Research Article

An Analysis on the Positive Solutions for a Fractional Configuration of the Caputo Multiterm Semilinear Differential Equation

Sh. Rezapour^{1,2}, B. Azzaoui,³ B. Tellab,³ S. Etemad¹, and H. P. Masiha⁴

¹Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran

²Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan

³Laboratory of Applied Mathematics, Kasdi Merbah University, B.P. 511.30000, Ouargla, Algeria

⁴Department of Mathematics, K. N. Toosi University of Technology, P.O. Box 16315-1618, Tehran 15418, Iran

Correspondence should be addressed to S. Etemad; sina.etemad@azaruniv.ac.ir

Received 22 April 2021; Accepted 30 May 2021; Published 23 June 2021

Academic Editor: Huseyin Isik

Copyright © 2021 Sh. Rezapour et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we consider a multiterm semilinear fractional boundary value problem involving Caputo fractional derivatives and investigate the existence of positive solutions by terms of different given conditions. To do this, we first study the properties of Green's function, and then by defining two lower and upper control functions and using the wellknown Schauder's fixed-point theorem, we obtain the desired existence criteria. At the end of the paper, we provide a numerical example based on the given boundary value problem and obtain its upper and lower solutions, and finally, we compare these positive solutions with exact solution graphically.

1. Introduction

At a vast level, it is understood that the hereditary properties and the memory of most processes, phenomena, and materials are predictable with the help of different modelings under the fractional operators. In this regard, differential equations involving fractional derivatives have recently been confirmed to be a useful tool in modeling of a considerable variety of structures in miscellaneous branches of sciences. For the sake of the increasing acceleration and advancements of studies and researches in the field of fractional calculus, several works have been done; see [1, 2]. Since theoretical findings are used to achieve a deep understanding for the fractional models, a large number of mathematicians have also assigned their focus on studying the existence aspects of solutions for several structures of fractional equations by means of different techniques and methods. For instance, see [3–10].

In the next periods, a large number of researchers studied the notion of positive solutions for nonlinear fractional differential equations, and accordingly, many papers have been

published in this direction. In 2003, Zhang [11] investigated the multiple and infinitely solvability of positive solutions for a nonlinear generalized fractional differential equation by relying on fixed point methods on cones. In 2007, El-Shahed [12] investigated the existence and nonexistence of positive solutions for a nonlinear fractional boundary value problem in the Riemann-Liouville settings. The author used Krasnoselskii's fixed point theorem on cone preserving operators for deriving some required criteria. In [13], Guezane-Lakoud et al. presented a fourth-order mathematical model of elastic beam in three separate points of domain and studied the existence of positive solutions with the help of fixed point techniques.

In [14], Tian et al. turned to investigate positive solutions for a new class of fourpoint boundary value problem of fractional differential equations with p -laplacian operator and used the Leggett-Williams fixed point theorem on a cone to prove the multiplicity results of such solutions. More recently, Seemab et al. [15] established the existence results of positive solutions for a boundary value problem defined within generalized Riemann-Liouville and Caputo fractional

operators by studying the properties of Green functions in three different types. Along with these, some other researchers investigated numerical methods and nonsingular fractional operators for obtaining numerical solutions of different fractional differential equations such as [16, 17].

More specifically, in [18], Zhang studied the multiplicity and existence of positive solutions for the fractional nonlinear boundary value problem given by

$$\begin{cases} \mathfrak{D}^\nu w(z) = \Xi(z, w(z)), (1 < \nu < 2, 0 \leq z \leq 1) \\ w(0) + w'(0) = 0, w(1) + w'(1) = 0, \end{cases} \quad (1)$$

where $\mathfrak{D}^\nu$ stands for the Caputo fractional derivative. To obtain the existence conditions, Zhang applied a method based on cones. Bai and Lu [19] also employed some nonlinear methods to establish the multiplicity and existence of positive solutions of the given problem as

$$\begin{cases} {}^R\mathfrak{D}^\nu w(z) = \Xi(z, w(z)), (0 \leq z \leq 1, 1 < \nu \leq 2), \\ w(0) = 0, w(1) = 0, \end{cases} \quad (2)$$

where $\mathfrak{D}^\nu$ denotes the Riemann-Liouville fractional derivative and $\Xi : [0, 1] \times \mathbb{R}^{>0} \rightarrow \mathbb{R}^{>0}$ is a continuous function. Their method is based upon the reduction of the given boundary value problem to the equivalent Fredholm integral equation of the second kind.

Inspired by the above works, in this paper, we derive some sufficient conditions to establish our main results on the existence of positive solutions to multiterm semilinear fractional boundary value problem given by

$$\begin{cases} \mathfrak{D}^r w(z) = \Xi(z, w(z), \mathfrak{D}^p w(z)), (z \in \bar{J} = [0, 1]), \\ w(0) = 0, \mathfrak{D}^{r-1} w(1) = 0, \end{cases} \quad (3)$$

where $1 < r < 2$, $0 < p < 1$ and Ξ is a continuous positive function on $[0, 1] \times \mathbb{R} \times \mathbb{R}$ and $\mathfrak{D}^{(\cdot)}$ denotes the Caputo fractional derivative. Note that the main method of this paper is to convert our multiterm semilinear boundary value problem (3) to an equivalent integral equation which allows us to convert it to a fixed point problem. In the following path, the criteria establishing the existence of positive solutions are guaranteed by imposing several sufficient conditions. In fact, we introduce two upper and lower control functions to achieve such aims, and after surveying some properties of the Green's function, the fundamental theorems in relation to the existence results are derived. To prove the existence results, by applying lower and upper control functions, we use the standard Schauder's fixed point theorem to obtain lower and upper solutions. In contrast to other works, we generalize and consider two terminal boundary conditions in the context of the Caputo fractional derivative of the unknown function and show our findings graphically. In other words, by plotting the graphs of lower and upper solutions, one can compare our results with the exact positive solution. It is notable that in the suggested problem (3), we have considered a multiterm semilinear boundary value problem, and

in the future researches, one can implement this technique for the complicated boundary conditions and one can also cover other generalized nonlinear fractional boundary problems arising in real-world phenomena.

The structure of the article is presented as follows: Sect. 2 recall the auxiliary and preliminary notions in relation to the fractional calculus and also some basic lemmas are provided. Sect. 3 is assigned to derive sufficient conditions to obtain positive solutions of the multiterm semilinear boundary value problem (3). Finally, a special example is provided to validate our theoretical findings according to the method implemented in theorems.

