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Research Article

Extended Jensen's Functional for Diamond Integral via Hermite Polynomial

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In this paper, with the help of Hermite interpolating polynomial, extension of Jensen's functional for n-convex function is deduced from Jensen's inequality involving diamond integrals. Special Hermite conditions, including Taylor two-point formula and Lagrange's interpolation, are also deployed to find further extensions of Jensen's functional. The paper also includes discussion on bounds for Grüss-type inequality, Ostrowski-type inequality, and Čebyšev functional associated with newly defined Jensen's functional.

1. Introduction

John Jensen proved Jensen's inequality in [1]. It serves as a tool in discrete and continuous analysis for generating classical inequalities. A discrete variant is as below:

$$\chi\left(\frac{\sum_{m=1}^{n} g_m z_m}{\sum_{m=1}^{n} g_m}\right) \le \frac{\sum_{m=1}^{n} g_m \chi(z_m)}{\sum_{m=1}^{n} g_m}, \tag{1}$$

where $(z_1, \cdots, z_n) \in S$, S is interval in $\mathbb{R}(g_1, \cdots, g_n) \in \mathbb{R}_+^n$ (i.e., nonnegative weights are taken into account in this inequality), and function $\chi : S \longrightarrow \mathbb{R}$ is a convex on S. Steffensen in [2] extended it by using negative weights.

Integral representation of Jensen's inequality in [3] is as follows: If $\tau \in C([a_1, a_2], (a_3, a_4))$ and $\chi \in C((a_3, a_4), \mathbb{R})$ are convex, then

$$\chi\left(\frac{\int_{a_1}^{a_2} \tau(\mathfrak{s}) d\mathfrak{s}}{a_2 - a_1}\right) \le \int_{a_1}^{a_2} \frac{\chi(\tau(\mathfrak{s})) d\mathfrak{s}}{a_2 - a_1}. \tag{2}$$

The researchers have devised several new functions for refinements of Jensen's discrete/integral inequalities. For instance, in [4–7], improvements of the operated version of Jensen's inequality are given. In [8], Aras-Gazic et al. generalized Jensen's inequality via the Hermite polynomial.

Several researchers discussed and applied these inequalities on time scales. In [9], Anwar et al. proved Jensen's inequality for delta integrals.

Suppose $a_1, a_2 \in \mathbb{T}$ s.t. $a_1 < a_2$. Let $\eth \in C_{rd}([a_1, a_2]_{\mathbb{T}}, \mathbb{R})$ assures $\int_{a_1}^{a_2} |\eth(\mathfrak{s})| \Delta\mathfrak{s} > 0$. If $\chi \in C(S, \mathbb{R})$ is convex, an interval $S \subset \mathbb{R}$ and $\hbar \in C_{rd}([a_1, a_2]_{\mathbb{T}}, S)$, then

$$\chi\left(\frac{\int_{a_1}^{a_2} |\eth(\mathfrak{s})| \hbar(\mathfrak{s}) \Delta\mathfrak{s}}{\int_{a_1}^{a_2} |\eth(\mathfrak{s})| \Delta\mathfrak{s}}\right) \le \frac{\int_{a_1}^{a_2} |\eth(\mathfrak{s})| \chi(\hbar(\mathfrak{s})) \Delta\mathfrak{s}}{\int_{a_1}^{a_2} |\eth(\mathfrak{s})| \Delta\mathfrak{s}}. \tag{3}$$

Under a similar hypothesis, in [10], by replacing the delta integral with the nabla integral, the same results are obtained.

Sheng et al. in [11] presented the convex combination of the delta and nabla integrals named as diamond alpha integrals, where $\alpha \in [0, 1]$. For $\alpha = 1$, we get the usual delta integral and nabla integral for $\alpha = 0$. In [12], following Jensen's inequality for the diamond alpha integral is given.

Suppose a time scale $\mathbb{T}a_1, a_2 \in \mathbb{T}$ s.t. $a_1 < a_2$, and $E \in \mathbb{R}$ is an interval $\hbar \in C_{rd}([a_1, a_2]_{\mathbb{T}}, E)$, and $\eth \in C([a_1, a_2], \mathbb{R})$ so that

$$\int_{a_1}^{a_2} |\eth(\mathfrak{s})| \Diamond_\alpha \mathfrak{s} > 0, \quad (4)$$

if $\chi \in C(E, \mathbb{R})$ is convex, then

$$\chi\left(\frac{\int_{a_1}^{a_2} |\eth(\mathfrak{s})| \hbar(\mathfrak{s}) \Diamond_\alpha \mathfrak{s}}{\int_{a_1}^{a_2} |\eth(\mathfrak{s})| \Diamond_\alpha \mathfrak{s}}\right) \leq \frac{\int_{a_1}^{a_2} |\eth(\mathfrak{s})| \chi(\hbar(\mathfrak{s})) \Diamond_\alpha \mathfrak{s}}{\int_{a_1}^{a_2} |\eth(\mathfrak{s})| \Diamond_\alpha \mathfrak{s}}. \quad (5)$$

In [13], the authors introduced the more generalized variant of diamond-alpha integrals; termed as the diamond integral, those are of special concern even for $\mathbb{T} = \mathbb{R}$. These integrals get us nearer in building a true symmetric integral on time scales.

In [14], Jensen's inequality for diamond integrals is proved.

Let $a_1, a_2 \in \mathbb{T}$ with $a_1 < a_2$, $\eth \in C([a_1, a_2]_{\mathbb{T}}, \mathbb{R}^+)$ and $\hbar \in C([a_1, a_2]_{\mathbb{T}}, S)$. Suppose a convex function $\chi \in C(S, \mathbb{R})$ assuring $\int_{a_1}^{a_2} \eth(u) \Diamond u > 0$, where $S = [m_1, m_2]$ and $m_1 = \min_{\mathfrak{s} \in [a_1, a_2]_{\mathbb{T}}} \hbar(\mathfrak{s})$, $m_2 = \max_{\mathfrak{s} \in [a_1, a_2]_{\mathbb{T}}} \hbar(\mathfrak{s})$, then

$$\chi\left(\frac{\int_{a_1}^{a_2} \eth(\mathfrak{s}) \hbar(\mathfrak{s}) \Diamond \mathfrak{s}}{\int_{a_1}^{a_2} \eth(\mathfrak{s}) \Diamond \mathfrak{s}}\right) \leq \frac{\int_{a_1}^{a_2} \eth(\mathfrak{s}) \chi(\hbar(\mathfrak{s})) \Diamond \mathfrak{s}}{\int_{a_1}^{a_2} \eth(\mathfrak{s}) \Diamond \mathfrak{s}}. \quad (6)$$

Jensen's-type linear functional defined on $\mathbb{T}$ is given as the following.

