Research Article

Fourth Hankel Determinant for a Subclass of Starlike Functions Based on Modified Sigmoid

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In our present investigation, we obtain the improved third-order Hankel determinant for a class of starlike functions connected with modified sigmoid functions. Further, we investigate the fourth-order Hankel determinant, Zalcman conjecture, and also evaluate the fourth-order Hankel determinants for 2-fold, 3-fold, and 4-fold symmetric starlike functions.

1. Introduction and Motivation

Denoted by $\mathcal{A}$, the class of functions f which are analytic in

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}, \quad (1)$$

and are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n (\forall z \in \mathbb{U}). \quad (2)$$

Also, let $\mathcal{S}$ be a subclass of class $\mathcal{A}$, containing all univalent functions in $\mathbb{U}$, and be normalized by the conditions

$$\begin{aligned} f(0) &= 0, \\ f'(0) &= 1. \end{aligned} \quad (3)$$

In 1916, working on the coefficients a_n of class $\mathcal{S}$, Bieberbach conjectured that

$$|a_n| \leq n (n = 2, 3, \dots), \quad (4)$$

which was proved by De Branges in 1984 (see [1]). From 1916 to 1984, for some subclasses of $\mathcal{S}$, many researchers used different techniques and established a number of results. The classes $\mathcal{S}^*$, $\mathcal{K}$, and $\mathcal{R}$, namely, the classes of convex, starlike, and bounded turning functions, respectively, are some major subclasses of the class $\mathcal{S}$. These classes are defined as follows:

$$\begin{aligned}\mathcal{S}^* &= \left\{ f \in \mathcal{S} : \Re \left(\frac{zf'(z)}{f(z)} \right) > 0 \ (\forall z \in \mathbb{U}) \right\}, \\ \mathcal{K} &= \left\{ f \in \mathcal{S} : \Re \left(\frac{(zf'(z))'}{f'(z)} \right) > 0 \ (\forall z \in \mathbb{U}) \right\}, \\ \mathcal{R} &= \left\{ f \in \mathcal{S} : \Re [f'(z)] > 0 \ (\forall z \in \mathbb{U}) \right\}.\end{aligned}\quad (5)$$

Furthermore, we say that an analytic function $f_1(z)$ is subordinated to $f_2(z)$ in $\mathbb{U}$ and is symbolically written as

$$f_1(z) < f_2(z) \ (\forall z \in \mathbb{U}), \quad (6)$$

if there exists a Schwartz function $u(z)$ with properties that

$$\begin{aligned}|u(z)| &\leq 1, \\ u(0) &= 1,\end{aligned}\quad (7)$$

such that

$$f_1(z) = f_2(u(z)). \quad (8)$$

Moreover, if $f_2(z)$ is in the class $\mathcal{S}$. Due to [2, 3], we get the following equivalence relation

$$\begin{aligned}f_1(0) &= f_2(0), \\ f_1(\mathbb{U}) &\subseteq f_2(\mathbb{U}).\end{aligned}\quad (9)$$

Now, by using the principle of subordination, a generalized set of the classes $\mathcal{S}^*$, $\mathcal{K}$, and $\mathcal{R}$ are given, respectively, as follows:

$$\mathcal{S}^*(\psi) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} < \psi(z) = \frac{1+z}{1-z} \ (\forall z \in \mathbb{U}) \right\}, \quad (10)$$

$$\mathcal{K}(\psi) = \left\{ f \in \mathcal{S} : \frac{(zf'(z))'}{f'(z)} < \psi(z) = \frac{1+z}{1-z} \ (\forall z \in \mathbb{U}) \right\}, \quad (11)$$

$$\mathcal{R}(\psi) = \left\{ f \in \mathcal{S} : f'(z) < \psi(z) = \frac{1+z}{1-z} \ (\forall z \in \mathbb{U}) \right\}. \quad (12)$$

By changing the right-hand side in (10), several familiar classes can be obtained such as if we keep $\psi(z) = (1 + Az)/(1 + Bz)$, we get the class of starlike functions associated with the Janowski functions (see [4]). Moreover, if we take

$$\begin{aligned}A &= 1, \\ B &= 1 - 2\alpha \ (0 < \alpha < 1),\end{aligned}\quad (13)$$

then, we have a class of starlike functions of order α (see [5]). Also, for the choice of $\psi(z) = 1 + (2/\pi^2)(\log((1 + \sqrt{z})/(1 - \sqrt{z})))^2$, a corresponding class of starlike

functions is obtained, which was introduced by Ronning (see [6]). Furthermore, if we take $\psi(z) = \sqrt{1+z}$, the class of starlike functions related with the lemniscate of Bernoulli domain is resulted which was introduced and investigated by Jangteng et al. [7, 8]. Next, if we take $\psi(z) = 1 + \sin(z)$, the family of starlike functions connected with the sine function is obtained (see [9]). Mendiratta et al. [10] obtained a subclass of strongly starlike functions associated with exponential functions for the choice of $\psi(z) = e^z$. Sharma et al. [11] derived a class of starlike functions associated with a cardioid domain by taking $\psi(z) = 1 + (4/3)z + (2/3)z^2$.

