left(\frac{1}{(1/\theta)+(1/\Theta)-((x+y)/2xy)}\right)\\ &\leq \phi(\theta)+\phi(\Theta)-\frac{\phi(xy/(\zeta x+(1-\zeta)y))+\phi(xy/(\zeta y+(1-\zeta)a))}{2}.\end{aligned} \tag{17}$$

Conducting product on both sides of (17) by $\zeta^{\alpha-1}$ and then integrating the obtained inequality w.r.t ζ over $[0,1]$, we have

$$\begin{aligned}\frac{1}{\alpha}&\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-((x+y)/2xy)}\right)\\ &\leq \frac{1}{\alpha}[\phi(\theta)+\phi(\Theta)]-\frac{1}{2}\int_0^1\zeta^{\alpha-1}\left[\phi\left(\frac{xy}{\zeta x+(1-\zeta)y}\right)+\phi\left(\frac{xy}{\zeta y+(1-\zeta)a}\right)\right]d\zeta\\ &= \frac{1}{\alpha}[\phi(\theta)+\phi(\Theta)]-\frac{1}{2}\left(\frac{xy}{y-x}\right)^{\alpha}\\ &\quad\times\left[\int_{1/y}^{1/x}\left(\frac{1}{x}-u\right)^{\alpha-1}\phi\left(\frac{1}{u}\right)du+\int_{1/y}^{1/x}\left(u-\frac{1}{y}\right)^{\alpha-1}\phi\left(\frac{1}{u}\right)du\right]\\ &= \frac{1}{\alpha}[\phi(\theta)+\phi(\Theta)]-\frac{\Gamma(\alpha)}{2}\left(\frac{xy}{y-x}\right)^{\alpha}\left\{J^{\alpha}_{1/y^+}(\phi\circ\hbar)\left(\frac{1}{x}\right)+J^{\alpha}_{1/x^-}(\phi\circ\hbar)\left(\frac{1}{y}\right)\right\}.\end{aligned} \tag{18}$$

That is

$$\begin{aligned}\phi&\left(\frac{1}{(1/\theta)+(1/\Theta)-((x+y)/2xy)}\right)\\ &\leq \phi(\theta)+\phi(\Theta)-\frac{\Gamma(\alpha+1)}{2}\\ &\quad\times\left(\frac{xy}{y-x}\right)^{\alpha}\left\{J^{\alpha}_{1/y^+}(\phi\circ\hbar)\left(\frac{1}{x}\right)+J^{\alpha}_{1/x^-}(\phi\circ\hbar)\left(\frac{1}{y}\right)\right\}, \text{where } \hbar(r)=\frac{1}{r}.\end{aligned} \tag{19}$$

Thus, the first inequality of (14) is proved. Now, we prove the second inequality in (14), since $\phi \in H_K(I)$; then, for $\zeta \in [0,1]$, it yields

$$\begin{aligned}\phi\left(\frac{2xy}{x+y}\right) &= \phi\left(\frac{2}{(\zeta/x)+((1-\zeta)/y)+((1-\zeta)/x)+(\zeta/y)}\right)\\ &\leq \frac{1}{2}\left[\phi\left(\frac{xy}{\zeta y+(1-\zeta)x}\right)+\phi\left(\frac{xy}{\zeta x+(1-\zeta)y}\right)\right].\end{aligned} \tag{20}$$

Conducting product on both sides of (20) by $\zeta^{\alpha-1}$ and then integrating the obtained inequality w.r.t ζ over $[0,1]$, we obtain

$$\begin{aligned}\frac{1}{\alpha}\phi\left(\frac{2xy}{x+y}\right) &\leq \frac{1}{2}\left[\int_0^1\zeta^{\alpha-1}\phi\left(\frac{xy}{\zeta y+(1-\zeta)x}\right)+\phi\left(\frac{xy}{\zeta x+(1-\zeta)y}\right)\right]d\zeta\\ &= \frac{\Gamma(\alpha)}{2}\left(\frac{xy}{y-x}\right)^{\alpha}\left\{J^{\alpha}_{1/y^+}(\phi\circ\hbar)\left(\frac{1}{x}\right)+J^{\alpha}_{1/x^-}(\phi\circ\hbar)\left(\frac{1}{y}\right)\right\},\end{aligned} \tag{21}$$

and then

$$-\phi\left(\frac{2xy}{x+y}\right) \geq -\frac{\Gamma(\alpha+1)}{2}\left(\frac{xy}{y-x}\right)^{\alpha}\left\{J^{\alpha}_{1/y^+}(\phi\circ\hbar)\left(\frac{1}{x}\right)+J^{\alpha}_{1/x^-}(\phi\circ\hbar)\left(\frac{1}{y}\right)\right\}. \tag{22}$$

Adding $\phi(\theta)+\phi(\Theta)$ to both sides of (22), we find the second inequality of (14).

Next for the proof of the inequality (15), take $\phi \in H_K(I)$; we have for any $x, y \in [\theta, \Theta]$

$$\begin{aligned}\phi\left(\frac{2xy}{x+y}\right) &= \phi\left(\frac{1}{1/2((\zeta/x)+((1-\zeta)/y)+((1-\zeta)/x)+(\zeta/y))}\right)\\ &\leq \frac{1}{2}\left[\phi\left(\frac{1}{(\zeta/x)+((1-\zeta)/y)}\right)+\phi\left(\frac{1}{((1-\zeta)/x)+(\zeta/y)}\right)\right]\\ &\leq \frac{\phi(x)+\phi(y)}{2}.\end{aligned} \tag{23}$$

Replacing x and y by $1/(1/\theta)+(1/\Theta)-(1/x)$ and $1/(1/\theta)+(1/\Theta)-(1/y)$, respectively, in (23), we get

$$\begin{aligned}\phi&\left(\frac{1}{(1/\theta)+(1/\Theta)-((x+y)/2xy)}\right)\\ &\leq \frac{1}{2}\left[\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-((\zeta/x)+((1-\zeta)/y))}\right)\right.\\ &\quad\left.+\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-(((1-\zeta)/x)+(\zeta/y))}\right)\right]\\ &\leq \frac{1}{2}\left[\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-(1/x)}\right)+\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right].\end{aligned} \tag{24}$$

Conducting product on both sides of (24) by $\zeta^{\alpha-1}$ and then integrating the obtained inequality w.r.t ζ over $[0,1]$, we have

$$\begin{aligned}\frac{1}{\alpha}&\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-((x+y)/2xy)}\right)\\ &\leq \frac{1}{2}\left[\int_0^1\zeta^{\alpha-1}\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-((\zeta/x)+((1-\zeta)/y))}\right)d\zeta\right.\\ &\quad\left.+\int_0^1\zeta^{\alpha-1}\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-(((1-\zeta)/x)+(\zeta/y))}\right)d\zeta\right]\\ &\leq \frac{1}{2\alpha}\left[\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-(1/x)}\right)+\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right].\end{aligned} \tag{25}$$

