Research Article

Analytical Solution for Differential and Nonlinear Integral Equations via F_{ω_e} -Suzuki Contractions in Modified ω_e -Metric-Like Spaces

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The aim of this manuscript is to present a new space, namely, a modified ω_e -metric-like space, and we establish some related fixed point results using extended F_{ω_e} -Suzuki and generalized F_{ω_e} -Suzuki contractions on the mentioned space. Here, we support our theoretical consequences in two ways: the first one consists of presenting illustrative examples and the second one consists of finding analytical solutions for some integral and differential equations in the context of the mentioned space.

1. Introduction and Elementary Discussions

Solving second-order differential equations after converting to integral equations using Green's function has become commonplace for many academic researchers because of its importance in theoretical and practical applications. One aspect of the solution of the differential and integral equations is the analytical solution, which in turn mainly supports the numerical solution used in solving dynamical systems.

There are many methods for obtaining analytical solutions, including the fixed point technique. For instance, many works have focused their attention on solving the Fredholm integral equation [1] analytically and numerically by this technique [2, 3].

Among the generalizations of the Banach principle [4], the notion of F -contractions was initiated by Wardkowski [5].

After two years, Piri and Kuman [6] made a slight change in the principle of Wardkowski and called it an F -Suzuki

contraction. It has contributed significantly to upholding the reputation of the fixed point theory in many fields and has become a great weight in the functional analysis.

Definition 1. A mapping $A : \mathcal{T} \rightarrow \mathcal{T}$ defined on the metric space $(\mathcal{T}, d)$ is named as an

(i) F -contraction if there are $F \in \mathcal{I}$ and $\vartheta > 0$ so that

$$d(A\ell, A\hbar) > 0 \Rightarrow \vartheta + F(d(A\ell, A\hbar)) \leq F(d(\ell, \hbar)) \text{ for all } \ell, \hbar \in \mathcal{T} \quad (1)$$

(ii) F -Suzuki contraction if there are $F \in \mathcal{I}$ and $\vartheta > 0$ so that

$$\frac{1}{2}d(\ell, A\ell) < d(\ell, h) \Rightarrow \vartheta + F(d(A\ell, Ah)) \leq F(d(\ell, h)) \text{ for all } \ell, h \in \mathcal{T}, \quad (2)$$

where Σ is the class of functions $F : (0, +\infty) \rightarrow \mathbb{R}$ so that
 (F_1) for all $\mathcal{U}, \Omega \in \mathbb{R}^+$ such that $\mathcal{U} < \Omega, F(\mathcal{U}) < F(\Omega)$
 (F_2) for each positive real sequence $\{\mathcal{U}_p\}$, $\lim_{p \rightarrow \infty} \mathcal{U}_p = 0$
iff $\lim_{n \rightarrow \infty} F(\mathcal{U}_p) = -\infty$
 (F_3) there is $\omega \in (0, 1)$ so that $\lim_{\mathcal{U} \rightarrow 0^+} \mathcal{U}^\omega F(\mathcal{U}) = 0$

There are developments to translate fixed point theorems into nonlinear integral equations and differential equations (for related works and developments, see [7–20]).

The nonlinear mapping in Banach contraction needs to be continuous. It is not applicable in the discontinuous case. In the past, Kannan [21]. was able to overcome this shortcoming by giving a fixed point result without the mapping being continuous. Variant works appeared to resolve this problem by adding conditions to the spaces (see [22–25]).

Among modern spaces, a b -metric-like space was introduced by Alghamdi et al. [26] as an extension of a b -metric, which was presented by Bakhtin [27], and a metric-like, which was presented by Amini-Harandi [28]. In [26], some fixed point theorems have been provided. In recent works, many contributions on fixed point results involving different contractive conditions are given (see [29–36]).

At the beginning of 2019, Parvaneh and Kadelburg [37] generalized the b -metric-like space by replacing the coefficient located in the third condition by a strictly increasing continuous function. They named it an extended b -metric-like space and studied on it some fixed point sequences for JSHR-contractive type mappings with some applications.

It is noted that in this section, we did not address definitions and mathematical theorems for two reasons: first, there are large basics related to the mentioned spaces, and second, access to the main results is direct.