Let $a_1, a_2 \in \mathbb{T}$ with $a_1 < a_2$ and $\eth \in C([a_1, a_2]_{\mathbb{T}}, \mathbb{R})$, $\hbar \in C([a_1, a_2]_{\mathbb{T}}, S)$ and $\zeta \in C(S, \mathbb{R})$ satisfying $\int_{a_1}^{a_2} \eth(\mathfrak{s}) \Diamond \mathfrak{s} \neq 0$, where $S = [m_1, m_2]$ and $m_1 = \min_{\mathfrak{s} \in [a_1, a_2]_{\mathbb{T}}} \hbar(\mathfrak{s})$, $m_2 = \max_{\mathfrak{s} \in [a_1, a_2]_{\mathbb{T}}} \hbar(\mathfrak{s})$; then,

$$J(\zeta) = \frac{\int_{a_1}^{a_2} |\eth(\mathfrak{s})| \zeta(\hbar(\mathfrak{s})) \Diamond \mathfrak{s}}{\int_{a_1}^{a_2} |\eth(\mathfrak{s})| \Diamond \mathfrak{s}} - \zeta\left(\frac{\int_{a_1}^{a_2} |\eth(\mathfrak{s})| \hbar(\mathfrak{s}) \Diamond \mathfrak{s}}{\int_{a_1}^{a_2} |\eth(\mathfrak{s})| \Diamond \mathfrak{s}}\right). \quad (7)$$

Remark 1. Inequality (6) implies that $J(\zeta) \geq 0$ for the family of convex mappings and $J(\zeta) = 0$ for identity or constant functions.

The present study is aimed at extending (7) for the n-convex function with some types of interpolations introduced by Hermite. In the next section, after defining diamond derivative and integral, we recall the Hermite interpolating polynomial along with some of its special forms. Section 3 consists of the paper's main results, and finally, concluding remarks are given in the last section.

2. Preliminaries

2.1. Some Essentials Form Diamond Calculus. A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. It may be connected or not; keeping the time scales' disconnection under consideration, *forward and backward jump operators* $\sigma, \rho : \mathbb{T} \longrightarrow \mathbb{T}$, are defined by $\sigma(\hbar) = \inf \{s \in \mathbb{T} : s > \hbar\}$, and $\rho(\hbar) = \sup \{s \in \mathbb{T} : s < \hbar\}$. In general, $\sigma(\hbar) \geq y$ and $\rho(\hbar) \leq \hbar$. The mappings μ, $\nu : \mathbb{T} \longrightarrow [0, +\infty)$ defined by $\mu(\hbar) = \sigma(\hbar) - \hbar$, and $\nu(\hbar) = \hbar - \rho(\hbar)$ are called in the sequal the forward and backward graininess functions. For the classification of points on time scales, for any $\hbar \varepsilon \mathbb{T}$,

(i) if $\rho(\hbar) = \hbar$, then $\hbar$ is left dense

(ii) if $\sigma(\hbar) = \hbar$, then $\hbar$ is right dense

(iii) if $\rho(\hbar) < \hbar$, then $\hbar$ is left scattered

(iv) if $\sigma(\hbar) > \hbar$, then right scattered

(v) if $\rho(\hbar) = \hbar$ and $\sigma(\hbar) = \hbar$, then $\hbar$ is dense

(vi) if $\rho(\hbar) < \hbar$ and $\sigma(\hbar) > \hbar$, then $\hbar$ is isolated

A mapping $\rho : \mathbb{T} \longrightarrow \mathbb{R}$ is said to be rd-continuous if

(i) it is continuous $\forall \hbar \in \mathbb{T}$ s.t. $\sigma(\hbar) = \hbar$

(ii) left-sided limit is finite $\forall \hbar \in \mathbb{T}$ s.t. $\rho(\hbar) = \hbar$

A set of such functions is denoted by C_{rd}.

Definition 2. Let $\Lambda : \mathbb{T} \longrightarrow \mathbb{R}$ be a mapping and $\hbar \in \mathbb{T}_k^k$ define $\Lambda^\Diamond(\hbar)$ (presumed it as a finite positive number) having characteristic that, for given $\varepsilon > 0$, there exists neighbourhood W of $\hbar$ (i.e. $W = \hbar - s, \hbar + s) \cap T$) for some $\delta > 0$ such that

$$\left| [\Lambda^\sigma(\hbar) - \Lambda(s) + \Lambda(2\hbar - s) - \Lambda^\rho(\hbar)] - \Lambda^\Diamond(\hbar)[\sigma(\hbar) + 2\hbar - 2s - \rho(\hbar)] \right| \leq \varepsilon |\sigma(\hbar) + 2\hbar - 2s - \rho(\hbar)| \quad (8)$$

holds. For all $s \in W$ for which $2\hbar - s \in W$. Then $\Lambda^\Diamond(r)$ is known as diamond derivative of Λ at $\hbar$.

Definition 3. Let $\varrho : \mathbb{T} \longrightarrow \mathbb{R}$ and $a_1, a_2 \in \mathbb{T}$ be a function. The diamond integral of ϱ from a_1 *to* a_2 is given by

$$\int_{a_1}^{a_2} \varrho(\hbar) \Diamond \hbar := \int_{a_1}^{a_2} \gamma(\hbar) \varrho(\hbar) \Delta \hbar + \int_{a_1}^{a_2} (1 - \gamma(\hbar)) \varrho(\hbar) \nabla(\hbar), \quad (9)$$

for all $\hbar \in \mathbb{T}$.