2. Basic and Auxiliary Concepts

Before proving some preliminary lemmas, we need to present some definitions and properties on fractional calculus which are useful throughout our research work.

Definition 1 (see [20]). Let $\nu > 0$ and $\phi : (0, +\infty) \rightarrow \mathbb{R}$ be continuous. The integral

$$\mathcal{I}^\nu \phi(z) = \frac{1}{\Gamma(\nu)} \int_0^z (z-s)^{\nu-1} \phi(s) ds, \quad (4)$$

is called the fractional integral in the Riemann-Liouville settings of order ν such that it has finite values.

Definition 2 (see [20]). Let $k-1 < \nu < k$ with $k = [\nu] + 1$ and $\phi : (0, +\infty) \rightarrow \mathbb{R}$ belongs to $AC(k)((0, \infty), \mathbb{R})$. Then

$$\mathfrak{D}^\nu \phi(z) = \frac{1}{\Gamma(k-\nu)} \int_0^z (z-s)^{k-\nu-1} \phi^{(k)}(s) ds, \quad (5)$$

is called the Caputo fractional derivative of order ν such that it exists.

Remark 3. We have the following:

(RE1) For $0 \leq \nu < \gamma$, the equality $\mathfrak{D}^\nu \mathcal{I}^\gamma \phi(z) = \mathcal{I}^{\gamma-\nu} \phi(z)$ holds

(RE2) For $\gamma > -1$ with $\gamma \neq \nu - j$ ($j = 1, 2, \dots, n$) and for each $z \geq 0$, we have

$$\mathfrak{D}^\nu z^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(\gamma-\nu+1)} z^{\gamma-\nu} \text{ and } \mathfrak{D}^\nu z^{\nu-1} = 0, (j = 1, 2, \dots, n). \quad (6)$$

Proposition 4 (see [21]). Suppose that ϕ is contained in the space $\mathcal{L}(0, 1) \cap \mathcal{C}(0, 1)$ and $k = [\nu] + 1$. Then

$$\mathcal{I}^\nu \mathfrak{D}^\nu w(z) = w(z) + c_0 + c_1 z + c_2 z^2 + \dots + c_{k-1} z^{k-1}, \quad (7)$$

such that $c_0, c_2, \dots, c_{k-1} \in \mathbb{R}$.

The following proposition is important and specifies the structure of the equivalent solution of the integral equation arising in the multiterm semilinear boundary value problem (3).

Proposition 5. Consider $Q \in \mathcal{C}([0, 1], \mathbb{R})$ and $1 < r < 2$. Then, the solution of the linear problem

$$\begin{cases} \mathfrak{D}^r w(z) = Q(z), & z \in \bar{J} \\ w(0) = 0, \mathfrak{D}^{r-1} w(1) = 0, \end{cases} \quad (8)$$

is given by the following integral equation

$$w(z) = \int_0^1 H(z, s) Q(s) ds, \quad (9)$$

where

$$H(z, s) = \begin{cases} \frac{(z-s)^{r-1}}{\Gamma(r)} - \Gamma(3-r)z, & 0 \leq s \leq z \leq 1 \\ -\Gamma(3-r)z, & 0 \leq z \leq s \leq 1. \end{cases} \quad (10)$$

Proof. If w is a solution of the linear boundary value problem (8), then from Proposition (7), it is followed that

$$w(z) = c_0 + c_1 z + I^r Q(z) = c_0 + c_1 z + \frac{1}{\Gamma(r)} \int_0^z (z-s)^{r-1} Q(s) ds. \quad (11)$$

Then, the first boundary condition gives $c_0 = 0$. By applying the operator $\mathfrak{D}^{r-1}$ to both sides of (11) and using (6), we find that

$$\mathfrak{D}^{r-1} w(z) = \frac{c_1}{\Gamma(3-r)} z^{2-r} + I_{0+}^1 Q(z), \quad (12)$$

which in view of the second boundary condition, gives

$$c_1 = -\Gamma(3-r) \int_0^1 Q(s) ds. \quad (13)$$

By substituting c_0 and c_1 in (11), we get

$$\begin{aligned} w(z) &= \frac{1}{\Gamma(r)} \int_0^z (z-s)^{r-1} Q(s) ds - \Gamma(3-r)z \int_0^1 Q(s) ds \\ &= \int_0^1 H(z, s) Q(s) ds, \end{aligned} \quad (14)$$

where $H(z, s)$ is given by (10). In this case, we follow that w will be a solution of (9). This completes the proof. $\square$

Remark 6. It is easy to show by a simple computation that the function H satisfies

$$\int_0^1 |H(z, s)| ds \leq \frac{1}{\Gamma(r-1)} + \Gamma(3-r). \quad (15)$$

Lemma 7. The function $|\partial H(z, s)/\partial z|$ is integrable for each $z \in [0, 1]$.

Proof. We have

$$\frac{\partial H(z, s)}{\partial z} = \begin{cases} \frac{(z-s)^{r-2}}{\Gamma(r-1)} - \Gamma(3-r), & 0 \leq s \leq z \leq 1, \\ -\Gamma(3-r), & 0 \leq z \leq s \leq 1. \end{cases} \quad (16)$$

Then,

$$\begin{aligned} \int_0^1 \left| \frac{\partial H(z, s)}{\partial z} \right| ds &\leq \int_0^z \frac{(z-s)^{r-2}}{\Gamma(r-1)} ds + \int_0^z \Gamma(3-r) ds + \int_z^1 \Gamma(3-r) ds \\ &= \frac{z^{r-1}}{\Gamma(r)} + \Gamma(3-r)z + \Gamma(3-r)(1-z) \\ &\leq \frac{1}{\Gamma(r)} + \Gamma(3-r) < \infty. \end{aligned} \quad (17)$$

This completes the proof. $\square$

Remark 8. Consider the space $\mathbb{X} = \mathcal{C}^1([0, 1], \mathbb{R})$. For $0 < p < 1$ and $w \in \mathbb{X}$, define the norm of w by

$$\|w\|_{\mathbb{X}} = \max_{z \in [0, 1]} |w(z)| + \max_{z \in [0, 1]} |w'(z)| + \max_{z \in [0, 1]} |\mathfrak{D}^p w(z)|. \quad (18)$$

Then, clearly, $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is a Banach space.