It is obvious that

$$\begin{aligned}
&\frac{1}{2}\left[\int_0^1 \zeta^{\alpha-1}\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-((\zeta/x)+((1-\zeta)/y))}\right)d\zeta\right.\\
&\quad\left.+\int_0^1 \zeta^{\alpha-1}\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-(((1-\zeta)/x)+(\zeta/y))}\right)d\zeta\right]\\
&=\frac{\Gamma(\alpha)}{2}\left(\frac{xy}{y-x}\right)^{\alpha}\left[\int_{(1/\theta)+(1/\Theta)-(1/x)}^{(1/\theta)+(1/\Theta)-(1/y)}\left(\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right)-u\right)^{\alpha-1}\phi\left(\frac{1}{u}\right)du\right.\\
&\quad\left.+\int_{(1/\theta)+(1/\Theta)-(1/x)}^{(1/\theta)+(1/\Theta)-(1/y)}\left(u-\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{x}\right)\right)^{\alpha-1}\phi\left(\frac{1}{u}\right)du\right]\\
&=\frac{\Gamma(\alpha)}{2}\left(\frac{xy}{y-x}\right)^{\alpha}\left\{J^{\alpha}_{((1/\theta)+(1/\Theta)-(1/x))^+}(\phi\circ\hbar)\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right)\right.\\
&\quad\left.+J^{\alpha}_{((1/\theta)+(1/\Theta)-(1/y))^-}(\phi\circ\hbar)\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{x}\right)\right\}.
\end{aligned}\tag{26}$$

Using (J-M) inequality for $\phi\in H_K(I)$, we conclude that

$$\begin{aligned}
&\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-((x+y)/2xy)}\right)\\
&\leq\frac{\Gamma(\alpha+1)}{2}\left(\frac{xy}{y-x}\right)^{\alpha}\left\{J^{\alpha}_{((1/\theta)+(1/\Theta)-(1/x))^+}(\phi\circ\hbar)\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right)\right.\\
&\quad\left.+J^{\alpha}_{((1/\theta)+(1/\Theta)-(1/y))^-}(\phi\circ\hbar)\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{x}\right)\right\}\\
&\leq\frac{1}{2}\left[\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-(1/x)}\right)+\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right]\\
&\leq\phi(\theta)+\phi(\Theta)-\frac{\phi(x)+\phi(y)}{2}.
\end{aligned}\tag{27}$$

So, the inequality (15) is proved.□ □

Remark 12. Under the assumptions of Theorem 11 with $\alpha = 1$, one has

$$\begin{aligned}
&\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-((x+y)/2xy)}\right)\leq\phi(\theta)+\phi(\Theta)-\int_0^1\phi\left(\frac{xy}{\zeta x+(1-\zeta)y}\right)d\zeta\\
&\leq\phi(\theta)+\phi(\Theta)-\phi\left(\frac{2xy}{x+y}\right),
\end{aligned}$$

$$\begin{aligned}
&\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-((x+y)/2xy)}\right)\\
&\leq\frac{xy}{y-x}\int_x^y\frac{1}{\zeta^2}\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-(1/\zeta)}\right)d\zeta\\
&\leq\phi(\theta)+\phi(\Theta)-\frac{\phi(x)+\phi(y)}{2},
\end{aligned}\tag{28}$$

for all $x, y\in[\theta,\Theta]$. The proof of remark is proved by Baloch et al. in [18], Theorem 3.5, and [19], Theorem 2.1.

3. Related Variants of (H-H-M) Type Inequalities for $\phi\in H_K(I)$ via (RL) Fractional Integrals

Throughout the paper, we assumed the following assumption.

A_1=Let $\phi : I=[\theta,\Theta]\subseteq(0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function on (θ,Θ) with $0<\theta<\Theta$,

$$\begin{aligned}
I_\phi(\hbar;\alpha,x,y)&=\frac{1}{2}\left[\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-(1/x)}\right)+\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right]\\
&\quad-\frac{\Gamma(\alpha+1)}{2}\left(\frac{xy}{y-x}\right)^{\alpha}\left\{J^{\alpha}_{((1/\theta)+(1/\Theta)-(1/x))^+}(\phi\circ\hbar)\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right)\right.\\
&\quad\left.+J^{\alpha}_{((1/\theta)+(1/\Theta)-(1/y))^-}(\phi\circ\hbar)\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{x}\right)\right\},
\end{aligned}\tag{29}$$

where $x, y\in I$ with $x<y$, $\alpha>0$, and $\hbar(r)=1/r$, $r\in[1/\Theta, 1/\theta]$.

We give the new following lemma for our results.

Lemma 13. *If $\phi'\in L[\theta,\Theta]$ along with assumption A_1, then the following equality for fractional integrals holds:*

$$\begin{aligned}
I_\phi(\hbar;\alpha,x,y)&=\frac{y-x}{2xy}\int_0^1\frac{[\zeta^\alpha-(1-\zeta)^\alpha]}{((1/\theta)+(1/\Theta)-((\zeta/x)+((1-\zeta)/y)))^2}\phi'\\
&\quad\cdot\left(\frac{1}{(1/\theta)+(1/\Theta)-((\zeta/x)+((1-\zeta)/y))}\right)d\zeta.
\end{aligned}\tag{30}$$

Proof. Let $A_\zeta=(1/\theta)+(1/\Theta)-((\zeta/x)+((1-\zeta)/y))$. It suffices to note that

$$\begin{aligned}
I_\phi(\hbar;\alpha,x,y)&=\frac{y-x}{2xy}\int_0^1\frac{[\zeta^\alpha-(1-\zeta)^\alpha]}{A_\zeta^2}\phi'\left(\frac{1}{A_\zeta}\right)d\zeta\\
&=\frac{1}{2}\left[\frac{y-x}{xy}\int_0^1\frac{\zeta^\alpha}{A_\zeta^2}\phi'\left(\frac{1}{A_\zeta}\right)d\zeta-\frac{y-x}{xy}\int_0^1\frac{(1-\zeta)^\alpha}{A_\zeta^2}\phi'\left(\frac{1}{A_\zeta}\right)d\zeta\right]\\
&=\frac{1}{2}[I_1-I_2].
\end{aligned}\tag{31}$$