Let $\gamma\varrho$ and $(1 - \gamma)\varrho$ be delta and nabla integrable on $[r, s]_{\mathbb{T}}$, respectively. It is to be noted that the antiderivative is absent for diamond combined derivatives. For $\hbar \in \mathbb{T}_k^k$, $(\int_b^s \varrho(\hbar) \Diamond)^\Diamond \neq \varrho(s)$, in general. The fundamental theorem of calculus also does not hold for diamond integrals.

2.2. Results on Hermite Interpolating Polynomial. Let $-\infty < \mu < \nu < \infty$ and $\mu = a_1 < \cdots < a_r = \nu (r \geq 2)$ be the given r points. For $\mathfrak{I} \in C^n[\mu, \nu]$, there is a $(n-1)^{\text{th}}$ degree polynomial $\sqsubseteq_H(\mathbf{t})$, defined as

$$\sqsubseteq_H(\mathbf{t}) = \sum_{v=1}^{r} \sum_{u=0}^{k_v} H_{uv}(\mathbf{t}) \zeta^{(u)}(a_v). \quad (10)$$

It satisfies the following Hermite condition:

$$\sqsubseteq_H{}^{(u)}(a_v) = \zeta^{(u)}(a_v), 0 \le u \le k_v, 1 \le v \le r, \sum_{v=1}^{r} k_v + r = n. \quad (11)$$

H_{uv} represents essential polynomials of the Hermite basis which satisfy the relations:

$$\begin{aligned} &H_{uv}^{(p)}(a_d) = 0, d \ne v, p = 0, \cdots, k_d, \\ &H_{uv}^{(p)}(a_v) = \delta_{up}, p = 0, \cdots, k_v, \text{for } u = 0, \cdots, k_v, \end{aligned} \quad (12)$$

with $d, v = 1, \cdots, r$ and

$$\delta_{up} = \begin{cases} 1u = p, \\ 0u \ne p. \end{cases} \quad (13)$$

$H_{uv}(\mathfrak{t})$ is given by

$$H_{uv}(\mathfrak{t}) = \frac{1}{u!} \frac{\omega(\mathfrak{t})}{(\mathfrak{t} - a_v)^{k_v+1-u}} \sum_{k=0}^{k_v-u} \frac{1}{k!} \frac{d^k}{d\mathfrak{t}^k} \left(\frac{(\mathfrak{t} - a_v)^{k_v+1}}{\omega(\mathfrak{t})} \right) \Bigg|_{\mathfrak{t}=a_v} (\mathfrak{t} - a_v)^k, \quad (14)$$

with

$$\omega(\mathfrak{t}) = \prod_{v=1}^{r} (\mathfrak{t} - a_v)^{k_v+1}. \quad (15)$$

Hermite conditions encompass the following specific cases.

2.2.1. Lagrange Conditions. Let $r = n, k_v = 0$ for all v, where $1 < v < r$. Then, we have Lagrange polynomial $\sqsubseteq_L(\mathfrak{t})$, satisfying

$$\sqsubseteq_L(a_v) = \zeta(a_v), 1 \le v \le n. \quad (16)$$

2.2.2. Conditions for Type $(\mathfrak{z}, n - \mathfrak{z})$. Let $r = 2, 1 \le \mathfrak{z} \le n - 1$, $k_1 = \mathfrak{z} - 1, k_2 = n - \mathfrak{z} - 1$. Then we have $\sqsubseteq_{(\mathfrak{z},n)}(\mathfrak{t})$ polynomial, satisfying

$$\begin{aligned} &\sqsubseteq_{(\mathfrak{z},n)}^{(u)}(\mu) = \zeta^{(u)}(\mu), 0 \le u \le \mathfrak{z} - 1, \\ &\sqsubseteq_{(\mathfrak{z},n)}^{(u)}(\nu) = \zeta^{(u)}(\nu), 0 \le u \le n - \mathfrak{z} - 1. \end{aligned} \quad (17)$$

2.2.3. Conditions for Taylor's Two-Point Formula. For $n = 2\mathfrak{z}, r = 2, k_1 = k_2 = \mathfrak{z} - 1$, we have Taylor two-point interpolating polynomial $\sqsubseteq_{2T}(\mathfrak{t})$, satisfying

$$\sqsubseteq_{2T}^{(i)}(\mu) = \zeta^{(u)}(\mu), \sqsubseteq_{2T}^{(u)}(\nu) = \zeta^{(u)}(\nu), 0 \le u \le \mathfrak{z} - 1. \quad (18)$$

The next theorem is useful for our results and is given in [15].

Theorem 4. *Suppose we have $-\infty < \mu < \nu < \infty$ and $\mu = a_1 < \cdots < a_r = \nu (r \ge 2)$, and $\mathfrak{I} \in C^n[\mu, \nu]$. Then we have*

$$\zeta(\mathfrak{t}) = \sqsubseteq_H(\mathfrak{t}) + R_H(\zeta, \mathfrak{t}), \quad (19)$$

where $\sqsubseteq_H(\mathfrak{t})$ is the Hermite interpolating polynomial as defined in (10) and $R_H(\zeta, \mathfrak{t})$ denotes the remainder given by

$$R_H(\zeta, \mathfrak{t}) = \int_{\mu}^{\nu} G_{H,n}(\mathfrak{t}, s) \zeta^{(n)}(s) ds; \quad (20)$$

$G_{H,n}(\mathfrak{t}, s)$ is

$$G_{H,n}(\mathfrak{t}, s) = \begin{cases} \displaystyle\sum_{v=1}^{b} \sum_{u=0}^{k_v} \frac{(a_v - h)^{n-u-1}}{(n - u - 1)!} H_{uv}(\mathfrak{t}), & s \le \mathfrak{t}, \\ \displaystyle -\sum_{v=b+1}^{r} \sum_{u=0}^{k_v} \frac{(a_v - h)^{n-u-1}}{(n - u - 1)!} H_{uv}(\mathfrak{t}), & s \ge \mathfrak{t}, \end{cases} \quad (21)$$

for all $a_b \le s \le a_{b+1}, b = 0, \cdots, r$ with $a_0 = \mu$ and $a_{r+1} = \nu$.