According to the previous results, in this paper, we present fixed point consequences by using F_{ω_e} -Suzuki contractions in the class of modified ω_e -metric-like spaces. Under the framework of the mentioned space, we apply the theoretical results to find an analytical solution for nonlinear integral equations. On the other hand, some important examples to justify our theorems are discussed.

2. An Extended F_{ω_e} -Suzuki Contraction

We begin this section with definitions of metric-like and b -metric-like spaces.

Definition 2 (see [26]). Let $\mathcal{T}$ be a nonempty set. A function $\Delta : \mathcal{T}^2 \rightarrow \mathbb{R}^+$ is named as metric-like on $\mathcal{T}$, if for all $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3 \in \mathcal{T}$:

$$\begin{aligned} (\mathcal{T}_1) \Delta(\mathcal{U}_1, \mathcal{U}_2) &= 0 \Rightarrow \mathcal{U}_1 = \mathcal{U}_2 \\ (\mathcal{T}_2) \Delta(\mathcal{U}_1, \mathcal{U}_2) &= \Delta(\mathcal{U}_2, \mathcal{U}_1) \\ (\mathcal{T}_3) \Delta(\mathcal{U}_1, \mathcal{U}_3) &\leq \Delta(\mathcal{U}_1, \mathcal{U}_2) + \Delta(\mathcal{U}_2, \mathcal{U}_3) \end{aligned}$$

Definition 3 (see [26]). A b -metric-like on a nonempty set $\mathcal{T}$ is a function $\omega : \mathcal{T}^2 \rightarrow \mathbb{R}^+$ so that for all $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3 \in \mathcal{T}$ and a constant $s \geq 1$:

$$\begin{aligned} (\omega_1) \omega(\mathcal{U}_1, \mathcal{U}_2) &= 0 \Rightarrow \mathcal{U}_1 = \mathcal{U}_2 \\ (\omega_2) \omega(\mathcal{U}_1, \mathcal{U}_2) &= \omega(\mathcal{U}_2, \mathcal{U}_1) \\ (\omega_3) \omega(\mathcal{U}_1, \mathcal{U}_3) &\leq s[\omega(\mathcal{U}_1, \mathcal{U}_2) + \omega(\mathcal{U}_2, \mathcal{U}_3)] \end{aligned}$$

Here, $(\mathcal{T}, \omega)$ is named as a b -metric-like space (with constant s).

For examples about metric-like and b -metric-like spaces, see [33–35].

Now, we will generalize Definition 3 as follows.

Definition 4. Let $\mathcal{T}$ be a nonempty set and $s : \mathcal{T} \times \mathcal{T} \rightarrow [1, \infty)$. A function $\omega_e : \mathcal{T}^2 \rightarrow [0, \infty)$ is called a modified ω_e -metric-like if, for all $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3 \in \mathcal{T}$:

$$\begin{aligned} (\omega_{e1}) \omega_e(\mathcal{U}_1, \mathcal{U}_2) &= 0 \Rightarrow \mathcal{U}_1 = \mathcal{U}_2 \\ (\omega_{e2}) \omega_e(\mathcal{U}_1, \mathcal{U}_2) &= \omega_e(\mathcal{U}_2, \mathcal{U}_1) \\ (\omega_{e3}) \omega_e(\mathcal{U}_1, \mathcal{U}_3) &\leq s(\mathcal{U}_1, \mathcal{U}_3)[\omega_e(\mathcal{U}_1, \mathcal{U}_2) + \omega_e(\mathcal{U}_2, \mathcal{U}_3)] \end{aligned}$$

Here, $(\mathcal{T}, \omega_e)$ is named as a modified extended b -metric-like space (simply, a modified ω_e -metric-like space).

Note that the class of modified ω_e -metric-like spaces is larger than the class of b -metric-like spaces by replacing the constant $s \geq 1$ of Definition 3 by a nonconstant function $s : \mathcal{T} \times \mathcal{T} \rightarrow [1, \infty)$ of Definition 4.