Integrating by parts, we get

$$\begin{aligned}
I_1&=\frac{y-x}{xy}\int_0^1\frac{\zeta^\alpha}{A_\zeta^2}\phi'\left(\frac{1}{A_\zeta}\right)d\zeta=\zeta^\alpha\phi\left(\frac{1}{A_\zeta}\right)\Bigg|_0^1-\alpha\int_0^1\zeta^{\alpha-1}\phi\left(\frac{1}{A_\zeta}\right)d\zeta\\
&=\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-(1/x)}\right)-\alpha\left(\frac{xy}{y-x}\right)^{\alpha}\int_{(1/\theta)+(1/\Theta)-(1/x)}^{(1/\theta)+(1/\Theta)-(1/y)}\\
&\quad\times\left(\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right)-u\right)^{\alpha-1}\phi\left(\frac{1}{u}\right)du\\
&=\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-(1/x)}\right)-\Gamma(\alpha+1)\left(\frac{xy}{y-x}\right)^{\alpha}J^{\alpha}_{((1/\theta)+(1/\Theta)-(1/x))^+}(\phi\circ\hbar)\\
&\quad\times\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right).
\end{aligned}\tag{32}$$

Similarly, we get

$$I_2=\frac{y-x}{xy}\int_0^1\frac{(1-\zeta)^\alpha}{A_\zeta^2}\phi'\left(\frac{1}{A_\zeta}\right)d\zeta=(1-\zeta)^\alpha\phi\left(\frac{1}{A_\zeta}\right)\Bigg|_0^1+\alpha\int_0^1(1-\zeta)^{\alpha-1}\phi\left(\frac{1}{A_\zeta}\right)d\zeta$$
$$=-\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-(1/y)}\right)+\alpha\left(\frac{xy}{y-x}\right)^\alpha\int_{(1/\theta)+(1/\Theta)-(1/x)}^{(1/\theta)+(1/\Theta)-(1/y)}$$
$$\times\left(u-\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{x}\right)\right)^{\alpha-1}\phi\left(\frac{1}{u}\right)du$$
$$=-\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-(1/y)}\right)+\Gamma(\alpha+1)$$
$$\times\left(\frac{xy}{y-x}\right)^\alpha J^\alpha_{((1/\theta)+(1/\Theta)-(1/y))^-}(\phi\circ\hbar)\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{x}\right). \tag{33}$$

Using (32) and (33) in (31), we get inequality (30).□ □

Corollary 14. *If we choose $\alpha=1$ in Lemma 13, then we have the following equality:*

$$\frac{1}{2}\left[\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-(1/x)}\right)+\phi\left(\frac{1}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right]$$
$$-\frac{xy}{y-x}\int_{(1/\theta)+(1/\Theta)-(1/x)}^{(1/\theta)+(1/\Theta)-(1/y)}\phi\left(\frac{1}{u}\right)du$$
$$=\frac{y-x}{2xy}\int_0^1\frac{(2\zeta-1)}{((1/\theta)+(1/\Theta)-((\zeta/x)+((1-\zeta)/y)))^2}\phi'$$
$$\times\left(\frac{1}{(1/\theta)+(1/\Theta)-((\zeta/x)+((1-\zeta)/y))}\right)d\zeta. \tag{34}$$

Remark 15. If we take $x=\theta$ and $y=\Theta$ in Corollary 14, then the equality (34) reduces to the equality

$$\frac{\phi(\theta)+\phi(\Theta)}{2}-\frac{\theta\Theta}{\Theta-\theta}\int_\theta^\Theta\frac{\phi(x)}{x^2}dx$$
$$=\frac{\theta\Theta(\Theta-\theta)}{2}\int_0^1\frac{(2\zeta-1)}{(\zeta\theta+(1-\zeta)\Theta)^2}\phi'\left(\frac{\theta\Theta}{\zeta\theta+(1-\zeta)\Theta}\right)d\zeta, \tag{35}$$

which is proved by İşcan in [12].

Using Lemma 13, we present the following fractional integral inequality for $|\phi'|^q\in H_K(I)$ as follows.

Theorem 16. *If $|\phi'|^q\in H_K([\theta,\Theta])$ for some fixed $q\geq 1$ and $\phi'\in L[\theta,\Theta]$ along with assumption A_1, then the following inequality for fractional integrals holds:*

$$\left|I_\phi(\hbar;\alpha,x,y)\right|\leq\frac{y-x}{2xy}K_1^{1-(1/q)}(\alpha;x,y)\left(K_1(\alpha;x,y)\left[\left|\phi'(\theta)\right|^q+\left|\phi'(\Theta)\right|^q\right]\right.$$
$$\left.-\left(K_2(\alpha;x,y)\left|\phi'(x)\right|^q+K_3(\alpha;x,y)\left|\phi'(y)\right|^q\right)\right)^{1/q}, \tag{36}$$

where

$$K_1(\alpha;x,y)=\frac{((1/\theta)+(1/\Theta)-(1/y))^{-2}}{\alpha+1}$$
$$\cdot\left[{}_2F_1\left(2,1;\alpha+2;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right.$$
$$\left.+{}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right],$$

$$K_2(\alpha;x,y)=\frac{((1/\theta)+(1/\Theta)-(1/y))^{-2}}{\alpha+2}$$
$$\cdot\left[\frac{1}{\alpha+1}{}_2F_1\left(2,2;\alpha+3;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right.$$
$$\left.+{}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right],$$

$$K_3(\alpha;x,y)=\frac{((1/\theta)+(1/\Theta)-(1/y))^{-2}}{\alpha+2}$$
$$\cdot\left[{}_2F_1\left(2,1;\alpha+3;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right.$$
$$\left.+\frac{1}{\alpha+1}{}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right]. \tag{37}$$