Remark 5. By imposing the Lagrange conditions, Theorem 4 takes the form

$$\zeta(\mathfrak{t}) = \sqsubseteq_L(\mathfrak{t}) + R_L(\zeta, \mathfrak{t}). \quad (22)$$

Here, $\sqsubseteq_L(\mathfrak{t})$ represents the Lagrange polynomial, which is

$$\sqsubseteq_L(\mathfrak{t}) = \sum_{v=1}^{n} \prod_{\substack{k=1 \\ k \ne v}}^{n} \left(\frac{\mathfrak{t} - a_k}{a_v - a_k} \right) \zeta(a_v), \quad (23)$$

and $R_L(\zeta, \mathfrak{t})$ is the remainder, defined by

$$R_L(\zeta, \mathfrak{t}) = \int_{\mu}^{\nu} G_L(\mathfrak{t}, s) \zeta^{(n)}(s) ds, \quad (24)$$

with

$$G_L(\mathfrak{t}, s) = \frac{1}{(n-1)!} \begin{cases} \displaystyle\sum_{v=1}^{b} (a_v - s)^{n-1} \prod_{\substack{k=1 \\ k \ne v}}^{n} \left(\frac{\mathfrak{t} - a_k}{a_v - a_k} \right) s \le \mathfrak{t}, \\ \displaystyle -\sum_{v=1}^{b} (a_v - s)^{n-1} \prod_{\substack{k=1 \\ k \ne v}}^{n} \left(\frac{\mathfrak{t} - a_k}{a_v - a_k} \right) s \le \mathfrak{t}, \end{cases} \quad (25)$$

$a_b \le s \le a_{b+1}, b = 1, \cdots, n - 1$ along $a_1 = \mu$ and $a_n = \nu$.

Remark 6. Similarly, by imposing $(\mathfrak{z}, n - \mathfrak{z})$ conditions on Theorem 4, one gets

$$\zeta(\mathfrak{t}) = \sqsubseteq_{(\mathfrak{z},n)}(\mathfrak{t}) + R_{(\mathfrak{z},n)}(\zeta, \mathfrak{t}), \quad (26)$$

where

$$\sqsubseteq_{(\mathfrak{z},n)}(\mathbf{t}) = \sum_{u=0}^{\mathfrak{z}-1} \xi_u(\mathbf{t})\zeta^{(u)}(\mu) + \sum_{u=0}^{n-\mathfrak{z}-1} \eta_u(\mathbf{t})\zeta^{(u)}(\nu), \tag{27}$$

with

$$\xi_u(\mathbf{t}) = \frac{1}{u!}(\mathbf{t}-\mu)^u \left(\frac{\mathbf{t}-\nu}{\mu-\nu}\right)^{(n-\mathfrak{z})} \sum_{k=0}^{\mathfrak{z}-1-u} \binom{n-\mathfrak{z}+k-1}{k} \left(\frac{\mathbf{t}-\mu}{\nu-\mu}\right)^k, \tag{28}$$

$$\eta_u(\mathbf{t}) = \frac{1}{u!}(\mathbf{t}-\nu)^u \left(\frac{\mathbf{t}-\mu}{\nu-\mu}\right)^{\mathfrak{z}} \sum_{k=0}^{n-\mathfrak{z}-1-u} \binom{\mathfrak{z}+k-1}{k} \left(\frac{\mathbf{t}-\nu}{\mu-\nu}\right)^k. \tag{29}$$

The remainder $R_{(\mathfrak{z},n)}(\zeta,\mathbf{t})$ is

$$R_{(\mathfrak{z},n)}(\zeta,\mathbf{t}) = \int_\mu^\nu G_{(m,n)}(\mathbf{t},s)\zeta^{(n)}(s)ds, \tag{30}$$

with

$$G_{(\mathfrak{z},n)}(\mathbf{t},s) = \begin{cases} \sum_{\nu=0}^{\mathfrak{z}-1}\left[\sum_{l=0}^{\mathfrak{z}-1-\nu}\binom{n-\mathfrak{z}+l-1}{l}\left(\frac{\mathbf{t}-\nu}{\nu-\mu}\right)^l\right] \times \frac{(\mathbf{t}-\mu)^\nu(\mu-s)^{n-\nu-1}}{\nu!(n-\nu-1)!}\left(\frac{\nu-\mathbf{t}}{\nu-\mu}\right)^{n-\mathfrak{z}} & \mu \le s \le \mathbf{t} \le \nu, \\ -\sum_{u=0}^{n-\mathfrak{z}-1}\left[\sum_{\mathfrak{r}=0}^{n-\mathfrak{z}-u-1}\binom{\mathfrak{z}+\mathfrak{r}-1}{\mathfrak{r}}\left(\frac{\nu-\mathbf{t}}{\nu-\mu}\right)^{\mathfrak{r}}\right] \times \frac{(\mathbf{t}-\nu)^u(\nu-s)^{n-u-1}}{u!(n-u-1)!}\left(\frac{\mathbf{t}-\mu}{\nu-\mu}\right)^e & \mu \le \mathbf{t} \le s \le \nu. \end{cases} \tag{31}$$