Example 5. Let $\mathcal{T} = [0, \infty)$. Define $\omega_e : \mathcal{T}^2 \rightarrow [0, \infty)$ by

$$\omega_e(\kappa, \mu) = \begin{cases} 0, & \text{if } \kappa = \mu = 0, \\ \frac{\mu}{1 + \mu}, & \text{if } \kappa = 0, \mu \neq 0, \\ \frac{\kappa}{1 + \kappa}, & \text{if } \mu = 0, \kappa \neq 0, \\ \kappa + \mu, & \text{if } \kappa \neq 0, \mu \neq 0. \end{cases} \quad (3)$$

Consider $s : \mathcal{T}^2 \rightarrow [1, \infty)$ as $s(\kappa, \mu) = 2 + 2\kappa + 2\mu$.

First, (ω_{e1}) and (ω_{e2}) are obvious. We need to prove (ω_{e3}) . For this, let $\mathcal{U}_1, \mathcal{U}_2$, and $\mathcal{U}_3$ in $\mathcal{T}$. We state the following cases.

Case 1. $\mathcal{U}_1 = \mathcal{U}_3 = 0$. Here, (ω_{e3}) holds.

Case 2. $\mathcal{U}_1 = 0$ and $\mathcal{U}_3 \neq 0$. Then,

$$\begin{aligned} \omega_e(\mathcal{U}_1, \mathcal{U}_3) &= \frac{\mathcal{U}_3}{1 + \mathcal{U}_3}, \\ s(\mathcal{U}_1, \mathcal{U}_3) &= 2 + 2\mathcal{U}_3. \end{aligned} \quad (4)$$

Subcase 1. $\mathcal{U}_2 = 0$. We have

$$\begin{aligned} \omega_e(\mathcal{U}_1, \mathcal{U}_3) &= \frac{\mathcal{U}_3}{1 + \mathcal{U}_3} \leq (2 + 2\mathcal{U}_3) \left[0 + \frac{\mathcal{U}_3}{1 + \mathcal{U}_3} \right] \\ &= s(\mathcal{U}_1, \mathcal{U}_3)[\omega_e(\mathcal{U}_1, \mathcal{U}_2) + \omega_e(\mathcal{U}_2, \mathcal{U}_3)]. \end{aligned} \quad (5)$$

Subcase 2. $\mathfrak{U}_2 \neq 0$. We have

$$\begin{aligned}\bar{\omega}_e(\mathfrak{U}_1, \mathfrak{U}_3) &= \frac{\mathfrak{U}_3}{1 + \mathfrak{U}_3} \leq (2 + 2\mathfrak{U}_3) \left[\frac{\mathfrak{U}_2}{1 + \mathfrak{U}_2} + \mathfrak{U}_2 + \mathfrak{U}_3 \right] \\ &= s(\mathfrak{U}_1, \mathfrak{U}_3) [\bar{\omega}_e(\mathfrak{U}_1, \mathfrak{U}_2) + \bar{\omega}_e(\mathfrak{U}_2, \mathfrak{U}_3)].\end{aligned}\quad (6)$$

Case 3. $\mathfrak{U}_3 = 0$ and $\mathfrak{U}_1 \neq 0$. Proceeding similarly as in Case 2, $(\bar{\omega}_{e3})$ holds.

Case 4. $\mathfrak{U}_1 \neq 0$ and $\mathfrak{U}_3 \neq 0$. Then,

$$\begin{aligned}\bar{\omega}_e(\mathfrak{U}_1, \mathfrak{U}_3) &= \mathfrak{U}_1 + \mathfrak{U}_3, \\ s(\mathfrak{U}_1, \mathfrak{U}_3) &= 2 + 2\mathfrak{U}_1 + 2\mathfrak{U}_3.\end{aligned}\quad (7)$$

Subcase 1. $\mathfrak{U}_2 = 0$. We have

$$\begin{aligned}\bar{\omega}_e(\mathfrak{U}_1, \mathfrak{U}_3) &= \mathfrak{U}_1 + \mathfrak{U}_3 \leq (2 + 2\mathfrak{U}_1 + 2\mathfrak{U}_3) \left[\frac{\mathfrak{U}_1}{1 + \mathfrak{U}_1} + \frac{\mathfrak{U}_3}{1 + \mathfrak{U}_3} \right] \\ &= s(\mathfrak{U}_1, \mathfrak{U}_3) [\bar{\omega}_e(\mathfrak{U}_1, \mathfrak{U}_2) + \bar{\omega}_e(\mathfrak{U}_2, \mathfrak{U}_3)].\end{aligned}\quad (8)$$