Proof. Let $A_\zeta=(1/\theta)+(1/\Theta)-((\zeta/x)+((1-\zeta)/y))=\zeta((1/\theta)+(1/\Theta)-(1/x))+(1-\zeta)((1/\theta)+(1/\Theta)-(1/y))$. From Lemma 13 and Lemma 9, using the properties of the modulus, the power mean inequality, and $|\phi'|^q\in H_K(I)$, we find that

$$\left|I_\phi(\hbar;\alpha,x,y)\right|\leq\frac{y-x}{2xy}\int_0^1\frac{\left|(1-\zeta)^\alpha-\zeta^\alpha\right|}{A_\zeta^2}\left|\phi'\left(\frac{1}{A_\zeta}\right)\right|d\zeta$$
$$\leq\frac{y-x}{2xy}\left(\int_0^1\frac{\left|(1-\zeta)^\alpha-\zeta^\alpha\right|}{A_\zeta^2}d\zeta\right)^{1-(1/q)}$$
$$\cdot\left(\int_0^1\frac{\left|(1-\zeta)^\alpha-\zeta^\alpha\right|}{A_\zeta^2}\left|\phi'\left(\frac{1}{A_\zeta}\right)\right|^q d\zeta\right)^{(1/q)}$$
$$\leq\frac{y-x}{2xy}\left(\int_0^1\frac{\left[(1-\zeta)^\alpha+\zeta^\alpha\right]}{A_\zeta^2}d\zeta\right)^{1-(1/q)}$$
$$\cdot\left(\int_0^1\frac{\left[(1-\zeta)^\alpha+\zeta^\alpha\right]}{A_\zeta^2}\left[\left|\phi'(\theta)\right|^q+\left|\phi'(\Theta)\right|^q\right.\right.$$
$$\left.\left.-\left(\zeta\left|\phi'(x)\right|^q+(1-\zeta)\left|\phi'(y)\right|^q\right)\right]d\zeta\right)^{1/q}$$
$$\leq\frac{y-x}{2xy}K_1^{1-(1/q)}(\alpha;x,y)\left(K_1(\alpha;x,y)\left[\left|\phi'(\theta)\right|^q+\left|\phi'(\Theta)\right|^q\right]\right.$$
$$\left.-\left(K_2(\alpha;x,y)\left|\phi'(x)\right|^q+K_3(\alpha;x,y)\left|\phi'(y)\right|^q\right)\right)^{1/q}. \tag{38}$$

Calculating $K_1(\alpha;x,y)$, $K_2(\alpha;x,y)$, and $K_3(\alpha;x,y)$, we have

$$
\begin{aligned}
K_1(\alpha;x,y) &= \int_0^1 \frac{\left[(1-\zeta)^\alpha+\zeta^\alpha\right]}{A_\zeta^2}d\zeta \\
&= \left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right)^{-2}\left\{\int_0^1 (1-\zeta)^\alpha\left(1-\zeta\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2}d\zeta\right. \\
&\quad \left.+\int_0^1 \zeta^\alpha\left(1-\zeta\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2}d\zeta\right\} \\
&= \frac{((1/\theta)+(1/\Theta)-(1/y))^{-2}}{\alpha+1}\left[{}_2F_1\left(2,1;\alpha+2;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right. \\
&\quad \left.+{}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right],
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
K_2(\alpha;x,y) &= \int_0^1 \frac{\left[(1-\zeta)^\alpha+\zeta^\alpha\right]}{A_\zeta^2}\zeta d\zeta \\
&= \left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right)^{-2}\left\{\int_0^1 \zeta(1-\zeta)^\alpha\left(1-\zeta\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2}d\zeta\right. \\
&\quad \left.+\int_0^1 \zeta^{\alpha+1}\left(1-\zeta\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2}d\zeta\right\} \\
&= \frac{((1/\theta)+(1/\Theta)-(1/y))^{-2}}{\alpha+2} \\
&\quad \cdot\left[\frac{1}{\alpha+1}{}_2F_1\left(2,2;\alpha+3;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right. \\
&\quad \left.+{}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right],
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
K_3(\alpha;x,y) &= \int_0^1 \frac{\left[(1-\zeta)^\alpha+\zeta^\alpha\right]}{A_\zeta^2}(1-\zeta)d\zeta \\
&= \left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right)^{-2}\left\{\int_0^1 (1-\zeta)^{\alpha+1}\left(1-\zeta\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2}d\zeta\right. \\
&\quad \left.+\int_0^1 \zeta^\alpha(1-\zeta)\left(1-\zeta\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2}d\zeta\right\} \\
&= \frac{((1/\theta)+(1/\Theta)-(1/y))^{-2}}{\alpha+2}\left[{}_2F_1\left(2,1;\alpha+3;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right. \\
&\quad \left.+\frac{1}{\alpha+1}{}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right].
\end{aligned}
\tag{41}
$$

Using (39), (40), and (41) in (38), we get the inequality of (36).□ □

Remark 17. If we take $x=\theta$ and $y=\Theta$ in Theorem 16, then it becomes Theorem 5 proved by Íşcan and Whu in [27].

When $0<\alpha\le 1$, using Lemma 9 and Lemma 13, we can obtain another results for $|\phi'|^q \in H_K(I)$ via fractional integral as follows.

Theorem 18. *If $|\phi'|^q\in H_K([\theta,\Theta])$ for some fixed $q\ge 1$ and $\phi'\in L[\theta,\Theta]$ along with assumption A_1 and $0<\alpha\le 1$, then the following inequality for fractional integrals holds:*

$$
\begin{aligned}
\left|I_\phi(\hbar;\alpha,x,y)\right| &\le \frac{y-x}{2xy}K_1^{1-(1/q)}(\alpha;x,y)\left(K_1(\alpha;x,y)\left[\left|\phi'(\theta)\right|^q+\left|\phi'(\Theta)\right|^q\right]\right. \\
&\quad \left.-\left(K_2(\alpha;x,y)\left|\phi'(x)\right|^q+K_3(\alpha;x,y)\left|\phi'(y)\right|^q\right)\right)^{1/q},
\end{aligned}
\tag{42}
$$

where

$$
\begin{aligned}
K_1(\alpha;x,y) &= \frac{((1/\theta)+(1/\Theta)-(1/y))^{-2}}{\alpha+1} \\
&\quad \cdot\left[{}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right. \\
&\quad -{}_2F_1\left(2,1;\alpha+2;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right) \\
&\quad \left.+{}_2F_1\left(2,1;\alpha+2;\frac{1}{2}\left(1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right)\right], \\
K_2(\alpha;x,y) &= \frac{((1/\theta)+(1/\Theta)-(1/y))^{-2}}{\alpha+2} \\
&\quad \cdot\left[{}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right. \\
&\quad -\frac{1}{\alpha+1}{}_2F_1\left(2,2;\alpha+3;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right) \\
&\quad \left.+\frac{1}{2(\alpha+1)}{}_2F_1\left(2,2;\alpha+3;\frac{1}{2}\left(1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right)\right], \\
K_3(\alpha;x,y) &= \frac{((1/\theta)+(1/\Theta)-(1/y))^{-2}}{\alpha+2} \\
&\quad \cdot\left[\frac{1}{\alpha+1}{}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right. \\
&\quad -{}_2F_1\left(2,1;\alpha+3;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right) \\
&\quad +{}_2F_1\left(2,1;\alpha+3;\frac{1}{2}\left(1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right) \\
&\quad \left.+\frac{1}{2(\alpha+1)}{}_2F_1\left(2,2;\alpha+3;\frac{1}{2}\left(1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right)\right].
\end{aligned}
\tag{43}
$$