3. Existence Criteria for Positive Solutions

In this section, several conditions are derived for which the existence of positive solutions to the multiterm semilinear boundary value problem (3) is guaranteed. Let $\alpha_1, \alpha_3 \in \mathbb{R}^+$ and $\alpha^2, \alpha^4 \in \mathbb{R}$ with $\alpha_1 < \alpha_3$ and $\alpha_2 < \alpha_4$. The upper control function

$$\widehat{\Delta} : [0, 1] \times [\alpha_1, +\infty) \times [\alpha_2, +\infty) \longrightarrow \mathbb{R}^+, \quad (19)$$

and the lower control function $\widehat{\delta} : [0, 1] \times [-\infty, \alpha_3) \times [-\infty, \alpha_4) \longrightarrow \mathbb{R}^+$ is defined by

$$\begin{aligned} \widehat{\Delta}(z, u, v) &= \sup_{\substack{\alpha_1 \leq \theta \leq u \\ \alpha_2 \leq \mu \leq v}} \Xi(z, \theta, \mu) \text{ and} \\ \widehat{\delta}(z, u, v) &= \inf_{\substack{u \leq \theta \leq \alpha_3 \\ v \leq \mu \leq \alpha_4}} \Xi(z, \theta, \mu), \end{aligned} \quad (20)$$

respectively. We clearly have

$$\begin{aligned} \widehat{\delta}(z, u, v) &\leq \Xi(z, u, v) \leq \widehat{\Delta}(z, u, v), \text{ for } 0 \\ &\leq z \leq 1, \alpha_1 \leq u \leq \alpha_3, \alpha_2 \leq v \leq \alpha_4. \end{aligned} \quad (21)$$

In addition to these, define the set

$$\tilde{\Lambda} = \{w \in \mathbb{X} : w(z) \geq 0, 0 \leq z \leq 1\}, \quad (22)$$

which is used in the sequel. Here, we mean by a positive solution, each function w satisfies $w \in X$, $w(0) = 0$ and $w(z) > 0$ for each $0 < z \leq 1$; in other words, $w \in \tilde{\Lambda}$.

3.1. Required Assumptions. Now, for our main results, we need some assumptions given as follows:

(A1) There are $w^*, w_* \in \tilde{\Lambda}$ which satisfy $\alpha_1 \leq w_*(z) \leq w^*(z) \leq \alpha_3$ and $\alpha_2 \leq \mathfrak{D}^p w_*(z) \leq \mathfrak{D}^p w^*(z) \leq \alpha_4$ along with

$$w^*(z) \geq \int_0^1 |H(z, s)| \hat{\Delta}(s, w^*(s), \mathfrak{D}^p w^*(s)) ds,$$

$$w_*(z) \leq \int_0^1 |H(z, s)| \hat{\delta}(s, w_*(s), \mathfrak{D}^p w_*(s)) ds,$$

$$\begin{aligned} \mathfrak{D}^p w^*(z) &\geq -\frac{1}{\Gamma(r-p)} \int_0^z (z-s)^{r-p-1} \hat{\Delta}(s, w^*(s), \mathfrak{D}^p w^*(s)) ds \\ &\quad + \frac{\Gamma(3-r)}{\Gamma(2-p)} z^{1-p} \int_0^1 \hat{\delta}(s, w^*(s), \mathfrak{D}^p w^*(s)) ds, \end{aligned}$$

$$\begin{aligned} \mathfrak{D}^p w_*(z) &\leq -\frac{1}{\Gamma(r-p)} \int_0^z (z-s)^{r-p-1} \hat{\delta}(s, w_*(s), \mathfrak{D}^p w_*(s)) ds \\ &\quad + \frac{\Gamma(3-r)}{\Gamma(2-p)} z^{1-p} \int_0^1 \hat{\Delta}(s, w_*(s), \mathfrak{D}^p w_*(s)) ds. \end{aligned} \quad (23)$$

(A2) There exist $\xi > 0$ and nonnegative function $\theta \in \mathcal{L}^1(0, 1)$ such that

$$\Xi(z, w, v) \leq \theta(z) + \xi(|w| + |v|), \quad 0 \leq z \leq 1, w, v \in \mathbb{R}. \quad (24)$$

(A3) There exists $\zeta > 0$ such that

$$\begin{aligned} A + B + \frac{B}{\Gamma(2-p)} + \xi \zeta \left(\frac{1}{\Gamma(r)} + \frac{1}{\Gamma(r+1)} \right) \\ + 2\Gamma(3-r) + \frac{1}{\Gamma(2-p)\Gamma(r)} + \frac{\Gamma(3-r)}{\Gamma(2-p)} \leq \zeta. \end{aligned} \quad (25)$$

with

$$A = \max_{z \in [0,1]} \int_0^1 |H(z, s)| \theta(s) ds \text{ and } B = \max_{z \in [0,1]} \int_0^1 \left| \frac{\partial H(z, s)}{\partial z} \right| \theta(s) ds. \quad (26)$$

At this moment, we are ready to present the first existence theorem.

Theorem 9. Suppose that the assumptions (A1)–(A3) hold. Then, the multiterm semilinear boundary value problem (3) has at least a positive solution w in $\mathbb{X}$ such that all inequalities $w_*(z) \leq w(z) \leq w^*(z)$ and $\mathfrak{D}^p w_*(z) \leq \mathfrak{D}^p w(z) \leq \mathfrak{D}^p w^*(z)$ hold for each $0 \leq z \leq 1$.

Proof. For each $\zeta > 0$, define the set Γ_ζ as

$$\begin{aligned} \Gamma_\zeta = \left\{ w \in \tilde{\Lambda} : \|w\|_\mathbb{X} \leq \zeta, w_*(z) \leq w(z) \leq w^*(z), \mathfrak{D}^p w_*(z) \leq \mathfrak{D}^p w(z) \leq \mathfrak{D}^p w^*(z), 0 \leq z \leq 1 \right\}. \end{aligned} \quad (27)$$

Obviously, Γ_ζ is a convex, closed, and bounded set in $\mathbb{X}$. Consider the operator $\mathfrak{P} : \Gamma_\zeta \rightarrow \mathbb{X}$ under the following rule

$$\begin{aligned} (\mathfrak{P}w)(z) &= -\frac{1}{\Gamma(r)} \int_0^z (z-s)^{r-1} \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \\ &\quad + \Gamma(3-r) \int_0^1 \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \\ &= \int_0^1 H(z, s) \Xi(s, w(s), \mathfrak{D}^p w(s)) ds. \end{aligned} \quad (28)$$

To prove Theorem 9, we will show that the hypotheses of Schauder's fixed point theorem hold. So, the process of proof will be done in several steps.