Remark 7. Theorem 4 in the form of the Taylor two-point formula becomes

$$\zeta(\mathbf{t}) = \sqsubseteq_{2T}(\mathbf{t}) + R_{2T}(\zeta,\mathbf{t}), \tag{32}$$

where the Taylor two-point interpolating polynomial $\sqsubseteq_{2T}(\mathbf{t})$ is defined by

$$\begin{aligned}\sqsubseteq_{2T}(\mathbf{t}) = \sum_{u=0}^{\mathfrak{z}-1}\sum_{k=0}^{\mathfrak{z}-1-u}\binom{\mathfrak{z}+k-1}{k}\Bigg[&\frac{(\mathbf{t}-\mu)^u}{u!}\left(\frac{\mathbf{t}-\nu}{\mu-\nu}\right)^{\mathfrak{z}}\left(\frac{\mathbf{t}-\mu}{\nu-\mu}\right)^k\zeta^{(u)}(\mu) \\ &+ \frac{(\mathbf{t}-\nu)^u}{u!}\left(\frac{\mathbf{t}-\mu}{\nu-\mu}\right)^{\mathfrak{z}}\left(\frac{\mathbf{t}-\nu}{\mu-\nu}\right)^k\zeta^{(u)}(\nu)\Bigg],\end{aligned} \tag{33}$$

and $R_{2T}(\zeta,\mathbf{t})$ is

$$R_{2T}(\zeta,\mathbf{t}) = \int_\mu^\nu G_{2T}(\mathbf{t},s)\zeta^{(n)}(s)ds, \tag{34}$$

with

$$G_{2T}(\mathbf{t},s) = \begin{cases} \frac{(-1)^{\mathfrak{z}}}{(2\mathfrak{z}-1)!} l^{\mathfrak{z}}(\mathbf{t},s)\sum_{\nu=0}^{\mathfrak{z}-1}\binom{\mathfrak{z}-1+\nu}{\nu}(\mathbf{t}-s)^{\mathfrak{z}-1-\nu}\mathfrak{r}^\nu(\mathbf{t},s), & s \le \mathbf{t}, \\ \frac{(-1)^{\mathfrak{z}}}{(2\mathfrak{z}-1)!}\mathfrak{r}^{\mathfrak{z}}(\mathbf{t},s)\sum_{\nu=0}^{\mathfrak{z}-1}\binom{\mathfrak{z}-1+\nu}{\nu}(s-\mathbf{t})^{\mathfrak{z}-1-\nu}l^\nu(\mathbf{t},s) & \mathbf{t} \le s, \end{cases} \tag{35}$$

where $l(\mathbf{t},s) = (s-\mu)(\nu-\mathbf{t})/\nu-\mu$, $\mathfrak{r}(\mathbf{t},s) = l(\mathbf{t},s)$, for all $\mathbf{t}, s \in [\mu,\nu]$.

3. Main Results

3.1. Extension of Jensen's Functional via Hermite Polynomial. Here, we prove our key identity regarding the extension of Jensen's functional.

Theorem 8. *Let $n \in \mathbb{N}$ and $\zeta \in C^n[\mu,\nu]$ be a convex function. Then for all $\mathbf{t} \in [\mu,\nu]$, we have*

$$J(\zeta) = \sum_{\nu=1}^{r}\sum_{u=0}^{k_\nu} J(H_{u\nu}(\mathbf{t}))\zeta^{(u)}(a_\nu) + \int_\mu^\nu J(G_{H,n}(\mathbf{t},s))\zeta^n(s)ds, \tag{36}$$

where $J(G_{H,n}(\mathbf{t},s))$ is defined as

$$J(G_{H,n}(\mathbf{t},s)) = \frac{\int_{a_1}^{a_2}|h(\mathbf{t})|G_{H,n}(f(\mathbf{t}),s)\Diamond\mathbf{t}}{\int_{a_1}^{a_2}|h(\mathbf{t})|\Diamond\mathbf{t}} - G_{H,n}\left(\frac{\int_{a_1}^{a_2}|h(\mathbf{t})|f(\mathbf{t})\Diamond\mathbf{t}}{\int_{a_1}^{a_2}|h(\mathbf{t})|\Diamond\mathbf{t}}, s\right). \tag{37}$$

Proof. Substitute (19) in (7), then the linearity of $J(\cdot)$ gives us (36). □

Remark 9. Instead of using (19) in (7), if we use (22), (26), or (32) in (7), then extended results similar to Theorem 8 are obtained for Lagrange conditions, type $(\mathfrak{z}, n-\mathfrak{z})$ conditions, and two-point Taylor conditions, respectively, in the following form:

$$J(\zeta) = \sum_{v=1}^{n} \zeta(a_v) J\left(\prod_{\substack{k=1 \\ k\neq v}}^{n} \left(\frac{\mathbf{t}-a_k}{a_v-a_k}\right)\right) + \int_{\mu}^{\nu} J(G_L(\mathbf{t},s))\zeta^{n}(s)ds,$$

$$J(\zeta) = \sum_{u=0}^{\mathfrak{z}-1} \zeta^{(u)}(\mu) J(\xi_u(t)) + \sum_{u=o}^{n-\mathfrak{z}-1} \zeta^{(u)}(\nu) J(\eta_u(\mathbf{t})) + \int_{\mu}^{\nu} J\left(G_{(\mathfrak{z},n)}(\mathbf{t},s)\right)\zeta^{(n)}(s)ds,$$

$$J(\zeta) = \sum_{u=0}^{\mathfrak{z}-1}\sum_{k=0}^{\mathfrak{z}-1-u} \binom{\mathfrak{z}+k-1}{k}\left[J\left(\frac{(\mathbf{t}-\mu)^u}{u!}\left(\frac{\mathbf{t}-\nu}{\mu-\nu}\right)^{\mathfrak{z}}\left(\frac{\mathbf{t}-\mu}{\nu-\mu}\right)^{k}\right)\cdot\zeta^{(u)}(\mu) + J\left(\frac{(\mathbf{t}-\nu)^u}{u!}\left(\frac{\mathbf{t}-\mu}{\nu-\mu}\right)^{\mathfrak{z}}\left(\frac{\mathbf{t}-\nu}{\mu-\nu}\right)^{k}\right)\zeta^{(u)}(\nu)\right] + \int_{\mu}^{\nu} J\left(G_{(2T)}(\mathbf{t},s)\right)\zeta^{(n)}(s)ds. \tag{38}$$

Theorem 10. *Presume n be a natural number and $G_{H,n}$ be a Green function given in (21) satisfying*

$$J(G_{H,n}(\mathbf{t},s)) \geq 0, \tag{39}$$

for all $\mathbf{t}\in[\mu,\nu]$. *Then for every n-convex* $\zeta:[\mu,\nu]\longrightarrow\mathbb{R}$,

$$J(\zeta) \geq J(B) \tag{40}$$

holds, where

$$B(\mathbf{t}) = \sum_{v=1}^{r}\sum_{u=0}^{k_v} \zeta^{(u)}(a_v) H_{uv}(\mathbf{t}). \tag{41}$$