Subcase 2. $\mathfrak{U}_2 \neq 0$. We have

$$\begin{aligned}\bar{\omega}_e(\mathfrak{U}_1, \mathfrak{U}_3) &= \mathfrak{U}_1 + \mathfrak{U}_3 \leq (2 + 2\mathfrak{U}_1 + 2\mathfrak{U}_3) [\mathfrak{U}_1 + \mathfrak{U}_2 + \mathfrak{U}_2 + \mathfrak{U}_3] \\ &= s(\mathfrak{U}_1, \mathfrak{U}_3) [\bar{\omega}_e(\mathfrak{U}_1, \mathfrak{U}_2) + \bar{\omega}_e(\mathfrak{U}_2, \mathfrak{U}_3)].\end{aligned}\quad (9)$$

On the other hand, $(\mathfrak{T}, \bar{\omega}_e)$ is not a b -metric-like space. We argue by contradiction by assuming that $(\mathfrak{T}, \bar{\omega}_e)$ is a b -metric-like space with a coefficient $s \geq 1$ (a constant). Then, for any real $\mu > 0$, we have

$$\bar{\omega}_e(\mu, \mu + 1) \leq s[\bar{\omega}_e(\mu, 0) + \bar{\omega}_e(0, \mu + 1)].\quad (10)$$

That is,

$$2\mu + 1 \leq s \left[\frac{\mu}{1 + \mu} + \frac{\mu + 1}{2 + \mu} \right].\quad (11)$$

Letting $\mu \rightarrow \infty$, we get $+\infty \leq 2s$, which is a contradiction.

Example 6. Let $\mathfrak{T} = \{0, 1, 2\}$. Define $\bar{\omega}_e : \mathfrak{T}^2 \rightarrow [0, \infty)$ and $s : \mathfrak{T} \times \mathfrak{T} \rightarrow [1, \infty)$ as follows:

$$\begin{aligned}\bar{\omega}_e(0, 0) &= \bar{\omega}_e(1, 1) = \bar{\omega}_e(2, 2) = 0, \\ \bar{\omega}_e(0, 1) &= \bar{\omega}_e(1, 0) = 12, \\ \bar{\omega}_e(0, 2) &= \bar{\omega}_e(2, 0) = 1, \\ \bar{\omega}_e(1, 2) &= \bar{\omega}_e(2, 1) = 3,\end{aligned}\quad (12)$$

and $s(\kappa, \mu) = 2 + \kappa + \mu$.

First, we show that $\bar{\omega}_e$ is a modified $\bar{\omega}_e$ -metric-like space. Trivially, the conditions $(\bar{\omega}_{e1})$ and $(\bar{\omega}_{e2})$ hold. For $(\bar{\omega}_{e3})$, we get

$$\bar{\omega}_e(0, 1) = 12 ; s(0, 1) [\bar{\omega}_e(0, 2) + \bar{\omega}_e(2, 1)] = 12.\quad (13)$$

Thus,

$$\bar{\omega}_e(0, 1) \leq s(0, 1) [\bar{\omega}_e(0, 2) + \bar{\omega}_e(2, 1)].\quad (14)$$

Again,

$$\begin{aligned}\bar{\omega}_e(1, 2) &= 3 ; s(1, 2) [\bar{\omega}_e(1, 0) + \bar{\omega}_e(0, 2)] = 65, \\ \bar{\omega}_e(0, 2) &= 1 ; s(0, 2) [\bar{\omega}_e(0, 1) + \bar{\omega}_e(1, 2)] = 60.\end{aligned}\quad (15)$$