Step 1. $\mathfrak{P}$ is continuous in X . To prove such a claim, we consider a sequence $\{w_n\}$ which converges to w in X . We have

$$\begin{aligned} |\mathfrak{P}w_n(z) - \mathfrak{P}w(z)| &= \left| \int_0^1 H(z, s) (\Xi(s, w_n(s), \mathfrak{D}^p w_n(s)) - \Xi(s, w(s), \mathfrak{D}^p w(s))) ds \right| \\ &\leq \max_{z \in [0,1]} |\Xi(z, w_n(z), \mathfrak{D}^p w_n(z)) - \Xi(z, w(z), \mathfrak{D}^p w(z))| \int_0^1 |H(z, s)| ds \\ &\leq \left(\frac{1}{\Gamma(r+1)} + \Gamma(3-r) \right) \max_{z \in [0,1]} |\Xi(z, w_n(z), \mathfrak{D}^p w_n(z)) - \Xi(z, w(z), \mathfrak{D}^p w(z))|, \end{aligned} \quad (29)$$

$$\begin{aligned} |\mathfrak{D}^p \mathfrak{P}w_n(z) - \mathfrak{D}^p \mathfrak{P}w(z)| &= \left| \frac{1}{\Gamma(1-p)} \int_0^z (z-s)^{-p} \left((\mathfrak{P}w_n)'(s) - (\mathfrak{P}w)'(s) \right) ds \right| \\ &\leq \Gamma(1-p) \int_0^z (z-s)^{-p} \left(\int_0^1 \left| \frac{\partial H(s, \lambda)}{\partial s} \right| [\Xi(\lambda, w_n(\lambda), \mathfrak{D}^p w_n(\lambda)) - \Xi(\lambda, w(\lambda), \mathfrak{D}^p w(\lambda))] d\lambda \right) ds \\ &\leq \max_{z \in [0,1]} |\Xi(z, w_n(z), \mathfrak{D}^p w_n(z)) - \Xi(z, w(z), \mathfrak{D}^p w(z))| \\ &\quad \cdot \frac{1}{\Gamma(1-p)} \int_0^z (z-s)^{-p} \left(\int_0^1 \left| \frac{\partial H(s, \lambda)}{\partial s} \right| d\lambda \right) ds \\ &\leq \max_{z \in [0,1]} |\Xi(z, w_n(z), \mathfrak{D}^p w_n(z)) - \Xi(z, w(z), \mathfrak{D}^p w(z))| \\ &\quad \cdot \frac{1}{\Gamma(1-p)} \left(\frac{1}{\Gamma(r)} + \frac{\Gamma(3-r)}{\Gamma(2)} \right) \int_0^z (z-s)^{-p} ds \\ &\leq \frac{1}{\Gamma(2-p)} \left(\frac{1}{\Gamma(r)} + \Gamma(3-r) \right) \max_{z \in [0,1]} |\Xi(z, w_n(z), \mathfrak{D}^p w_n(z)) - \Xi(z, w(z), \mathfrak{D}^p w(z))|, \end{aligned} \quad (30)$$

$$\begin{aligned}
& |(\mathfrak{P}w_n)'(z) - (\mathfrak{P}w)'(z)| \\
&= \left| \int_0^1 \frac{\partial H(z, s)}{\partial z} (\Xi(s, w_n(s), \mathfrak{D}^p w_n(s)) \right. \\
&\quad \left. - \Xi(s, w(s), \mathfrak{D}^p w(s))) ds \right| \\
&\leq \max_{z \in [0,1]} |\Xi(z, w_n(z), \mathfrak{D}^p w_n(z)) \\
&\quad - \Xi(z, w(z), \mathfrak{D}^p w(z))| \int_0^1 \left| \frac{\partial H(z, s)}{\partial z} \right| ds \\
&\leq \left(\frac{1}{\Gamma(r)} + \Gamma(3-r) \right) w_{z \in [0,1]} |\Xi(z, w_n(z), \mathfrak{D}^p w_n(z)) \\
&\quad - \Xi(z, w(z), \mathfrak{D}^p w(z))|. \tag{31}
\end{aligned}$$

By tending $n \rightarrow \infty$ and from the inequalities (29), (30), and (31), we follow that $\mathfrak{P}$ is continuous in $\mathbb{X}$.

Step 2. Now, we show that $\mathfrak{P} : \Gamma_\zeta \rightarrow \Gamma_\zeta$ is a selfmap on Γ_ζ . Let $w \in \Gamma_\zeta$. By inequalities (15) and (17) along with the assumptions (A2) and (A3), we get

$$\begin{aligned}
|\mathfrak{P}w(z)| &= \left| \int_0^1 H(z, s) \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \right| \\
&\leq \int_0^1 |H(z, s) \Xi(s, w(s), \mathfrak{D}^p w(s))| ds \\
&\leq \int_0^1 |H(z, s) [\theta(s) + \xi(|w(s)| + |\mathfrak{D}^p w(s)|)]| ds \\
&\leq \int_0^1 |H(z, s) \theta(s)| ds + \xi \zeta \int_0^1 |H(z, s)| ds \\
&\leq B + \xi \zeta \left(\frac{1}{\Gamma(r)} + \Gamma(3-r) \right), \tag{32}
\end{aligned}$$

$$\begin{aligned}
|(\mathfrak{P}w)'(z)| &= \left| \int_0^1 \frac{\partial H(z, s)}{\partial z} \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \right| \\
&\leq \int_0^1 \left| \frac{\partial H(z, s)}{\partial z} [\theta(s) + \xi(|w(s)| + |\mathfrak{D}^p w(s)|)] \right| ds \\
&\leq \int_0^1 \left| \frac{\partial H(z, s)}{\partial z} \theta(s) \right| ds + \xi \zeta \int_0^1 \left| \frac{\partial H(z, s)}{\partial z} \right| ds \\
&\leq B + \xi \zeta \left(\frac{1}{\Gamma(r)} + \Gamma(3-r) \right), \tag{33}
\end{aligned}$$