Moreover, several more subclasses of starlike functions have recently been presented in [12–15] through selecting specific functions for ψ , like functions associated with conic domains, shell-like curves associated with Fibonacci numbers, and functions related with Bell numbers.

Lately, based on the techniques of Ma and Minda [16], Goel and Kumar in [17] defined the class $\mathcal{S}_{SG}^*$, based on the subordination principle, as follows:

$$\frac{zf'(z)}{f(z)} < \frac{2}{1+e^{-z}} \ (\forall z \in \mathbb{U}), \quad (14)$$

and studied its various important geometric properties.

For a function f of the form (2), Pommerenke [18, 19] defined Hankel determinant $H_{q,n}(f)$, parameter q , with $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, as follows:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \quad (15)$$

The growth of $H_{q,n}(f)$ for fixed integer q and n was evaluated for different subfamilies of univalent functions. Jangteng et al. [7, 20] investigated the sharp bound of the determinant $H_{2,2}(f) = |a_2a_4 - a_3^2|$ for each of the classes $\mathcal{K}$, $\mathcal{S}^*$, and $\mathcal{R}$, while sharp estimation for the family of close-to-convex functions is still unknown (see [21]). On the other hand, Krishna et al. [22] proved the best estimate of $|H_{2,2}(f)|$ for the class of Bazilevič functions. More detailed work on $H_{2,2}(f)$ can be seen in [23–27] and also the references cited therein.

The determinant,

$$H_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}, \quad (16)$$

is known as third-order Hankel determinant and the estimation of this determinant $|H_{3,1}(f)|$ is more difficult and hence a potential attraction for a lot of researchers who focus on this field. In 1966–1967, Pommerenke defined this Hankel determinant but was an open problem till 2010. In 2010,

Babalola [28] was the first researcher who worked on $H_{3,1}(f)$ and successfully obtained the upper bound of $|H_{3,1}(f)|$ for the functions belonging to the classes $\mathcal{S}^*$, $\mathcal{K}$, and $\mathcal{R}$. A few mathematicians further expanded on this work to include other subclasses of holomorphic and univalent functions (see for example [29–35]). Zaprawa [36] enhanced their work in 2017 by demonstrating

$$|H_{3,1}(f)| \leq \begin{cases} 1, & (f \in \mathcal{S}^*), \\ \frac{49}{540}, & (f \in \mathcal{K}), \\ \frac{41}{60}, & (f \in \mathcal{R}), \end{cases} \quad (17)$$

and asserted that these inequalities are still nonsharp. Additionally, for the sharpness, he thought about the subfamilies of $\mathcal{S}^*$, $\mathcal{K}$, and $\mathcal{R}$ comprising of functions with m -fold symmetry and acquired the sharp bounds. Recently, in 2018, Kowalczyk et al. [37] and Lecko et al. [38] got the sharp inequalities which are

$$\begin{aligned} |H_{3,1}(f)| &\leq \frac{4}{135}, \\ |H_{3,1}(f)| &\leq \frac{1}{9}, \end{aligned} \quad (18)$$

for the classes $\mathcal{K}$ and $\mathcal{S}^*(1/2)$, respectively, where the symbol $\mathcal{S}^*(1/2)$ indicates the family of starlike functions of order $1/2$.

The main goal of this paper is to investigate the necessary and sufficient conditions for functions to get into the class $\mathcal{S}_{SG}^*$ in the form of coefficient inequality, convolution results, and the essential third-order Hankel determinant for this class in (6) and also for its 2-, 3-, and 4-fold symmetric functions.

2. A Set of Lemmas

Let $\mathcal{P}$ be the family of functions p that are holomorphic in $\mathbb{U}$ with $\Re(p(z)) > 0$ and its series form is as follow:

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (\forall z \in \mathbb{U}). \quad (19)$$

Lemma 1. If $p \in \mathcal{P}$ and it is of the form ((19)), then,

$$|c_n| \leq 2(n \geq 1), \quad (20)$$

$$|c_{n+k} - \delta c_n c_k| \leq 2(0 \leq \delta \leq 1), \quad (21)$$

$$|c_2 - \xi c_1^2| \leq 2 \max \{1; |2\xi - 1|\} \quad (\xi \in \mathbb{C}). \quad (22)$$

Further results related to Lemma 1 can be found in [39, 40].