Proof. Let $A_\zeta=(1/\theta)+(1/\Theta)-((\zeta/x)+((1-\zeta)/y))=\zeta((1/\theta)+(1/\Theta)-(1/x))+(1-\zeta)((1/\theta)+(1/\Theta)-(1/y))$. From Lemma 13 and Lemma 9, using the properties of the modulus, the power mean inequality, and $|\phi'|^q\in H_K([\theta,\Theta])$, we find that

$$
\begin{aligned}
\left|I_\phi(\hbar;\alpha,x,y)\right| &\le \frac{y-x}{2xy}\int_0^1 \frac{\left|(1-\zeta)^\alpha-\zeta^\alpha\right|}{A_\zeta^2}\left|\phi'\left(\frac{1}{A_\zeta}\right)\right|d\zeta \\
&\le \frac{y-x}{2xy}\left(\int_0^1 \frac{\left|(1-\zeta)^\alpha-\zeta^\alpha\right|}{A_\zeta^2}d\zeta\right)^{1-(1/q)} \\
&\quad \cdot\left(\int_0^1 \frac{\left|(1-\zeta)^\alpha-\zeta^\alpha\right|}{A_\zeta^2}\left|\phi'\left(\frac{1}{A_\zeta}\right)\right|^q d\zeta\right)^{1/q} \\
&\le \frac{y-x}{2xy}\left(\int_0^1 \frac{\left|(1-\zeta)^\alpha-\zeta^\alpha\right|}{A_\zeta^2}d\zeta\right)^{1-(1/q)} \\
&\quad \cdot\left(\int_0^1 \frac{\left|(1-\zeta)^\alpha-\zeta^\alpha\right|}{A_\zeta^2}\left[\left|\phi'(\theta)\right|^q+\left|\phi'(\Theta)\right|^q\right.\right. \\
&\quad \left.\left.-\left(\zeta\left|\phi'(x)\right|^q+(1-\zeta)\left|\phi'(y)\right|^q\right)\right]d\zeta\right)^{1/q} \\
&\le \frac{y-x}{2xy}Z_1^{1-(1/q)}\left(Z_1\left[\left|\phi'(\theta)\right|^q+\left|\phi'(\Theta)\right|^q\right]\right. \\
&\quad \left.-\left(Z_2\left|\phi'(x)\right|^q+Z_3\left|\phi'(y)\right|^q\right)\right)^{1/q},
\end{aligned}
\tag{44}
$$

where

$$Z_1=\int_0^1 \frac{\left|(1-\zeta)^\alpha-\zeta^\alpha\right|}{A_\zeta^2}d\zeta,$$
$$Z_2=\int_0^1 \frac{\left|(1-\zeta)^\alpha-\zeta^\alpha\right|}{A_\zeta^2}\zeta d\zeta, \tag{45}$$
$$Z_3=\int_0^1 \frac{\left|(1-\zeta)^\alpha-\zeta^\alpha\right|}{A_\zeta^2}(1-\zeta)d\zeta.$$

Calculating Z_1, Z_2, and Z_3, by lemma, we have

$$\begin{aligned}
Z_1&=\int_0^1 \frac{\left|(1-\zeta)^\alpha-\zeta^\alpha\right|}{A_\zeta^2}d\zeta=\int_0^{1/2}\frac{(1-\zeta)^\alpha-\zeta^\alpha}{A_\zeta^2}d\zeta+\int_{1/2}^1\frac{\zeta^\alpha-(1-\zeta)^\alpha}{A_\zeta^2}d\zeta\\
&=\int_0^1\frac{\zeta^\alpha-(1-\zeta)^\alpha}{A_\zeta^2}d\zeta+2\int_0^{1/2}\frac{(1-\zeta)^\alpha-\zeta^\alpha}{A_\zeta^2}d\zeta\\
&\le\int_0^1\frac{\zeta^\alpha}{A_\zeta^2}d\zeta-\int_0^1\frac{(1-\zeta)^\alpha}{A_\zeta^2}d\zeta+2\int_0^{1/2}\frac{(1-2\zeta)^\alpha}{A_\zeta^2}d\zeta\\
&=\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right)^{-2}\left\{\int_0^1\zeta^\alpha\left(1-\zeta\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2}d\zeta-\int_0^1(1-\zeta)^\alpha\right.\\
&\quad\times\left(1-\zeta\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2}d\zeta+\int_0^1(1-u)^\alpha\\
&\quad\left.\cdot\left(1-\frac{u}{2}\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2}du\right\}\\
&=\frac{((1/\theta)+(1/\Theta)-(1/y))^{-2}}{\alpha+1}\left[{}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right.\\
&\quad-{}_2F_1\left(2,1;\alpha+2;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\\
&\quad\left.+{}_2F_1\left(2,1;\alpha+2;\frac{1}{2}\left(1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right)\right]=K_1(\alpha;x,y).
\end{aligned}\tag{46}$$

Similarly, we get

$$\begin{aligned}
Z_2&=\int_0^1 \frac{\left|(1-\zeta)^\alpha-\zeta^\alpha\right|}{A_\zeta^2}\zeta d\zeta\le\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right)^{-2}\\
&\quad\cdot\left\{\int_0^1\zeta^{\alpha+1}\left(1-\zeta\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2}d\zeta\right.\\
&\quad-\int_0^1\zeta(1-\zeta)^\alpha\times\left(1-\zeta\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2}d\zeta\\
&\quad\left.+\frac{1}{2}\int_0^1u(1-u)^\alpha\left(1-\frac{u}{2}\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2}du\right\}\\
&=\frac{((1/\theta)+(1/\Theta)-(1/y))^{-2}}{\alpha+2}\left[{}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right.\\
&\quad-\frac{1}{\alpha+1}\times{}_2F_1\left(2,2;\alpha+3;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\\
&\quad\left.+\frac{1}{2(\alpha+1)}{}_2F_1\left(2,2;\alpha+3;\frac{1}{2}\left(1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right)\right]=K_2(\alpha;x,y),
\end{aligned}\tag{47}$$