$$\begin{aligned}
|\mathfrak{D}^p \mathfrak{P}w(z)| &= \left| \frac{1}{\Gamma(1-p)} \int_0^z (z-s)^{-p} (\mathfrak{P}w)'(s) ds \right| \\
&\leq \frac{1}{\Gamma(1-p)} \int_0^z (z-s)^{-p} \\
&\quad \cdot \left(\int_0^1 \left| \frac{\partial H(s, \lambda)}{\partial s} \Xi(\lambda, w(\lambda), \mathfrak{D}^p w(\lambda)) \right| d\lambda \right) ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(1-p)} \int_0^1 (z-s)^{-p} \left(\int_0^1 \left| \frac{\partial H(s, \lambda)}{\partial s} \theta(\lambda) \right| d\lambda \right. \\
&\quad \left. + \int_0^1 \frac{\partial H(s, \lambda)}{\partial s} \xi(|x(\lambda)| + |\mathfrak{D}^p w(\lambda)|) d\lambda \right) ds \\
&\leq \frac{B}{\Gamma(1-p)} \int_0^z (z-s)^{-p} ds \\
&\quad + \frac{\xi \zeta}{\Gamma(1-p)} \int_0^z (z-s)^{-p} \left(\int_0^1 \left| \frac{\partial H(s, \lambda)}{\partial s} \right| d\lambda \right) ds \\
&\leq \frac{B}{\Gamma(2-p)} z^{1-p} + \frac{\xi \zeta}{\Gamma(2-p)} \left(\frac{1}{\Gamma(r)} + \Gamma(3-r) \right) z^{1-p} \\
&\leq \frac{B}{\Gamma(2-p)} + \frac{\xi \zeta}{\Gamma(2-p)} \left(\frac{1}{\Gamma(r)} + \Gamma(3-r) \right). \tag{34}
\end{aligned}$$

By virtue of inequalities (32), (33), (34), and the assumption (A3), we get $\|\mathfrak{P}x\|_\mathbb{X} \leq \zeta$.

In the sequel, we investigate the inequalities $w_*(z) \leq \mathfrak{P}w(z) \leq w^*(z)$ and also $\mathfrak{D}^p w_*(z) \leq \mathfrak{D}^p \mathfrak{P}w(z) \leq \mathfrak{D}^p w^*(z)$ for each $0 \leq z \leq 1$. Since w belongs to Γ_ζ , we obviously have $w_*(z) \leq w(z) \leq w^*(z)$. By using definitions of upper and lower control functions together with the assumption (A1), we get

$$\begin{aligned}
\mathfrak{P}w(z) &\leq \int_0^1 |H(z, s)| \widehat{\Delta}(s, w(s), \mathfrak{D}^p w(s)) ds \\
&\leq \int_0^1 |H(z, s)| \widehat{\Delta}(s, w^*(s), \mathfrak{D}^p w^*(s)) ds \leq w^*(z), \\
\mathfrak{P}w(z) &\geq \int_0^1 |H(z, s)| \widehat{\delta}(s, w(s), \mathfrak{D}^p w(s)) ds \\
&\geq \int_0^1 |H(z, s)| \widehat{\delta}(s, w_*(s), \mathfrak{D}^p w_*(s)) ds \geq w_*(z), \tag{35}
\end{aligned}$$

Hence, we obtain $w_*(z) \leq \mathfrak{P}w(z) \leq w^*(z)$. Now, we need to show that $\mathfrak{D}^p w_*(z) \leq \mathfrak{D}^p \mathfrak{P}w(z) \leq \mathfrak{D}^p w^*(z)$. We have

$$\begin{aligned}
D_{0+}^p \mathfrak{P}w(z) &= -\frac{1}{\Gamma(r-p)} \int_0^z (z-s)^{r-p-1} \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \\
&\quad + \frac{\Gamma(3-r)z^{1-p}}{\Gamma(2-p)} \int_0^1 \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \\
&\leq -\frac{1}{\Gamma(r-p)} \int_0^z (z-s)^{r-p-1} \widehat{\Delta}(s, w^*(s), \mathfrak{D}^p w^*(s)) ds \\
&\quad + \frac{\Gamma(3-r)z^{1-p}}{\Gamma(2-p)} \int_0^1 \widehat{\delta}(s, w^*(s), \mathfrak{D}^p w^*(s)) ds \\
&\leq \mathfrak{D}^p w^*(z). \tag{36}
\end{aligned}$$

Similarly, we showed that $\mathfrak{D}^p \mathfrak{P}w(z) \geq \mathfrak{D}^p w_*(z)$. Therefore, $\mathfrak{P}(\Gamma_\zeta) \subseteq \Gamma_\zeta$.

Step 3. At the final step, we aim to prove that P has the complete continuity property.

To see this, let $w \in \Gamma_\zeta$ and take $M = \max_{z \in [0,1]} \Xi(z, w(z), \mathfrak{D}^p w(z))$. We have

$$\begin{aligned}
|\mathfrak{P}w(z)| &= \left| \frac{1}{\Gamma(r)} \int_0^z (z-s)^{r-1} \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \right. \\
&\quad \left. - \Gamma(3-r)z \int_0^1 \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \right| \\
&\leq \frac{1}{\Gamma(r)} \int_0^z |(z-s)^{r-1} \Xi(s, w(s), \mathfrak{D}^p w(s))| ds \\
&\quad + \Gamma(3-r) \int_0^1 \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \\
&\leq \left(\frac{z^r}{\Gamma(r+1)} + \Gamma(3-r)z \right) M \\
&\leq \left(\frac{1}{\Gamma(r+1)} + \Gamma(3-r) \right) M, \\
|(\mathfrak{P}w)'(z)| &= \left| \int_0^1 \frac{\partial H(z, s)}{\partial z} \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \right| \\
&\leq M \int_0^1 \left| \frac{\partial H(z, s)}{\partial z} \right| ds \leq \left(\frac{1}{\Gamma(r)} + \Gamma(3-r) \right) M, \\
|\mathfrak{D}^p \mathfrak{P}w(z)| &= \left| \frac{1}{\Gamma(r-p)} \int_0^z (z-s)^{r-p-1} \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \right. \\
&\quad \left. - \frac{\Gamma(3-r)z^{1-p}}{\Gamma(2-p)} \int_0^1 \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \right| \\
&\leq \frac{1}{\Gamma(r-p)} \int_0^1 (z-s)^{r-p-1} \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \\
&\quad + \frac{\Gamma(3-r)z^{1-p}}{\Gamma(2-p)} \int_0^1 \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \\
&\leq \left(\frac{z^{r-p}}{\Gamma(r-p+1)} + \frac{\Gamma(3-r)z^{1-p}}{\Gamma(2-p)} \right) M \\
&\leq \left(\frac{z^{r-p}}{\Gamma(r-p+1)} + \frac{\Gamma(3-r)}{\Gamma(2-p)} \right) M.
\end{aligned} \tag{37}$$

Thus,

$$\begin{aligned}
\|\mathfrak{P}w\|_{\mathfrak{M}} &\leq \left(\frac{1}{\Gamma(r)} + \frac{1}{\Gamma(r+1)} + \frac{1}{\Gamma(r-p+1)} \right. \\
&\quad \left. + 2F(3-r) + \frac{\Gamma(3-r)}{\Gamma(2-p)} \right) M.
\end{aligned} \tag{38}$$