Lemma 2 (see [41]). Let $p \in \mathcal{P}$ have the series expansion of the form ((19)). Then, for $x, \sigma, \rho \in \mathbb{U} = \mathbb{U} \cup \{1\}$,

$$\begin{aligned} 2c_2 &= c_1^2 + x(4 - c_1^2), \\ 4c_3 &= c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)\sigma, \\ 8c_4 &= c_1^4 + (4 - c_1^2)x[c_1^2(x^2 - 3x + 3) + 4x] - 4(4 - c_1^2) \\ &\quad \cdot (1 - |x|^2)[c(x - 1)\sigma + \bar{x}\sigma^2 - (1 - |\sigma|^2)\rho]. \end{aligned} \quad (23)$$

Lemma 3 (see [42]). Let m, n, l satisfy the inequalities $0 < m < 1, 0 < r < 1$, and

$$\begin{aligned} 8r(1 - r)[(mn - 2l)^2 + (m(r + m) - n)^2] \\ + m(1 - m)(n - 2rm)^2 \leq 4m^2(1 - m)^2r(1 - r). \end{aligned} \quad (24)$$

If $p \in \mathcal{P}$ and has power series (19), then,

$$\left| lp_1^4 + rp_2^2 + 2mp_1p_3 - \frac{3}{2}np_1^2p_2 - p_4 \right| \leq 2. \quad (25)$$

3. Improve Upper Bound $H_{3,1}(f)$ for the Class $\mathcal{S}_{SG}^*$

To prove Theorem 6, we need the following two lemmas (Lemma 4 and Lemma 5).

Lemma 4 (see [43]). If $f \in \mathcal{S}_{SG}^*$ and is of the form (2), then,

$$|a_3 - a_2^2| \leq \frac{1}{4}. \quad (26)$$

Lemma 5 (see [43]). Let $f \in \mathcal{S}_{SG}^*$ be of the form (2), and then,

$$|a_2a_3 - a_4| \leq \frac{1}{6}. \quad (27)$$

We now state and prove Theorem 6.

Theorem 6. Let $f \in \mathcal{S}_{SG}^*$ be of the form (2), and then,

$$\begin{aligned} |a_6| &\leq \frac{21\,797}{57\,600} \approx 0.378\,42, \\ |a_7| &\leq \frac{1424\,429}{2073\,600} \approx 0.686\,94, \\ |a_2a_4 - a_3^2| &\leq \frac{1}{16}. \end{aligned} \quad (28)$$

The result is sharp for function

$$f(z) = z \exp \left(\int_0^z \frac{e^{t^2} - 1}{t(e^{t^2} + 1)} dt \right) = z + \frac{1}{4}z^3 + \dots \quad (29)$$

Proof. Since $f \in \mathcal{S}_{SG}^*$, then, there exists a Schwarz function $w(z)$, given in (7) such that

$$\frac{zf'(z)}{f(z)} = \frac{2}{1 + e^{-w(z)}}. \quad (30)$$

Let

$$\Psi(w(z)) = \frac{2}{1 + e^{-w(z)}}, \quad (31)$$

$$k(z) = 1 + c_1 z + c_2 z^2 + \dots = \frac{1 + w(z)}{1 - w(z)}. \quad (32)$$

Obviously, the function $k(z) \in \mathcal{P}$ and

$$w(z) = \frac{k(z) - 1}{k(z) + 1}. \quad (33)$$

This gives

$$w(z) = \frac{k(z) - 1}{k(z) + 1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \dots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \dots}. \quad (34)$$

From (31) and (34), we have

$$\begin{aligned} \frac{2}{1 + e^{-w(z)}} &= 1 + \frac{1}{4} c_1 z + \left(\frac{1}{4} c_2 - \frac{1}{8} c_1^2 \right) z^2 + \left(\frac{11}{192} c_1^3 - \frac{1}{4} c_2 c_1 + \frac{1}{4} c_3 \right) z^3 \\ &+ \left(-\frac{3}{128} c_1^4 + \frac{11}{64} c_1^2 c_2 - \frac{1}{4} c_3 c_1 - \frac{1}{8} c_2^2 + \frac{1}{4} c_4 \right) z^4 + \dots, \end{aligned} \quad (35)$$

while

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= 1 + a_2 z + (2a_3 - a_2^2) z^2 + (a_2^3 - 3a_2 a_3 + 3a_4) z^3 \\ &+ (-a_2^4 + 4a_2^2 a_3 - 4a_2 a_4 - 2a_3^2 + 4a_5) z^4 \\ &+ (5a_6 - 5a_2 a_5 + a_2^5 - 5a_3 a_4 - 5a_2^2 a_3 + 5a_2^2 a_4 + 5a_2 a_3^2) z^5 \\ &+ \left(6a_7 - 6a_2 a_6 + 6a_2^2 a_5 - 6a_3 a_5 + 12a_2 a_3 a_4 - a_2^6 - 6a_2^3 a_3 \right. \\ &\quad \left. - 3a_4^2 + 2a_3^3 - 9a_2^2 a_3^2 + 6a_2^4 a_3 \right) z^6 \\ &+ \dots. \end{aligned} \quad (36)$$