$$\begin{aligned}
Z_3&=\int_0^1 \frac{\left|(1-\zeta)^\alpha-\zeta^\alpha\right|}{A_\zeta^2}(1-\zeta)d\zeta\le\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right)^{-2}\\
&\quad\cdot\left\{\int_0^1\zeta^\alpha(1-\zeta)\left(1-\zeta\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2}d\zeta\right.\\
&\quad-\int_0^1(1-\zeta)^{\alpha+1}\times\left(1-\zeta\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2}d\zeta\\
&\quad+\int_0^1(1-u)^\alpha\left(1-\frac{u}{2}\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2}du\\
&\quad\left.+\frac{1}{2}\int_0^1u(1-u)^\alpha\left(1-\frac{u}{2}\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2}du\right\}\\
&=\frac{((1/\theta)+(1/\Theta)-(1/y))^{-2}}{\alpha+2}\\
&\quad\cdot\left[\frac{1}{\alpha+1}{}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right.\\
&\quad-{}_2F_1\left(2,1;\alpha+3;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)+{}_2F_1\\
&\quad\cdot\left(2,1;\alpha+3;\frac{1}{2}\left(1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right)\\
&\quad\left.+\frac{1}{2(\alpha+1)}{}_2F_1\left(2,2;\alpha+3;\frac{1}{2}\left(1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right)\right]\\
&=K_3(\alpha;x,y).
\end{aligned}\tag{48}$$

Using (46), (47), and (48) in (44), we get the inequality of (42).□ □

Remark 19. If we take $x=\theta$ and $y=\Theta$ in Theorem 18, then it becomes Theorem 6 proved by İşcan and Whu in [27].

Remark 20. If we take $\alpha=1$, $x=\theta$, and $y=\Theta$ in Theorem 18, then it becomes Theorem 2.6 proved by İşcan in [12].

Theorem 21. *If $|\phi'|^q\in H_K([\theta,\Theta])$ for some fixed $q>1$ and $\phi'\in L[\theta,\Theta]$ along with assumption A_1 and $0<\alpha\le 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
\left|I_\phi(\hbar;\alpha,x,y)\right|&\le\frac{y-x}{2xy}\left(\frac{1}{\alpha p+1}\right)^{1/p}\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right)^{-2}\\
&\quad\cdot\left(\left|\phi'(\theta)\right|^q+\left|\phi'(\Theta)\right|^q-\frac{\left|\phi'(x)\right|^q+\left|\phi'(y)\right|^q}{2}\right)^{1/q}\\
&\quad\times\left[{}_2F_1^{1/p}\left(2p,1;\alpha p+2;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right.\\
&\quad\left.+{}_2F_1^{1/p}\left(2p,\alpha p+1;\alpha p+2;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right)\right],
\end{aligned}\tag{49}$$

where $(1/p)+(1/q)=1$.

Proof. Let $A_\zeta=(1/\theta)+(1/\Theta)-((\zeta/x)+((1-\zeta)/y))=\zeta((1/\theta)+(1/\Theta)-(1/x))+(1-\zeta)((1/\theta)+(1/\Theta)-(1/y))$. From Lemma 13 and Lemma 9, using the Hölder inequality and $|\phi'|^q\in H_K([\theta,\Theta])$, we find

$$\begin{aligned}\left|I_\phi(\hbar;\alpha,x,y)\right| &\le \frac{y-x}{2xy}\left(\int_0^1 \frac{(1-\zeta)^\alpha}{A_\zeta^2}\left|\phi'\left(\frac{1}{A_\zeta}\right)\right|d\zeta+\int_0^1\frac{\zeta^\alpha}{A_\zeta^2}\left|\phi'\left(\frac{1}{A_\zeta}\right)\right|d\zeta\right)\\ &\le \frac{y-x}{2xy}\left\{\left(\int_0^1\frac{(1-\zeta)^{\alpha p}}{A_\zeta^{2p}}d\zeta\right)^{1/p}\left(\int_0^1\left|\phi'\left(\frac{1}{A_\zeta}\right)\right|^q d\zeta\right)^{1/q}\right.\\ &\left.\quad+\left(\int_0^1\frac{\zeta^{\alpha p}}{A_\zeta^{2p}}d\zeta\right)^{1/p}\left(\int_0^1\left|\phi'\left(\frac{1}{A_\zeta}\right)\right|^q d\zeta\right)^{1/q}\right\}\le\frac{y-x}{2xy}\left(Z_4^{1/p}+Z_5^{1/p}\right)\\ &\quad\cdot\left(\int_0^1\left[\left|\phi'(\theta)\right|^q+\left|\phi'(\Theta)\right|^q-\left(\zeta\left|\phi'(x)\right|^q+(1-\zeta)\left|\phi'(y)\right|^q\right)\right]d\zeta\right)^{1/q}\\ &\le\frac{y-x}{2xy}\left(Z_4^{1/p}+Z_5^{1/p}\right)\left(\left|\phi'(\theta)\right|^q+\left|\phi'(\Theta)\right|^q-\frac{\left|\phi'(x)\right|^q+\left|\phi'(y)\right|^q}{2}\right)^{1/q}.\end{aligned}\tag{50}$$

Calculating Z_4 and Z_5, we have

$$\begin{aligned}Z_4&=\int_0^1\frac{(1-\zeta)^{\alpha p}}{A_\zeta^{2p}}d\zeta=\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right)^{-2p}\int_0^1(1-\zeta)^{\alpha p}\\ &\quad\cdot\left(1-\zeta\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2p}d\zeta\\ &=\frac{((1/\theta)+(1/\Theta)-(1/y))^{-2p}}{\alpha p+1}{}_2F_1\\ &\quad\cdot\left(2p,1;\alpha p+2;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right),\end{aligned}\tag{51}$$

$$\begin{aligned}Z_5&=\int_0^1\frac{\zeta^{\alpha p}}{A_\zeta^{2p}}d\zeta=\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right)^{-2p}\int_0^1\zeta^{\alpha p}\\ &\quad\cdot\left(1-\zeta\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2p}d\zeta\\ &=\frac{((1/\theta)+(1/\Theta)-(1/y))^{-2p}}{\alpha p+1}{}_2F_1\\ &\quad\cdot\left(2p,\alpha p+1;\alpha p+2;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right).\end{aligned}\tag{52}$$

Using (51) and (52) in (50), we get the inequality of (49). This completes the proof.□ □

Remark 22. If we take $x=\theta$ and $y=\Theta$ in Theorem 21, then it becomes Theorem 7 proved by Íşcan and Whu in [27].