Hence, $\mathfrak{P}(\Gamma_\zeta)$ has the property of the uniform boundedness. Next, we show that $\mathfrak{P}w$ is equicontinuous. To do this, for each $w \in \Gamma_\zeta$ and $z_1, z_2 \in [0, 1]$ with $z_1 < z_2$, we have

$$\begin{aligned}
|\mathfrak{P}w(z_2) - \mathfrak{P}w(z_1)| &= \left| \frac{1}{\Gamma(r)} \int_0^{z_1} (z_2-s)^{r-1} \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \right. \\
&\quad \left. - \frac{1}{\Gamma(r)} \int_0^{z_1} (z_1-s)^{r-1} \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \right|
\end{aligned}$$

$$\begin{aligned}
&+ \frac{1}{\Gamma(r)} \int_{z_1}^{z_2} (z_2-s)^{r-1} \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \\
&- \frac{1}{\Gamma(r)} \int_0^{z_1} (z_1-s)^{r-1} \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \\
&- \Gamma(3-r)(z_2-z_1) \int_0^1 \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \\
&\leq \frac{1}{\Gamma(r)} \int_0^{z_1} [(z_2-s)^{r-1} - (z_1-s)^{r-1}] \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \\
&\quad + \frac{1}{\Gamma(r)} \int_0^{z_2} (z_2-s)^{r-1} \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \\
&\quad - \Gamma(3-r)(z_2-z_1) \int_0^1 \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \\
&= \frac{M}{\Gamma(r)} \int_0^{z_1} [(z_2-s)^{r-1} - (z_1-s)^{r-1}] ds \\
&\quad + \frac{M}{\Gamma(r)} \int_{z_1}^{z_2} (z_2-s)^{r-1} ds + M\Gamma(3-r)(z_2-z_1) \\
&= \frac{M(z_2^r - z_1^r)}{\Gamma(r+1)} + M\Gamma(3-r)(z_2-z_1).
\end{aligned} \tag{39}$$

It is seen that the right-hand side of (39) does not depend on w and tends to zero whenever $z_1 \rightarrow z_2$ which leads to $|Bw(z_2) - Bw(z_1)| \rightarrow 0$. Further, we have

$$\begin{aligned}
|(Pw)'(z_2) - (Pw)'(z_1)| &= \left| \int_0^1 \frac{\partial H(z_2, s)}{\partial z_2} \Xi(s, w(s), D^p w(s)) ds \right. \\
&\quad \left. - \int_0^1 \frac{\partial H(z_1, s)}{\partial z_1} \Xi(s, w(s), D^p w(s)) ds \right| \\
&\leq \int_0^1 \left| \frac{\partial H(z_2, s)}{\partial z_2} - \frac{\partial H(z_1, s)}{\partial z_1} \right| \Xi(s, w(s), D^p w(s)) ds \\
&\leq \frac{M}{\Gamma(r-1)} \left[\int_0^{z_1} [(z_2-s)^{r-2} - (z_1-s)^{r-2}] ds + \int_{z_1}^{z_2} (z_2-s)^{r-2} ds \right] \\
&= \frac{M}{\Gamma(r)} [z_2^{r-1} - z_1^{r-1}],
\end{aligned} \tag{40}$$

which tends to zero whenever $z_1 \rightarrow z_2$. In addition,

$$\begin{aligned}
|D^p Bw(z_2) - D^p Bw(z_1)| &= \frac{1}{\Gamma(1-p)} \left| \int_0^{z_2} (z_2-s)^{-p} (Bw)'(s) ds \right. \\
&\quad \left. - \int_0^{z_1} (z_1-s)^{-p} (Bw)'(s) ds \right| \\
&= \frac{1}{\Gamma(1-p)} \left| \int_0^{z_1} [(z_1-s)^{-p} - (z_2-s)^{-p}] \right. \\
&\quad \cdot \left(\int_0^1 \frac{\partial H(s, \lambda)}{\partial s} \Xi(\lambda, w(\lambda), D^p w(\lambda)) d\lambda \right) ds
\end{aligned}$$

$$\begin{aligned}
& + \int_{z_1}^{t_2} (z_2 - s)^{-p} \left(\int_0^1 \frac{\partial H(s, \lambda)}{\partial s} \Xi(\lambda, w(\lambda), D^p w(\lambda)) d\lambda \right) ds \\
& \leq \frac{1}{\Gamma(1-p)} \int_0^{z_1} [(z_1 - s)^{-p} - (z_2 - s)^{-p}] \\
& \quad \cdot \left(\int_0^1 \left| \frac{\partial H(s, \lambda)}{\partial s} \right| \Xi(\lambda, w(\lambda), D^p w(\lambda)) d\lambda \right) ds \\
& \quad + \frac{1}{\Gamma(1-p)} \int_{z_1}^{z_2} (z_2 - s)^{-p} \\
& \quad \cdot \left(\int_0^1 \left| \frac{\partial H(s, \lambda)}{\partial s} \right| \Xi(\lambda, w(\lambda), D^p w(\lambda)) d\lambda \right) ds \\
& \leq \frac{M}{\Gamma(1-p)} \int_0^{z_1} [(z_1 - s)^{-p} - (z_2 - s)^{-p}] \\
& \quad \cdot \left(\int_0^1 \left| \frac{\partial H(s, \lambda)}{\partial s} \right| d\lambda \right) ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{M}{\Gamma(1-p)} \int_{z_1}^{z_2} (z_2 - s)^{-p} \left(\int_0^1 \left| \frac{\partial H(s, \lambda)}{\partial s} \right| d\lambda \right) ds \\
& \leq \frac{M}{\Gamma(2-p)} \left(\frac{1}{\Gamma(r)} + (3-r) \right) \left(2(z_2 - z_1)^{1-p} + z_1^{1-p} - z_2^{1-p} \right),
\end{aligned} \tag{41}$$

which tends to zero as $z_1 \rightarrow z_2$. Therefore, inequalities (39), (40), and (41) imply that Pw is equicontinuous. Knowing that it is uniformly bounded, we find that P is completely continuous. Schauder's fixed point theorem implies that P has a fixed point $w \in \Gamma_\zeta$ which is a solution for the multiterm semilinear boundary value problem (3) and the proof is completed. $\square$

Corollary 10. Let Ξ be continuous on $[0, 1] \times \mathbb{R} \times \mathbb{R}$ and there exists $\xi > 0$ such that

$$\xi \leq \frac{1}{((1/\Gamma(r)) + (1/\Gamma(r+1)) + 2\Gamma(3-r) + (1/\Gamma(2-p)\Gamma(r)) + (\Gamma(3-r)/\Gamma(2-p)))}. \tag{42}$$

Then, a solution exists for the multiterm semilinear boundary value problem (3).