□

On equating coefficients of (35) and (36), we get

$$a_2 = \frac{1}{4} c_1, \quad (37)$$

$$a_3 = \frac{1}{8} c_2 - \frac{1}{32} c_1^2, \quad (38)$$

$$a_4 = \frac{7}{1152} c_1^3 - \frac{5}{96} c_2 c_1 + \frac{1}{12} c_3, \quad (39)$$

$$a_5 = -\frac{17}{18432} c_1^4 + \frac{7}{384} c_1^2 c_2 - \frac{1}{24} c_3 c_1 - \frac{3}{128} c_2^2 + \frac{1}{16} c_4, \quad (40)$$

$$\begin{aligned} a_6 &= -\frac{257}{1843200} c_1^5 - \frac{107}{23040} c_1^3 c_2 + \frac{31}{1920} c_3 c_1^2 + \frac{139}{7680} c_1 c_2^2 \\ &- \frac{11}{320} c_4 c_1 - \frac{19}{480} c_3 c_2 + \frac{1}{20} c_5, \end{aligned} \quad (41)$$

$$\begin{aligned} a_7 &= \frac{33599}{132710400} c_1^6 - \frac{73}{2211840} c_1^4 c_2 - \frac{79}{17280} c_1^3 c_3 \\ &- \frac{1451}{184320} c_1^2 c_2^2 + \frac{109}{7680} c_4 c_1^2 + \frac{47}{1440} c_1 c_2 c_3 - \frac{7}{240} c_5 c_1 \\ &+ \frac{55}{9216} c_2^3 - \frac{13}{384} c_4 c_2 - \frac{5}{288} c_3^2 + \frac{1}{24} c_6. \end{aligned} \quad (42)$$

Now from (41), we have

$$\begin{aligned} |a_6| &= \left| \frac{1}{96} \left(c_5 - \frac{9}{10} c_1 c_4 \right) + \frac{19}{480} (c_5 - c_1 c_4) + \frac{31}{1920} c_1^2 \left(c_3 - \frac{107}{372} c_1 c_2 \right) \right. \\ &\quad \left. - \frac{257}{1843200} c_1^5 + \frac{139}{7680} c_1 c_2 \right| \\ &\leq \frac{1}{96} \left| c_5 - \frac{9}{10} c_1 c_4 \right| + \frac{19}{480} |c_5 - c_1 c_4| + \frac{31}{1920} |c_1|^2 \left| c_3 - \frac{107}{372} c_1 c_2 \right| \\ &\quad + \frac{257}{1843200} |c_1|^5 + \frac{139}{7680} |c_1| |c_2|. \end{aligned} \quad (43)$$

By applying (20) and (21) to above we get

$$|a_6| \leq \frac{21797}{57600}. \quad (44)$$

Now from (42), we have

$$\begin{aligned} |a_7| &= \left| \frac{1}{24} \left(c_6 - \frac{5}{12} c_3^2 \right) - \frac{13}{384} c_2 \left(c_4 - \frac{55}{312} c_2^2 \right) - \frac{7}{240} c_1 \left(c_5 - \frac{109}{224} c_1 c_4 \right) \right. \\ &\quad \left. + \frac{47}{1440} c_1 c_2 \left(c_3 - \frac{1451}{6016} c_1 c_2 \right) + \frac{33599}{132710400} c_1^6 - \frac{73}{2211840} c_1^4 c_2 - \frac{79}{17280} c_1^3 c_3 \right. \\ &\leq \frac{1}{24} \left| c_6 - \frac{5}{12} c_3^2 \right| + \frac{13}{384} |c_2| \left| c_4 - \frac{55}{312} c_2^2 \right| + \frac{7}{240} |c_1| \left| c_5 - \frac{109}{224} c_1 c_4 \right| + \frac{47}{1440} |c_1| |c_2| \\ &\quad \cdot \left| c_3 - \frac{1451}{6016} c_1 c_2 \right| + \frac{33599}{132710400} |c_1|^6 + \frac{73}{2211840} |c_1|^4 |c_2| + \frac{79}{17280} |c_1|^3 |c_3|. \end{aligned} \quad (45)$$

By applying (20) and (21) to the above, we get

$$|a_7| \leq \frac{1424429}{2073600}. \quad (46)$$

Now, from (37)–(39), we have

$$|a_2 a_4 - a_3^2| = \left| \frac{5}{9216} c_1^4 - \frac{1}{192} c_1^2 c_2 + \frac{1}{48} c_3 c_1 - \frac{1}{64} c_2^2 \right|. \quad (47)$$

Using Lemma 2, we get

$$|a_2a_4 - a_3^2| = \left| -\frac{7}{9216}c_1^4 - \frac{1}{192}c_1^2x^2(4 - c_1^2) - \frac{1}{256}x^2(4 - c_1^2)^2 + \frac{1}{96}c_1(4 - c_1^2)(1 - |x|^2)z \right|. \quad (48)$$

Let $|x| = y, y \in [0, 1], c_1 = c, c \in [0, 2]$, and $|z| = 1$, along with triangle inequality, and we have

$$|a_2a_4 - a_3^2| \leq \frac{7}{9216}c^4 + \frac{1}{192}c^2y^2(4 - c^2) + \frac{1}{256}y^2(4 - c^2)^2 + \frac{1}{96}c(4 - c^2)(1 - y^2) = G(c, y). \quad (49)$$