Proof. As the function ζ is n-convex, $\zeta^{n}(\mathbf{t}) \geq 0$ for all $\mathbf{t}\in[\mu,\nu]$; hence,

$$J(G_{H,n}(\mathbf{t},s))\zeta^{n}(\mathbf{t}) \geq 0. \tag{42}$$

Substituting (42) in (36), we have

$$J(\zeta) - \sum_{v=1}^{r}\sum_{u=0}^{k_v} \zeta^{(u)}(a_v) J(H_{uv}(\mathbf{t})) \geq 0. \tag{43}$$

Since J is linear, (43) can be written as

$$J(\zeta) - J(B) \geq 0. \tag{44}$$

Hence,

$$J(\zeta) \geq J(B). \tag{45}$$

□

Remark 11. (40) is reversed if (39) is reversed.

Utilization of the Lagrange condition produced the result given below.

Corollary 12. *Presume n is a natural number and G_L defined as in (25) satisfying*

$$J(G_L(\mathbf{t},s)) \geq 0, \tag{46}$$

for all $\mathbf{t}\in[\mu,\nu]$. *Then for every n-convex function* $\zeta:[\mu,\nu]\longrightarrow\mathbb{R}$,

$$J(\zeta) \geq J(B), \tag{47}$$

where

$$B(\mathbf{t}) = \sum_{v=1}^{n} \zeta(a_v) \prod_{\substack{k=1 \\ k\neq v}}^{n}\left(\frac{\mathbf{t}-a_k}{a_v-a_k}\right). \tag{48}$$

The use of type $(\mathfrak{z}, n-\mathfrak{z})$ condition yields the following result.

Corollary 13. *Let n be a natural number and $G_{(\mathfrak{z},n)}$ be defined in (31). Suppose ξ_u,η_u are defined as in (28) and (29), respectively, satisfying*

$$J\left(G_{(\mathfrak{z},n)}(\mathbf{t},s)\right) \geq 0, \tag{49}$$

for all $\mathbf{t}\in[\mu,\nu]$. *Then for every n-convex function* $\zeta:[\mu,\nu]\longrightarrow\mathbb{R}$, *we have*

$$J(\zeta) \geq J(B), \tag{50}$$

where

$$B(\mathbf{t}) = \sum_{u=0}^{\mathfrak{z}-1} \zeta^{(u)}(\mu)\xi_u(\mathbf{t}) + \sum_{u=o}^{n-\mathfrak{z}-1} \zeta^{(u)}(\nu)\eta_u(\mathbf{t}). \tag{51}$$

The application of two-point Taylor conditions gives a result as below.

Corollary 14. *Let $n\in\mathbb{N}$ and G_{2T} be the Green function defined as in (35), satisfying*

$$J(G_{2T}(\mathbf{t},s)) \geq 0, \tag{52}$$

for all $\mathbf{t}\in[\mu,\nu]$. *Then for n-convex function* $\zeta:[\mu,\nu]\longrightarrow\mathbb{R}$,

$$J(\zeta) \geq J(B), \tag{53}$$

where

$$B(\mathfrak{t}) = \sum_{u=0}^{\mathfrak{z}-1}\sum_{k=0}^{\mathfrak{z}-1-u}\binom{\mathfrak{z}+k-1}{k}\left[\frac{(\mathfrak{t}-\mu)^u}{u!}\left(\frac{\mathfrak{t}-\nu}{\mu-\nu}\right)^{\mathfrak{z}} \cdot \left(\frac{\mathfrak{t}-\mu}{\nu-\mu}\right)^k \zeta^{(u)}(\mu) + \frac{(\mathfrak{t}-\nu)^u}{u!}\left(\frac{\mathfrak{t}-\mu}{\nu-\mu}\right)^{\mathfrak{z}}\left(\frac{\mathfrak{t}-\nu}{\mu-\nu}\right)^k \zeta^{(u)}(\nu)\right]. \tag{54}$$

Remark 15. Technique to prove Corollaries 12–14 is the same as proof of Theorem 10, where we use n convexity of ζ and linearity of J.

Theorem 16. *In addition to presumptions of Theorem 10, consider $h \in C([a_1, a_2]_{\mathbb{T}}, \mathbb{R}^+)$ and $B : [\mu, \nu] \longrightarrow \mathbb{R}$ is convex, then*

$$J(\zeta(\mathfrak{t})) \geq 0, \tag{55}$$

for all $\mathfrak{t} \in [\mu, \nu]$.

Proof. As $B(\mathfrak{t})$ is convex for all $\mathfrak{t} \in [\mu, \nu]$, so Remark 1, inferred $J(B(\mathfrak{t})) \geq 0$. Hence, (40) implies $J(\zeta(\mathfrak{t})) \geq 0$. □

Remark 17. Relation (55) in Theorem 16 is an extension of Jensen's inequality (6).

3.2. Bounds for Identities Associated to Generalization of Jensen's Functional. Here, we utilize Čebyšev functional and Grüss-type inequalities to present few important results. Let two functions $g_1, g_2 : [\imath, \jmath] \longrightarrow \mathbb{R}$ be Lebesgue integrable, and the Čebyšev functional is

$$|Y(g_1, g_2)| = \frac{1}{\jmath-\imath}\int_{\imath}^{\jmath} g_1(\mathfrak{t})g_2(\mathfrak{t})d\mathfrak{t} - \frac{1}{\jmath-\imath}\int_{i}^{\jmath} g_1(\mathfrak{t})d\mathfrak{t} \cdot \frac{1}{\jmath-\imath}\int_{\imath}^{\jmath} g_2(\mathfrak{t})d\mathfrak{t}. \tag{56}$$

The next two theorems are given in [16].