Proof. We choose $\theta(z) = 0$ and $\max_{z \in [0,1]} \Xi(z, w, v) = \xi(|w| + |v|)$. So, the assumption (42) allows us to apply Theorem 9 which affirms the existence of a solution for the mentioned multiterm semilinear problem (3). $\square$

Corollary 11. Assume that there exist two real numbers $\eta, v > 0$ such that

$$\eta \geq \sup_{\substack{0 < z \leq 1 \\ w \in \mathbb{R}_+, v \in \mathbb{R}}} \Xi(z, w, v) \text{ and } v \leq \inf_{\substack{0 < z \leq 1 \\ w \in \mathbb{R}_+, v \in \mathbb{R}}} \Xi(z, w, v). \tag{43}$$

Then, the multiterm semilinear boundary value problem (3) has at least a positive solution on $[0, Z^*]$, where

$$Z^* = \left(\frac{v\Gamma(r+1)\Gamma(3-r)}{\eta} \right)^{1/r-1}. \tag{44}$$

$\square$

Proof. From definitions of the functions $\widehat{\delta}(z, u, v)$ and $\widehat{\Delta}(z, u, v)$, it is followed that

$$v \leq \widehat{\delta}(z, u, v) \leq \widehat{\Delta}(z, u, v) \leq \eta, \quad (0 \leq z \leq Z^*, w \in \mathbb{R}_+, v \in \mathbb{R}). \tag{45}$$

Define

$$\begin{aligned}
w^*(z) &= \Gamma(3-r)\eta z - \frac{vz^r}{\Gamma(r+1)}, \quad 0 \leq z \leq Z^*, \\
w_*(z) &= \Gamma(3-r)v - \frac{\eta z^r}{\Gamma(r+1)}, \quad 0 \leq z \leq Z^*,
\end{aligned} \tag{46}$$

So, we get $0 \leq w_*(z) \leq w^*(z)$ for $0 \leq z \leq Z^*$ and also

$$\begin{aligned}
w^*(z) &\geq \eta \int_0^1 |H(z, s)| ds \\
&\geq \int_0^1 |H(z, s)| \widehat{\Delta}(s, w^*(s), D^p w^*(s)) ds, \quad 0 \leq z \leq Z^*, \\
w_*(z) &\leq v \int_0^1 |H(z, s)| ds \\
&\leq \int_0^1 |H(z, s)| \widehat{\delta}(s, w_*(s), D^p w_*(s)) ds, \quad 0 \leq z \leq Z^*.
\end{aligned} \tag{47}$$

Moreover, by some direct computations, we get

$$\begin{aligned}
D^p w^*(z) &= \frac{\Gamma(3-r)\eta z^{1-p}}{\Gamma(2-p)} - \frac{vz^{r-p}}{\Gamma(r-p+1)}, \quad 0 \leq z \leq Z^*, \\
D^p w_*(z) &= \frac{\Gamma(3-r)v z^{1-p}}{\Gamma(2-p)} - \frac{\eta z^{r-p}}{\Gamma(r-p+1)}, \quad 0 \leq z \leq Z^*.
\end{aligned} \tag{48}$$

FIGURE 1: Graphs of w , w_* , and w^* .

Thus,

$$\begin{aligned}
 & \frac{\Gamma(3-r)z^{1-p}}{\Gamma(2-p)} \int_0^1 \widehat{\delta}(s, w^*(s), D^p w^*(s)) ds \\
 & - \frac{1}{\Gamma(r-p)} \int_0^z (z-s)^{r-p-1} \widehat{\Delta}(s, w^*(s), D^p w^*(s)) ds \\
 & \leq \frac{\Gamma(3-r)\eta z^{1-p}}{\Gamma(2-p)} - \frac{\nu z^{r-p}}{\Gamma(r-p+1)} = D_{0+}^p w^*(z), \\
 & \frac{\Gamma(3-r)z^{1-p}}{\Gamma(2-p)} \int_0^1 \widehat{\Delta}(s, w_*(s), D^p w_*(s)) ds \\
 & - \frac{1}{\Gamma(r-p)} \int_0^z (z-s)^{r-p-1} \widehat{\delta}(s, w_*(s), D^p w_*(s)) ds \\
 & \geq \frac{\Gamma(3-r)\nu z^{1-p}}{\Gamma(2-p)} - \frac{\eta z^{r-p}}{\Gamma(r-p+1)} = D_{0+}^p w_*(z),
 \end{aligned} \tag{49}$$

This means that the assumption (A1) is satisfied. Finally, if (A2) holds, then we can choose ζ such that

$$\begin{aligned}
 \zeta \geq & A + B + \frac{B}{\Gamma(2-p)} + \eta \left(\frac{1}{\Gamma(r)} + \frac{1}{\Gamma(r+1)} + 2\Gamma(3-r) \right. \\
 & \left. + \frac{1}{\Gamma(2-p)\Gamma(r)} + \frac{\Gamma(3-r)}{\Gamma(2-p)} \right).
 \end{aligned} \tag{50}$$

Now, all hypotheses of Theorem 9 hold. Consequently, the multiterm semilinear boundary value problem (3) has at least a positive solution $\mathbf{w} \in \Gamma_\zeta$, where $0 \leq w_*(z) \leq w(z) \leq w^*(z)$ and $D^p w_*(z) \leq D^p w(z) \leq D^p w^*(z)$ for each $z \in [0, Z^*]$ and the corollary is proved. $\square$

FIGURE 2: Graphs of $A = D^p w_*$, $B = D^p w^*$, and $C = D^p w$.

To validate the theoretical findings, we provide a special example corresponding to the suggested multiterm semilinear boundary value problem (3).