Differentiating (49) partially with respect to y , we have

$$\frac{\partial G(c, y)}{\partial y} = \frac{(c^2 - 8c + 12)(4 - c^2)y}{384} > 0, \quad (50)$$

showing that $G(c, y)$ is an increasing function in interval $y \in [0, 1], c \in [0, 2]$, so the maximum is attained at $y = 1$, that is,

$$\max G(c, y) = G(c, 1) = \frac{7}{9216}c^4 + \frac{1}{192}c^2(4 - c^2) + \frac{1}{256}(4 - c^2)^2 = F(c). \quad (51)$$

Now

$$\begin{aligned} F'(c) &= \frac{7}{2304}c^3 + \frac{1}{96}c(4 - c^2) - \frac{1}{96}c^3 - \frac{1}{64}c(4 - c^2) \\ &= \frac{7}{2304}c^3 - \frac{1}{192}c(4 - c^2) - \frac{1}{96}c^3, \end{aligned} \quad (52)$$

since $F'(c) = 0$ has root at $c = 0$ and also

$$F''(c) = -\frac{1}{48} < 0, \quad (53)$$

so the maximum is attained at $c = 0$; therefore, we have

$$F(c) = \frac{1}{16}. \quad (54)$$

Hence,

$$|a_2a_4 - a_3^2| \leq F(c) = \frac{1}{16}. \quad (55)$$

For the third Hankel determinant, we need the following result.

Lemma 7 (see [17]). *Let $f \in \mathcal{S}_{SG}^*$ be of the form (2). Then,*

$$\begin{aligned} |a_2| &\leq \frac{1}{2}, \\ |a_3| &\leq \frac{1}{4}, \\ |a_4| &\leq \frac{1}{6}, \\ |a_5| &\leq \frac{1}{8}. \end{aligned} \quad (56)$$

Theorem 8. *Let $f \in \mathcal{S}_{SG}^*$ be of the form (2). Then,*

$$|H_{3,1}(f)| \leq \frac{43}{576} \approx 0.07465. \quad (57)$$

Proof. Since from (16), we have

$$H_{3,1}(f) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2), \quad (58)$$

by applying triangle inequality, we obtain

$$|H_{3,1}(f)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|. \quad (59)$$

Now, using Lemmas 4–7 and Theorem 6 in conjunction with (59), we can get the required result. $\square$

4. Bounds of $H_{4,1}(f)$ for the Class $\mathcal{S}_{SG}^*$

In recent years, researchers have started to evaluate the fourth-order Hankel determinant for different subclasses of analytic functions. The trend of finding the fourth-order Hankel determinant in geometric function theory started in 2018, when Arif et al. [44] studied and obtained the upper bound for the class of bounded turning functions. Recently Zhang and Tang [31] studied the fourth-order Hankel determinant for a subclass of starlike functions associated with the sine function. Inspired from the recent research going on and from the above works, we discuss here the fourth-order Hankel determinant for the class $\mathcal{S}_{SG}^*$.

From (15), we can write $H_{4,1}(f)$ as

$$H_{4,1}(f) = a_7H_{3,1}(f) - a_6\delta_1 + a_5\delta_2 - a_4\delta_3, \quad (60)$$

where

$$\delta_1 = a_3(a_2a_5 - a_3a_4) - a_4(a_5 - a_2a_4) + a_6(a_3 - a_2^2), \quad (61)$$

$$\delta_2 = a_3(a_3a_5 - a_4^2) - a_5(a_5 - a_2a_4) + a_6(a_4 - a_2a_3), \quad (62)$$

$$\delta_3 = a_4(a_3a_5 - a_4^2) - a_5(a_2a_5 - a_3a_4) + a_6(a_4 - a_2a_3). \quad (63)$$

Theorem 9. Let $f \in \mathcal{S}_{SG}^*$ be of the form (2), and then,

$$|a_2 a_5 - a_3 a_4| \leq \frac{5}{36}. \quad (64)$$

Proof. From equations (37), (38), (39), and (40), we get

$$\begin{aligned} |a_2 a_5 - a_3 a_4| &= \left| -\frac{1}{24576} c_1^5 + \frac{5}{2304} c_1^3 c_2 - \frac{1}{128} c_3 c_1^2 + \frac{1}{1536} c_1 c_2^2 \right. \\ &\quad \left. + \frac{1}{64} c_4 c_1 - \frac{1}{96} c_3 c_2 \right| = \left| \frac{5}{2304} c_1^3 \left(c_2 - \frac{3}{360} c_1^2 \right) \right. \\ &\quad \left. + \frac{1}{64} c_1 \left(c_4 - \frac{1}{2} c_1 c_3 \right) - \frac{1}{96} c_2 \left(c_3 - \frac{1}{16} c_1 c_2 \right) \right| \\ &\leq \frac{5}{2304} |c_1|^3 \left| c_2 - \frac{3}{360} c_1^2 \right| + \frac{1}{64} |c_1| \left| c_4 - \frac{1}{2} c_1 c_3 \right| \\ &\quad + \frac{1}{96} |c_2| \left| c_3 - \frac{1}{16} c_1 c_2 \right|. \end{aligned} \quad (65)$$