Theorem 23. *If $|\phi'|^q\in H_K([\theta,\Theta])$ for some fixed $q>1$ and $\phi'\in L[\theta,\Theta]$ along with assumption A_1 and $0<\alpha\le 1$, then following inequality for fractional integrals holds:*

$$\begin{aligned}\left|I_\phi(\hbar;\alpha,x,y)\right|&\le\frac{y-x}{2xy}\left(\frac{1}{\alpha p+1}\right)^{1/p}\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right)^{-2}\left[{}_2F_1\left(2q,1;2;3-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right),\left[\left|\phi'(\theta)\right|^q+\left|\phi'(\Theta)\right|^q\right]\right.\\ &\left.\quad-\frac{{}_2F_1(2q,2;3;1-(((1/\theta)+(1/\Theta)-(1/x))/((1/\theta)+(1/\Theta)-(1/y))))\left|\phi'(x)\right|^q+{}_2F_1(2q,1;3;1-(((1/\theta)+(1/\Theta)-(1/x))/((1/\theta)+(1/\Theta)-(1/y))))\left|\phi'(y)\right|^q}{2}\right]^{1/q}.\end{aligned}\tag{53}$$

Proof. Let $A_\zeta=(1/\theta)+(1/\Theta)-((\zeta/x)+((1-\zeta)/y))=\zeta((1/\theta)+(1/\Theta)-(1/x))+(1-\zeta)((1/\theta)+(1/\Theta)-(1/y))$. From Lemma 13 and Lemma 9, using the Hölder inequality and $|\phi'|^q\in H_K([\theta,\Theta])$, we find

$$\begin{aligned}\left|I_\phi(\hbar;\alpha,x,y)\right|&\le\frac{y-x}{2xy}\int_0^1\frac{\left|(1-\zeta)^\alpha-\zeta^\alpha\right|}{A_\zeta^2}\left|\phi'\left(\frac{1}{A_\zeta}\right)\right|d\zeta\\ &\le\frac{y-x}{2xy}\left(\int_0^1\left|(1-\zeta)^\alpha-\zeta^\alpha\right|^p d\zeta\right)^{1/p}\left(\int_0^1\frac{1}{A_\zeta^{2q}}\left|\phi'\left(\frac{1}{A_\zeta}\right)\right|^q d\zeta\right)^{1/q}\\ &\le\frac{y-x}{2xy}\left(\int_0^1|1-2\zeta|^{\alpha p}d\zeta\right)^{1/p}\\ &\quad\cdot\left(\int_0^1\frac{1}{A_\zeta^{2q}}\left[\left|\phi'(\theta)\right|^q+\left|\phi'(\Theta)\right|^q\right.\right.\\ &\quad\left.\left.-\left(\zeta\left|\phi'(x)\right|^q+(1-\zeta)\left|\phi'(y)\right|^q\right)\right]d\zeta\right)^{1/q}\\ &\le\frac{y-x}{2xy}Z_6^{1/p}\left(Z_7\left[\left|\phi'(\theta)\right|^q+\left|\phi'(\Theta)\right|^q\right]\right.\\ &\quad\left.-\left(Z_8\left|\phi'(x)\right|^q+Z_9\left|\phi'(y)\right|^q\right)\right)^{1/q}.\end{aligned}\tag{54}$$

Calculating Z_6, Z_7, Z_8, and Z_9, we have

$$Z_6=\int_0^1|1-2\zeta|^{\alpha p}d\zeta=\frac{1}{\alpha p+1},\tag{55}$$

$$\begin{aligned}Z_7&=\int_0^1\frac{1}{A_\zeta^{2q}}d\zeta=\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right)^{-2q}\\ &\quad\cdot\int_0^1\left(1-\zeta\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2q}d\zeta\\ &=\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right)^{-2q}{}_2F_1\left(2q,1;2;1\right.\\ &\quad\left.-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right),\end{aligned}\tag{56}$$

$$Z_8=\int_0^1 \frac{\zeta}{A_\zeta^{2q}}d\zeta=\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right)^{-2q}\int_0^1 \zeta \cdot\left(1-\zeta\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2q}d\zeta =\frac{((1/\theta)+(1/\Theta)-(1/y))^{-2q}}{2}{}_2F_1\left(2q,2\,;3\,;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right), \tag{57}$$

$$Z_9=\int_0^1 \frac{(1-\zeta)}{A_\zeta^{2q}}d\zeta=\left(\frac{1}{\theta}+\frac{1}{\Theta}-\frac{1}{y}\right)^{-2q}\int_0^1 (1-\zeta) \cdot\left(1-\zeta\left[1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right]\right)^{-2q}d\zeta =\frac{((1/\theta)+(1/\Theta)-(1/y))^{-2q}}{2}{}_2F_1\left(2q,1\,;3\,;1-\frac{(1/\theta)+(1/\Theta)-(1/x)}{(1/\theta)+(1/\Theta)-(1/y)}\right). \tag{58}$$

Using (55), (56), (57), and (58) in (54), we get the inequality of (53). This completes the proof.□ □

Remark 24. If we take $x=\theta$ and $y=\Theta$ in Theorem 23, then it becomes Theorem 9 proved by Íşcan and Whu in [27].

Remark 25. If we take $\alpha=1$, $x=\theta$, and $y=\Theta$ in Theorem 23, then it becomes Theorem 2.7 proved by Íşcan in [12].

4. Application to Special Means

For positive numbers $\theta>0$ and $\Theta>0$ with $\theta\neq\Theta$.

(1) The arithmetic mean

$$A(\theta,\Theta)=\frac{\theta+\Theta}{2}. \tag{59}$$

(2) The geometric mean

$$G(\theta,\Theta)=\sqrt{\theta\Theta}. \tag{60}$$

(3) The harmonic mean

$$H(\theta,\Theta)=\frac{2\theta\Theta}{\theta+\Theta}. \tag{61}$$

(4) The p-logarithmic mean

$$L_p(\theta,\Theta)=\left[\frac{\Theta^{p+1}-\theta^{p+1}}{(p+1)(\Theta-\theta)}\right]^{1/p}, \tag{62}$$

where$p\in\mathbb{R}\setminus\{-1,0\}$.