Example 12. According to the multiterm semilinear boundary value problem (3), in the present example, we take $r = 1.5$, $p = 0.5$, $\eta = 1$, $\nu = 0.5$, $Z^* = 0.3$ and

$$\Xi(z, w, y) = \nu + (\eta - \nu)z = 0.5 + 0.5z. \tag{51}$$

By taking into account the definition of the function Ξ , we clearly have $\nu \leq \Xi(z, w, \nu) \leq \eta$. Now, we choose upper and lower control functions $\widehat{\Delta}(z, u, \nu) = \eta$ and $\widehat{\delta}(z, u, \nu) = \nu$, respectively, and then, we get

$$\begin{aligned}
 W^*(z) &= \Gamma(1.5)z - \frac{1}{2\Gamma(2.5)}z^{1.5} = 0.8862z - 0.3761z^{1.5}, \\
 W_*(z) &= 0.5 \times \Gamma(1.5)z - \frac{1}{\Gamma(2.5)}z^{1.5} = 0.4431z - 0.7522z^{1.5}, \\
 w(z) &= \frac{3\Gamma(1.5)}{4}z - \frac{1}{3\Gamma(1.5)}z^{1.5} - \frac{2}{15\Gamma(1.5)}z^{2.5} \\
 &= 0.6647z - 0.3761z^{1.5} - 0.1505z^{2.5}.
 \end{aligned} \tag{52}$$

Therefore, by some simple calculations, we obtain

$$\begin{aligned}
 D^p w^*(z) &= z^{0.5} - 0.5z, \\
 D^p w_*(z) &= 0.5z - z, \\
 D^p w(z) &= 0.75z^{0.5} - 0.5z - 0.25z^2.
 \end{aligned} \tag{53}$$

Note that in this example, we have taken the interval $[0, Z^*] \subset [0, 1]$ to hold the essential condition $0 \leq w_*(z) \leq w(z)$

$\leq w^*(z)$. The graphs of positive solutions and their derivatives are illustrated in Figures 1 and 2.

4. Conclusion

In this paper, we considered a new fractional class of the multiterm semilinear differential equation in the context of the standard Caputo differentiation operator. The main purpose here is to derive several criteria of the existence of positive solutions for mentioned multiterm boundary value problem. To achieve such an aim, we first obtained the relevant Green function of the equivalent integral equation and showed that the absolute value of the first-order partial derivative of this function is integrable. After introducing two lower and upper control functions, we defined an operator on the given Banach space and used Schauder's fixed point theorem for establishing the existence of positive solutions. With the help of a special numerical example at the end of the study, we validated our theoretical findings according to the method implemented here. By plotting upper and lower positive solutions and their derivatives, we compared our results with the exact solution graphically. The results and methods presented in this research can be widely applied in different complicated classes of boundary value problems involving modern integral conditions in the future works. Also, one can design such models by using new generalized fractional operators involving singular or nonsingular kernels for describing their dynamical behaviors in better settings.

Data Availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Acknowledgments

The authors would like to thank the dear respected reviewers for their constructive remarks to improve the last version of the paper.

References

- [1] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North Holland Mathematics Studis, Amsterdam, 2006.
- [2] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Amsterdam, 1993.
- [3] A. Alsaedi, B. Ahmad, M. Alghanmi, and S. K. Ntouyas, "On a generalized Langevin type nonlocal fractional integral multivalued problem," *Mathematics*, vol. 7, no. 11, p. 1015, 2019.
- [4] D. Baleanu, S. Etemad, and S. Rezapour, "On a fractional hybrid integro-differential equation with mixed hybrid integral boundary value conditions by using three operators," *Alexandria Engineering Journal*, vol. 59, no. 5, pp. 3019–3027, 2020.
- [5] A. Boulfoul, B. Tellab, N. Abdellouahab, and K. Zennir, "Existence and uniqueness results for initial value problem of nonlinear fractional integro-differential equation on an unbounded domain in a weighted Banach space," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 5, pp. 3509–3520, 2020.
- [6] S. T. M. Thabet, S. Etemad, and S. Rezapour, "On a new structure of the pantograph inclusion problem in the Caputo conformable setting," *Boundary Value Problems*, vol. 2020, no. 1, Article ID 171, 2020.
- [7] G. Wang, K. Pei, R. P. Agarwal, L. Zhang, and B. Ahmad, "Nonlocal Hadamard fractional boundary value problem with Hadamard integral and discrete boundary conditions on a half-line," *Journal of Computational and Applied Mathematics*, vol. 343, pp. 230–239, 2018.
- [8] M. Subramanian and A. Zada, "Existence and uniqueness of solutions for coupled systems of Liouville-Caputo type fractional integrodifferential equations with Erdélyi-Kober integral conditions," *International Journal of Nonlinear Sciences and Numerical Simulation*, 2020.
- [9] U. Riaz and A. Zada, "Analysis of (α, β) -order coupled implicit Caputo fractional differential equations using topological degree method," *International Journal of Nonlinear Sciences and Numerical Simulation*, 2020.
- [10] M. Ahmad, A. Zada, and X. Wang, "Existence, uniqueness and stability of implicit switched coupled fractional differential equations of ψ -Hilfer type," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 21, no. 3-4, pp. 327–337, 2020.
- [11] S. Zhang, "Existence of positive solution for some class of nonlinear fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 278, no. 1, pp. 136–148, 2003.
- [12] M. El-Shahed, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *Abstract and Applied Analysis*, vol. 2007, Article ID 10368, 8 pages, 2007.
- [13] A. Guezane-Lakoud and L. Zenkoui, "Existence of positive solutions for a fourth-order three-point boundary value problem," *Journal of Applied Mathematics and Computing*, vol. 50, no. 1-2, pp. 139–155, 2016.
- [14] Y. Tian, S. Sun, and Z. Bai, "Positive solutions of fractional differential equations with $-$ Laplacian," *Journal of Function Spaces*, vol. 2017, Article ID 3187492, 9 pages, 2017.
- [15] A. Seemab, M. Ur Rehman, J. Alzabut, and A. Hamdi, "On the existence of positive solutions for generalized fractional boundary value problems," *Boundary Value Problems*, vol. 2019, no. 1, Article ID 186, 2019.
- [16] H. M. Baskonus and H. Bulut, "On the numerical solutions of some fractional ordinary differential equations by fractional Adams-Bashforth-Moulton method," *Open Mathematics*, vol. 13, no. 1, pp. 457–556, 2015.
- [17] T. A. Sulaiman, M. Yavuz, H. Bulut, and H. M. Baskonus, "Investigation of the fractional coupled viscous Burgers'

- equation involving Mittag-Leffler kernel,” *Physica A*, vol. 527, article 121126, 2019.
- [18] S. Q. Zhang, “Positive solutions for boundary-value problems of nonlinear fractional differential equations,” *Electronic Journal of Differential Equations*, vol. 36, pp. 1–12, 2006.
 - [19] Z. Bai and H. Lu, “Positive solutions for boundary value problem of nonlinear fractional differential equation,” *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 495–505, 2005.
 - [20] I. Podlubny, *Fractional differential equations*, Academic Press, 1999.
 - [21] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, New Jersey, London and Hong Kong, 2014.