Now, making use of (20), (21), and (22) in conjunction with (65), we can get the required result. $\square$

Theorem 10. Let $f \in \mathcal{S}_{SG}^*$ be of the form (2), and then,

$$|a_5 - a_2 a_4| \leq \frac{1}{8}. \quad (66)$$

Proof. From equations (37), (39), and (40), we get

$$|a_5 - a_2 a_4| = \frac{1}{16} \left| \frac{5}{128} c_1^4 - \frac{1}{2} c_1^2 c_2 + c_3 c_1 + \frac{3}{8} c_2^2 - c_4 \right|. \quad (67)$$

Applying Lemma 3 to the last term, we get the required result. $\square$

Theorem 11. Let $f \in \mathcal{S}_{SG}^*$ be of the form (2), and then,

$$|a_3 a_5 - a_4^2| \leq \frac{7405}{82944}. \quad (68)$$

Proof. From equations (37), (39), and (40), we get

$$\begin{aligned} |a_3 a_5 - a_4^2| &= \left| -\frac{43}{5308416} c_1^6 - \frac{23}{442368} c_1^4 c_2 + \frac{1}{3456} c_1^3 c_3 \right. \\ &\quad \left. + \frac{11}{36864} c_1^2 c_2^2 - \frac{1}{512} c_4 c_1^2 + \frac{1}{288} c_1 c_2 c_3 - \frac{3}{1024} c_2^3 \right. \\ &\quad \left. + \frac{1}{128} c_4 c_2 - \frac{1}{144} c_3^2 \right| = \left| -\frac{43}{5308416} c_1^6 \right. \\ &\quad \left. - \frac{23}{442368} c_1^4 c_2 - \frac{1}{144} c_3 \left(c_3 - \frac{1}{2} c_1 c_2 \right) \right. \\ &\quad \left. - \frac{3}{1024} c_2^2 \left(c_2 - \frac{11}{108} c_1^2 \right) + \frac{1}{128} c_4 \left(c_2 - \frac{1}{4} c_1^2 \right) \right| \end{aligned}$$

$$\begin{aligned} &+ \frac{1}{3456} c_1^3 \left(c_3 - \frac{23}{128} c_1 c_2 \right) \Big| \leq \frac{43}{5308416} |c_1|^6 \\ &+ \frac{23}{442368} |c_1|^4 |c_2| + \frac{1}{144} |c_3| \left| c_3 - \frac{1}{2} c_1 c_2 \right| \\ &+ \frac{3}{1024} |c_2|^2 \left| c_2 - \frac{11}{108} c_1^2 \right| + \frac{1}{128} |c_4| \left| c_2 - \frac{1}{4} c_1^2 \right| \\ &+ \frac{1}{3456} |c_1|^3 \left| c_3 - \frac{23}{128} c_1 c_2 \right|. \end{aligned} \quad (69)$$

Now, making use of (20), (21), and (22) in conjunction with (69), we can get the required result. $\square$

Theorem 12. Let $f \in \mathcal{S}_{SG}^*$ be of the form (2), and then,

$$|H_{4,1}(f)| \leq \frac{16431024581}{119439360000} \approx 0.13757. \quad (70)$$

Proof. From (15), we have

$$H_{4,1}(f) = a_7 H_{3,1}(f) - a_6 \delta_1 + a_5 \delta_2 - a_4 \delta_3, \quad (71)$$

where δ_1, δ_2 , and δ_3 are defined in (61), (62), and (63), respectively. Now, using triangle inequalities, we have

$$|H_{4,1}(f)| \leq |a_7| |H_{3,1}(f)| + |a_6| |\delta_1| + |a_5| |\delta_2| + |a_4| |\delta_3|, \quad (72)$$

since

$$\begin{aligned} |\delta_1| &= |a_3(a_2 a_5 - a_3 a_4) - a_4(a_5 - a_2 a_4) + a_6(a_3 - a_2^2)| \\ &\leq |a_3| |a_2 a_5 - a_3 a_4| + |a_4| |a_5 - a_2 a_4| + |a_6| |a_3 - a_2^2|. \end{aligned} \quad (73)$$

By applying Lemmas 4 and 7 and Theorems 6, 9, and 10, we get

$$|\delta_1| \leq \frac{34597}{230400}. \quad (74)$$

And also,

$$\begin{aligned} |\delta_2| &= |a_3(a_3 a_5 - a_4^2) - a_5(a_5 - a_2 a_4) + a_6(a_4 - a_2 a_3)| \\ &\leq |a_3| |a_3 a_5 - a_4^2| + |a_5| |a_5 - a_2 a_4| + |a_6| |a_4 - a_2 a_3|. \end{aligned} \quad (75)$$