Proposition 26. *Let $0<\theta<\Theta$. Then, the following inequalities holds*

$$\left[\frac{1}{2A\left(\theta^{-1},\Theta^{-1}\right)-H^{-1}(x,y)}\right]^{r+2}\leq 2A\left(\theta^{r+2},\Theta^{r+2}\right)-G^2(x,y)L_r^r(x,y)\leq 2A\left(\theta^{r+2},\Theta^{r+2}\right)-H^{r+2}(x,y), \tag{63}$$

$$\left[\frac{1}{2A\left(\theta^{-1},\Theta^{-1}\right)-H^{-1}(x,y)}\right]^{r+2}\leq L_{-(r+2)}^{-(r+2)}\left(2A\left(\theta^{-1},\Theta^{-1}\right)-x^{-1},2A\left(\theta^{-1},\Theta^{-1}\right)-y^{-1}\right)\leq 2A\left(\theta^{r+2},\Theta^{r+2}\right)-A\left(x^{r+2},y^{r+2}\right). \tag{64}$$

Proof. Let $\alpha=1$ and $\phi(x)=x^{r+2}$ where $x>0$, $r\in(-1,\infty)\setminus\{0\}$ in Theorem 11 leads to the desired inequality (63) and (64) immediately, respectively.□ □

5. Conclusion

In this paper, we present the (H-H-M) inequalities involving (RL) fractional integrals for the class of harmonic convex function (instead of convex function) and established some integral inequalities connected with the right and left sides of fractional (H-H-M) type inequalities for differentiable mappings whose derivatives in absolute value are harmonically convex. Some applications to special means have also been presented. Our obtained results are an extension of previously known results. An interesting topic is whether we can use the techniques in this paper to establish the (H-H-M) inequalities for other kinds of convex functions via (RL) fractional integrals. Our ideas and approach may stimulate further research for the researchers working in this field.

Data Availability

Data sharing does not apply to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] N. Mehmood, S. I. Butt, D. Pečarić, and J. Pečarić, “Generalizations of cyclic refinements of Jensen’s inequality by Lidstone’s polynomial with applications in information theory,” *Journal of Mathematical Inequalities*, vol. 14, no. 1, pp. 249–271, 2007.

[2] S. I. Butt, M. K. Bakula, D. Pečarić, and J. Pečarić, “Jensen-Grüss inequality and its applications for the Zipf-Mandelbrot law,” *Mathematical Methods in the Applied Sciences*, vol. 44, no. 2, pp. 1664–1673, 2021.

[3] M. A. Khan, S. Khan, I. Ullah, K. A. Khan, and Y. M. Chu, “A novel approach to the Jensen gap through Taylor’s theorem,”

Mathematical Methods in the Applied Sciences, vol. 44, no. 5, pp. 3324–3333, 2021.

[4] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis, Mathematics and Its Applications (East European Series), 61*, Kluwer Academic Publishers Group, Dordrecht, 1993.

[5] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, Victoria University, RGMIA Monographs, 2000.

[6] A. M. D. Mercer, “A variant of Jensens inequality,” *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 4, 2003.

[7] M. Kian and M. S. Moslehian, “Refinements of the operator Jensen–Mercer inequality,” *Electronic Journal of Linear Algebra*, vol. 26, pp. 742–753, 2013.

[8] H. R. Moradi and S. Furuichi, “Improvement and generalization of some Jensen-Mercer-type inequalities,” *Journal of Mathematical Inequalities*, vol. 14, no. 2, pp. 377–383, 2007.

[9] M. Adil Khan, Z. Husain, and Y. M. Chu, “New estimates for Csiszár divergence and Zipf–Mandelbrot entropy via Jensen–Mercer’s inequality,” *Complexity*, vol. 2020, Article ID 8928691, 8 pages, 2020.

[10] H. N. Shi and J. Zhang, “Some new judgement theorems of Schur geometric and Schur harmonic convexities for a class of symmetric functions,” *Journal of Inequalities and Applications.*, vol. 2013, no. 1, 2013.

[11] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, “Generalized convexity and inequalities,” *Journal of Mathematical Analysis and Applications*, vol. 335, no. 2, pp. 1294–1308, 2007.

[12] Í. İşcan, “Hermite-Hadamard type inequalities for harmonically convex functions,” *Hacettepe Journal of Mathematics and Statistics*, vol. 6, no. 1014, pp. 935–942, 2014.

[13] F. Al-Azemi and O. Calin, “Asian options with harmonic average,” *Applied Mathematics and Information Sciences*, vol. 9, pp. 1–9, 2015.

[14] M. A. Noor, *Advanced Convex Analysis and Optimization*, Lecture Notes, CIIT, 2017.

[15] S. Dimitrijev, *Effective Mass in Semiconductors*, Bart J, Van Zeghbroeck, 1997, http://ece-www.colorado.edu/bart/book/.

[16] S. Rashid, D. Baleanu, and Y. M. Chu, “Some new extensions for fractional integral operator having exponential in the kernel and their applications in physical systems,” *Open Physics*, vol. 18, no. 1, pp. 478–491, 2020.

[17] S. S. Dragomir, “Inequalities of Jensen type for HA-convex functions,” *Analele Universitatii din Oradea Fascicola Matematica*, vol. 27, pp. 103–124, 2020.

[18] I. A. Baloch, A. A. Mughal, Y. M. Chu, A. U. Haq, and M. D. L. Sen, “A varient of Jensen-type inequality and related results for harmonic convex functions,” *Aims Mathematics*, vol. 5, no. 6, pp. 6404–6418, 2020.

[19] I. A. Baloch, A. A. Mughal, Y. M. Chu, A. U. Haq, and M. D. L. Sen, “Improvement and generalization of some results related to the class of harmonically convex functions and applications,” *Journal of Mathematics and Computer Science*, vol. 22, no. 3, pp. 282–294, 2010.

[20] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[21] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integral and Series. Elementary Functions, vol. 1*, Nauka, Moscow, 1981.

[22] J. Wang, C. Zhu, and Y. Zhou, “New generalized Hermite-Hadamard type inequalities and applications to special means,” *Journal of Inequalities and Applications*, vol. 2013, no. 1, 2013.

[23] H. H. Chu, S. Rashid, Z. Hammmouch, and Y. M. Chu, “New fractional estimates for Hermite–Hadamard–Mercer’s type inequalities,” *Alexandria Engineering Journal*, vol. 59, no. 5, pp. 3079–3089, 2020.

[24] S. I. Butt, M. Umar, S. Rashid, A. O. Akdemir, and Y. M. Chu, “New Hermite-Jensen-Mercer type inequalities via k-fractional integrals,” *Advances in Difference Equations*, vol. 2020, no. 1, 2020.

[25] J. Zhao, S. I. Butt, J. Nasir, Z. Wang, and I. Tlili, “Hermite–Jensen–Mercer type inequalities for Caputo fractional derivatives,” *Journal of Function Spaces*, vol. 2020, Article ID 7061549, 11 pages, 2020.

[26] Í. İşcan, “Weighted Hermite-Hadamard-Mercer type inequalities for convex functions,” *Numerical Methods for Partial Differential Equations*, vol. 37, pp. 118–130, 2021.

[27] Í. İşcan and S. Whu, “Hermite-Hadamard type inequalities for harmonically convex functions via fractional integral,” *Applied Mathematics and Computation*, vol. 238, no. 1, pp. 236–244, 2014.