Theorem 18. *Let $g_1, g_2 : [\imath, \jmath] \longrightarrow \mathbb{R}$ be functions such that g_1 is Lebesgue integrable and g_2 be the function which is absolutely continuous along $(\cdot-\imath)(\jmath-\cdot)[g_2']^2 \in L[\imath, \jmath]$. Then, we have*

$$|Y(g_1, g_2)| \leq \frac{1}{\sqrt{2}}[Y(g_1, g_1)]^{1/2}\frac{1}{\sqrt{j-i}}\left(\int_{\imath}^{\jmath}(y-\imath)(j-y)\left[g_2'(y)\right]^2 dy\right)^{1/2}. \tag{57}$$

$1/\sqrt{2}$ in (57) is possibly the best constant.

Theorem 19. *Let $g_1, g_2 : [\imath, \jmath] \longrightarrow \mathbb{R}$ be functions, such that g_1 is absolutely continuous with $g_1' \in L_\infty[\imath, \jmath]$ and g_2 is monotonic nondecreasing on $[\imath, \jmath]$, then*

$$|Y(g_1, g_2)| \leq \frac{1}{2(j-i)}\left\|g_1'\right\|_\infty \int_{\imath}^{\jmath}(y-i)(j-y)dg_2(y). \tag{58}$$

The best possible constant is 1/2 in (58).
We denote

$$\psi(s) = J(G_{H,n}(x, s)). \tag{59}$$

The Čebyšev functional is given as

$$Y(\psi, \psi) = \frac{1}{\nu-\mu}\int_{\mu}^{\nu}\psi^2(s)ds - \left(\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\psi(s)ds\right)^2. \tag{60}$$

Theorem 20. *Let $\zeta : [\mu, \nu] \longrightarrow \mathbb{R}$ be such that $\zeta \in C^n[\mu, \nu]$ for $n \in \mathbb{N}$ with $(\cdot-\mu)(\nu-\cdot)[\zeta^{(n+1)}]^2 \in L[\mu, \nu]$ and $\mathfrak{t} \in [\mu, \nu]$ and $G_{H,n}$, ψ and Y are defined in (21), (59), and (60), respectively. Then,*

$$J(\zeta) = \sum_{v=1}^{r}\sum_{u=0}^{k_v}\zeta^{(u)}(a_v)J(H_{uv}(x)) + \frac{\zeta^{(n-1)}(\nu) - \zeta^{(n-1)}(\mu)}{\nu-\mu}\int_{\mu}^{\nu}\psi(s)ds + \mathfrak{R}_n(\mu, \nu; \zeta), \tag{61}$$

where the remainder satisfies $\mathfrak{R}_n(\mu, \nu; \zeta)$ the estimation

$$|\mathfrak{R}_n(\mu, \nu; \zeta)| \leq [Y(\psi, \psi)]^{1/2}\sqrt{\frac{\nu-\mu}{2}}\left|\int_{\mu}^{\nu}(s-\mu)(\nu-s)\left[\zeta^{(n+1)}(s)\right]^2 ds\right|^{1/2}. \tag{62}$$

Proof. We use Theorem 18 for $g_1 \longrightarrow \psi$ and $g_2 \longrightarrow \zeta^n$, to obtain

$$\left|\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\psi(s)\zeta^n(s)ds - \frac{1}{\nu-\mu}\int_{\mu}^{\nu}\psi(s)ds \cdot \frac{1}{\nu-\mu}\int_{\mu}^{\nu}\zeta^n(s)ds\right| \leq \frac{1}{\sqrt{2}}[Y(\psi, \psi)]^{1/2}\frac{1}{\sqrt{\nu-\mu}}\left|\int_{\mu}^{\nu}(s-\mu)(\nu-s)\left[\zeta^{(n+1)}(s)\right]^2 ds\right|^{1/2}. \tag{63}$$

Therefore, we have

$$\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\psi(s)\zeta^n(s)ds = \frac{1}{(\nu-\mu)^2}\left(\zeta^{(n-1)}(\nu) - \zeta^{(n-1)}(\mu)\right)\int_{\mu}^{\nu}\psi(s)ds,$$

$$\int_{\mu}^{\nu}\psi(s)\zeta^n(s)ds = \frac{\zeta^{(n-1)}(\nu) - \zeta^{(n-1)}(\mu)}{\nu-\mu}\int_{\mu}^{\nu}\psi(s)ds + \mathfrak{R}(\mu, \nu; \zeta), \tag{64}$$

where the remainder $\mathfrak{R}(\mu,\nu;\zeta)$ satisfy the estimation (62). Now, from the identity (36),

$$J(\zeta)=\sum_{\nu=1}^{r}\sum_{u=0}^{k_\nu}\zeta^{(u)}(a_\nu)J(H_{u\nu}(\mathfrak{t}))+\int_\mu^\nu\psi(s)\zeta^{(n)}(s)ds,$$

$$J(\zeta)=\sum_{\nu=1}^{r}\sum_{u=0}^{k_\nu}\zeta^{(u)}(a_\nu)J(H_{u\nu}(\mathfrak{t}))+\frac{\zeta^{(n-1)}(\nu)-\zeta^{(n-1)}(\mu)}{\nu-\mu}\cdot\int_\mu^\nu\psi(s)ds+\mathfrak{R}(\mu,\nu;\zeta). \tag{65}$$

□

The Grüss-type inequality given below can be obtained by using Theorem 19.

Theorem 21. *Assume $\zeta:[\mu,\nu]\longrightarrow\mathbb{R}$ such that $\zeta\in C^{(n)}[\mu,\nu](n\in\mathbb{N})$ and $\zeta^{(2n+1)}\geq 0$ on $[\mu,\nu]$. Suppose ψ and Y are defined in (59) and (60), respectively. Then, we have (61) the remainder $\mathfrak{R}(\mu,\nu:\zeta)$ satisfies the bound*

$$|\mathfrak{R}(\mu,\nu;\zeta)|\leq(\nu-\mu)\|\psi'\|_\infty\times\left[\frac{\zeta^{(n-1)}(\nu)+\zeta^{(n-1)}(\mu)}{\nu-\mu}-\frac{\zeta^{(n-2)}(\nu)-\zeta^{(n-2)}(\mu)}{\nu-\mu}\right]. \tag{66}$$