Using Lemmas 5 and 7 and Theorems 6, 10, and 11 we get

$$|\delta_2| \leq \frac{837853}{8294400}. \quad (76)$$

Also, again,

$$\begin{aligned} |\delta_3| &= |a_4(a_3a_5 - a_4^2) - a_5(a_2a_5 - a_3a_4) + a_6(a_4 - a_2a_3)| \\ &\leq |a_4||a_3a_5 - a_4^2| + |a_5||a_2a_5 - a_3a_4| + |a_6||a_4 - a_2a_3|, \end{aligned} \quad (77)$$

Using Lemmas 5 and 7 and Theorems 6, 9, and 11, we get

$$|\delta_3| \leq \frac{1185\,817}{12\,441\,600}. \quad (78)$$

Now, using the values of (74), (76), and (78) along with Theorem 6 and Lemma 7 to (72), we get the desired the estimate. $\square$

5. Zalcman Conjecture for Class $\mathcal{S}_{SG}^*$

One of the main conjectures in the geometric function theory, suggested by Lawrence Zalcman in 1960, is that the coefficients of class $\mathcal{S}$ satisfy the inequality

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2. \quad (79)$$

Only the well-known Koebe function $k(z) = z/(1-z)^2$ and its rotations have equality in the above form. For the popular Fekete-Szegő inequality, when $n=2$, the equality holds. Recently, Khan et al. [43] evaluated the Zalcman conjecture for the class of starlike functions with respect to symmetric points associated with the sine function. Many researchers have studied the Zalcman function in the literature [45–47].

Theorem 13. Let $f \in \mathcal{S}_{SG}^*$ be of the form (2), and then,

$$|a_5 - a_3^2| \leq \frac{1}{8}. \quad (80)$$

The result is sharp for function

$$f(z) = z \exp \left(\int_0^z \frac{e^{t^4} - 1}{t(e^{t^4} + 1)} dt \right) = z + \frac{1}{8}z^5 + \dots \quad (81)$$

Proof. From equations (37) and (40), we get

$$|a_5 - a_3^2| = \frac{1}{16} \left| -\frac{35}{1152}c_1^4 + \frac{5}{12}c_1^2c_2 - \frac{2}{3}c_3c_1 - \frac{5}{8}c_2^2 + c_4 \right|. \quad (82)$$

Using Lemma 3 and equation (82), we can get the required result. $\square$

6. Bounds of $H_{4,1}(f)$ for 2-Fold, 3-Fold, and 4-Fold Symmetric Functions

Let $m \in \mathbb{N} = \{1, 2, 3, \dots\}$. It is called m -fold symmetric if a rotation of domain $\mathbb{U}$ about the origin through an angle 2π

/m carries itself on the domain $\mathbb{U}$. It is obvious that in $\mathbb{U}$, an analytic function f is m -fold symmetric if

$$f\left(e^{\frac{2\pi}{m}}z\right) = e^{\frac{2\pi}{m}}f(z) \quad (\forall z \in \mathbb{U}). \quad (83)$$

The set of m -fold symmetric univalent functions with the following series:

$$f(z) = z + \sum_{k=2}^{\infty} a_{mk+1} z^{mk+1} \quad (\forall z \in \mathbb{U}), \quad (84)$$

is referred to as $\mathcal{S}^{(m)}$.

The subclass $\mathcal{S}_{SG}^{*(m)}$ is a collection of m -fold symmetric starlike functions associated with the modified sigmoid function. More precisely, an analytic function f of the form (84) belongs to class $\mathcal{S}_{SG}^{*(m)}$ if and only if

$$\frac{zf'(z)}{f(z)} = \frac{2}{1 + e^{-(p(z)-1)/p(z)+1}} \quad (p \in \mathcal{P}^{(m)}), \quad (85)$$

where the set $\mathcal{P}^{(m)}$ is defined by

$$\mathcal{P}^{(m)} = \left\{ p \in \mathcal{P} : p(z) = 1 + \sum_{k=1}^{\infty} c_{mk} z^{mk} \quad (\forall z \in \mathbb{U}) \right\}. \quad (86)$$

Theorem 14. If $f \in \mathcal{S}_{SG}^{*(2)}$ and be of the form (84), then,

$$|a_3a_7 - a_5^2| \leq \frac{751}{21\,504}. \quad (87)$$

Proof. Since $f \in \mathcal{S}_{SG}^{*(2)}$, therefore, there exists a function $p \in \mathcal{P}^{(2)}$ such that

$$\frac{zf'(z)}{f(z)} = \frac{2}{1 + e^{-(p(z)-1)/p(z)+1}}. \quad (88)$$

Using the series forms (84) and (86), when $m=2$ in the above relation, we have

$$a_3 = \frac{1}{8}c_2, \quad (89)$$

$$a_5 = \frac{1}{16}c_4 - \frac{3}{128}c_2^2, \quad (90)$$

$$a_7 = \frac{55}{10\,752}c_2^3 - \frac{13}{448}c_4c_2 + \frac{1}{28}c_6. \quad (91)$$

Now, using (89), (90), and (91), we get

$$\begin{aligned}
 |a_3 a_7 - a_5^2| &= \left| \frac{31}{344\,064} c_2^4 - \frac{5}{7168} c_2^2 c_4 + \frac{1}{224} c_6 c_2 - \frac{1}{256} c_4^2 \right| \\
 &= \left| \frac{31}{344\,064} c_2^4 + \frac{1}{224} c_2 \left(c_6 - \frac{5}{32} c_2 c_4 \right) - \frac{1}{256} c_4^2 \right| \\
 &\leq \frac{31}{344\,064} |c_2|^4 + \frac{1}{224} |c_2| \left| c_6 - \frac{5}{32} c_2 c_4 \right| + \frac{1}{256} |c_4|^2.
 \end{aligned} \tag{92}$$

Now, using (20) and (21) to the above, we get the required result. $\square$

Theorem 15. If $f \in \mathcal{S}_{SG}^{*(2)}$ and be of the form (84), then,

$$|a_5 - a_3^2| \leq \frac{1}{8}. \tag{93}$$

Proof. Using (89) and (90), we have

$$|a_5 - a_3^2| = \frac{1}{16} \left| c_4 - \frac{5}{8} c_2^2 \right|. \tag{94}$$

Using (21) to the above, we get the required result. $\square$

Theorem 16. If $f \in \mathcal{S}_{SG}^{*(2)}$ and be of the form (84), then,

$$|H_{4,1}(f)| \leq \frac{751}{172\,032}. \tag{95}$$

Proof. Since $f \in \mathcal{S}_{SG}^{*(2)}$, therefore, $a_2 = a_4 = a_6 = 0$ and we have

$$H_{4,1}(f) = (a_5 - a_3^2)(a_3 a_7 - a_5^2). \tag{96}$$

Then,

$$|H_{3,1}(f)| = |a_5 - a_3^2| |a_3 a_7 - a_5^2|, \tag{97}$$

Using (87) and (93), we get the required result. $\square$

Theorem 17. If $f \in \mathcal{S}_{SG}^{*(3)}$ and be of the form (84), then,

$$|H_{4,1}(f)| \leq \frac{1}{432}. \tag{98}$$

Proof. Since $f \in \mathcal{S}_{SG}^{*(3)}$, therefore, there exists a function $p \in \mathcal{P}^{(3)}$ such that

$$\frac{zf'(z)}{f(z)} = \frac{2}{1 + e^{-(p(z)-1)/p(z)+1)}}. \tag{99}$$

Using the series forms (84) and (86), when $m = 3$ in the above relation, we have

$$a_4 = \frac{1}{12} c_3, \tag{100}$$

$$a_7 = \frac{1}{24} c_6 - \frac{5}{288} c_3^2.$$

Now,

$$H_{4,1}(f) = a_4^2 (a_4^2 - a_7). \tag{101}$$

Therefore,

$$|H_{4,1}(f)| = \left| -\frac{1}{3456} c_3^2 \left(c_6 - \frac{7}{12} c_3^2 \right) \right| = \frac{1}{3456} |c_3|^2 \left| c_6 - \frac{7}{12} c_3^2 \right|. \tag{102}$$

Using (20) and (21), we get the desired result. $\square$

Theorem 18. If $f \in \mathcal{S}_{SG}^{*(4)}$ and be of the form (84), then,

$$|H_{4,1}(f)| \leq \frac{1}{64}. \tag{103}$$

Proof. Since $f \in \mathcal{S}_{SG}^{*(4)}$, therefore, there exists a function $p \in \mathcal{P}^{(4)}$ such that

$$\frac{zf'(z)}{f(z)} = \frac{2}{1 + e^{-(p(z)-1)/p(z)+1)}}. \tag{104}$$

Using the series forms (84) and (86), when $m = 4$ in the above relation, we have

$$a_5 = \frac{1}{16} c_4. \tag{105}$$

Since $f \in \mathcal{S}_{SG}^{*(4)}$, therefore, $a_2 = a_3 = a_4 = a_6 = a_7 = 0$ and we have

$$\begin{aligned}
 H_{4,1}(f) &= a_5^2, \\
 H_{4,1}(f) &= \frac{1}{256} c_4^2.
 \end{aligned} \tag{106}$$

Now,

$$|H_{4,1}(f)| = \frac{1}{256} |c_4|^2. \tag{107}$$

Using (20) to the above, we get the required result. $\square$

7. Conclusion

In our present investigation, we have obtained the improved third-order Hankel determinant for a class of starlike functions connected with modified sigmoid functions. Furthermore, we have investigated the fourth-order Hankel determinant and Zalcman conjecture and also evaluated fourth-order Hankel determinants for 2-fold, 3-fold, and 4-fold symmetric starlike functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors equally contributed to this manuscript and approved the final version.

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