Proof. Applying Theorem 19 for $g_1\longrightarrow\psi$ and $g_2\longrightarrow\zeta^{(n)}$, we have

$$\left|\frac{1}{\nu-\mu}\int_\mu^\nu\psi(s)\zeta^n(s)ds-\frac{1}{\nu-\mu}\int_\mu^\nu\psi(s)ds\cdot\frac{1}{\nu-\mu}\int_\mu^\nu\zeta^n(s)ds\right|\leq\frac{1}{2(\nu-\mu)}\|\psi'\|_\infty\int_\mu^\nu(s-\mu)(\nu-s)\zeta^{(n+1)}(s)ds. \tag{67}$$

Since

$$\int_\mu^\nu(s-\mu)(\nu-s)\zeta^{(n+1)}(s)ds=\int_\mu^\nu[2s-(\mu+\nu)]\zeta^{(n)}(s)ds,$$

$$\int_\mu^\nu(s-\mu)(\nu-s)\zeta^{(n+1)}(s)ds=(\nu-\mu)\left[\zeta^{(n-1)}(\nu)+\zeta^{(n-1)}(\mu)\right]-2\left[\zeta^{(n-2)}(\nu)+\zeta^{(n-2)}(\mu)\right]; \tag{68}$$

using (36) and (67), we get (66). □

Theorem 22. *Let all the presumptions of Theorem 8 be satisfied. Suppose $(\breve{i},\breve{j})$ is a couple of numbers, i.e., $1\leq\breve{i},\breve{j}\leq\infty,1/\breve{i}+1/\breve{j}=1$. Suppose $|\zeta^{(n)}|^{\breve{i}}:[\mu,\nu]\longrightarrow\mathbb{R}$ is a function which is R -integrable for some $n\geq 2$. Then, we have*

$$\left|J(\zeta)-\sum_{\nu=1}^{r}\sum_{u=0}^{k_\nu}\zeta^{(u)}(a_\nu)J(H_{u\nu}(x))\right|\leq\left\|\zeta^{(n)}\right\|_{\breve{i}}\left(\int_\mu^\nu|J(G_H(x,s))|^{\breve{j}}ds\right)^{1/j}. \tag{69}$$

For $\breve{i}\in[1,\infty]$, the constant on the right of (69) is sharp and suitable when $\breve{i}$ is one.

Proof. Assume $\wp(s)=J(G_n(\mathfrak{t},s))$. Hölder's inequality and identity (36) give us

$$|J(\zeta)-\sum_{\nu=1}^{r}\sum_{u=0}^{k_\nu}J(H_{u\nu}(\mathfrak{t}))\zeta^{(u)}(a_\nu)|=\left|\int_\mu^\nu\wp(s)\zeta^{(n)}(s)ds\right|\leq\left\|\zeta^{(n)}\right\|_{\breve{i}}\left(\int_\mu^\nu|\wp(\mathfrak{t})|^{\breve{j}}ds\right)^{1/\breve{j}}. \tag{70}$$

Taking ζ for $\breve{i}\in(1,\infty)$, such that, $\zeta^n(s)=\text{sgn}\wp(s)|\wp(s)|^{1/\breve{i}-1}$. In case of $\breve{i}=\infty$, take $\zeta^{(n)}(s)=\text{sgn}\wp(s)$.

We prove that for $\breve{i}=1$,

$$\int_\mu^\nu\wp(s)\zeta^{(n)}(s)ds\leq\max_{s\in[\mu,\nu]}|\wp(s)|\left(\int_\mu^\nu\left|\zeta^{(n)}(s)\right|ds\right) \tag{71}$$

is the suitable inequality. Let $|\wp(s)|$ achieve its maximum at $\mathfrak{p}\in[\mu,\nu]$. We suppose firstly that $\wp(\ddot{d})>0$. We define $\zeta(s)$ for small enough δ, by

$$\zeta_\delta(s):=\begin{cases}0,\ \mu\leq s\leq\mathfrak{p},\\ \frac{1}{\varepsilon n!}(s-\mathfrak{p})^n,\ \mathfrak{p}\leq s\leq\mathfrak{p}+\delta,\\ \frac{1}{n!}(s-\mathfrak{p})^{n-1},\ \mathfrak{p}+\delta\leq s\leq\nu.\end{cases} \tag{72}$$

Then, for δ small enough,

$$\left|\int_\mu^\nu\wp(s)\zeta^{(n)}(s)ds\right|=\left|\int_\mathfrak{p}^{\mathfrak{p}+\delta}\wp(s)\frac{1}{\delta}ds\right|=\frac{1}{\delta}\int_\mathfrak{p}^{\mathfrak{p}+\delta}\wp(s)ds. \tag{73}$$

Now, from the inequality (71), we have

$$\frac{1}{\delta}\int_\mathfrak{p}^{\mathfrak{p}+\delta}\wp(s)ds\leq\wp(\mathfrak{p})\int_\mathfrak{p}^{\mathfrak{p}+\delta}\frac{1}{\delta}ds=\wp(\mathfrak{p}). \tag{74}$$

For $\wp(\mathfrak{p}) < 0$, we set $\zeta_\delta(s)$ as

$$\zeta_\delta(s) := \begin{cases} \frac{1}{n!}(s-\mathfrak{p}-\delta)^{n-1}, \mu \leq s \leq \mathfrak{p}, \\ -\frac{1}{\varepsilon n!}(s-\mathfrak{p}-\delta)^{n}, \mathfrak{p} \leq s \leq \mathfrak{p}+\epsilon, \\ 0, \mathfrak{p}+\delta \leq s \leq \nu. \end{cases} \quad (75)$$

The rest of the proof is the same as above. □

4. Conclusion

Jensen's functional for diamond integral (7) is generalized for n-convex functions using the Hermite polynomial in the present article. Different conditions of Hermite polynomial are utilized to describe respective refinements of the functional. In seek of applications, bounds for identities associated to constructed functional are also discussed. Moreover, by defining the functional as difference of right and left sides of extended inequality (40) (where B is defined in (41)), it is possible to study n-exponential convexity, exponential convexity, and applications to Stolarsky-type means as discussed by Aras-Gazic et al., in [17] (Sections 5 and 6). This article extends the results of [8] on time scales.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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