Hence, for all $\mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{U}_3 \in \mathfrak{T}$, $\bar{\omega}_e(\mathfrak{U}_1, \mathfrak{U}_3) \leq s(\mathfrak{U}_1, \mathfrak{U}_3) [\bar{\omega}_e(\mathfrak{U}_1, \mathfrak{U}_2) + \bar{\omega}_e(\mathfrak{U}_2, \mathfrak{U}_3)]$. Then, $(\mathfrak{T}, \bar{\omega}_e)$ is a modified $\bar{\omega}_e$ -metric-like space, but it is not a b -metric-like space because if we take $s = 2$ in the inequality (13), we get

$$\bar{\omega}_e(0, 1) = 12 ; 2[\bar{\omega}_e(0, 2) + \bar{\omega}_e(2, 1)] = 8.\quad (16)$$

Thus,

$$\bar{\omega}_e(\mathfrak{U}_1, \mathfrak{U}_3) \not\leq s[\bar{\omega}_e(\mathfrak{U}_1, \mathfrak{U}_2) + \bar{\omega}_e(\mathfrak{U}_2, \mathfrak{U}_3)].\quad (17)$$

Definition 7. Let $\{\mathfrak{U}_i\}$ be a sequence in the modified $\bar{\omega}_e$ -metric-like space $(\mathfrak{T}, \bar{\omega}_e)$.

- (a) If $\lim_{i \rightarrow \infty} \bar{\omega}_e(\mathfrak{U}_i, \mathfrak{U}) = \bar{\omega}_e(\mathfrak{U}, \mathfrak{U})$, then $\{\mathfrak{U}_i\}$ is convergent to $\mathfrak{U}$
- (b) $\{\mathfrak{U}_i\}$ is called Cauchy if $\lim_{i, j \rightarrow \infty} \bar{\omega}_e(\mathfrak{U}_i, \mathfrak{U}_j)$ exists and is finite
- (c) If for each Cauchy sequence $\{\mathfrak{U}_i\}$, there is $\mathfrak{U} \in \mathfrak{T}$, so that $\lim_{i, j \rightarrow \infty} \bar{\omega}_e(\mathfrak{U}_i, \mathfrak{U}_j) = \bar{\omega}_e(\mathfrak{U}, \mathfrak{U}) = \lim_{i \rightarrow \infty} \bar{\omega}_e(\mathfrak{U}_i, \mathfrak{U})$; therefore, $(\mathfrak{T}, \bar{\omega}_e)$ is said to be complete

Definition 8. A nonlinear self-mapping A on a modified $\bar{\omega}_e$ -metric-like space $(\mathfrak{T}, \bar{\omega}_e)$ is named as an extended $F_{\bar{\omega}_e}$ -Suzuki contraction if there are $F_{\bar{\omega}_e} \in \Pi$ and $\vartheta > 0$ so that for $\mathfrak{U}, \mu \in \mathfrak{T}$, the following condition holds:

$$\begin{aligned}\frac{1}{2} \bar{\omega}_e(\mathfrak{U}, A\mathfrak{U}) &< \bar{\omega}_e(\mathfrak{U}, \mu) \Rightarrow \vartheta + F_{\bar{\omega}_e}(\bar{\omega}_e(A\mathfrak{U}, A\mu)) \\ &\leq F_{\bar{\omega}_e}(\bar{\omega}_e(\mathfrak{U}, \mu)),\end{aligned}\quad (18)$$

such that $\lim_{n, m \rightarrow \infty} s(\mathfrak{U}_i, \mathfrak{U}_j) < 1/\eta$ for all $\mathfrak{U}_i \in \mathfrak{T}$, where $0 < \eta < 1$. We consider here $\mathfrak{U}_i = A^i \mathfrak{U}_0$, $i = 1, 2, \dots$, where Π is the set of continuous functions $F_{\bar{\omega}_e} : \mathbb{R}^+ \rightarrow \mathbb{R}$ so that

$$(\mathfrak{S}_1) \text{ For all } j, \ell \in \mathbb{R}^+ \text{ with } j < \ell, F_{\bar{\omega}_e}(j) < F_{\bar{\omega}_e}(\ell)$$

$(\mathfrak{S}_2)$ For each positive real sequence $\{j_p\}$, $\lim_{p \rightarrow \infty} j_p = 0$ iff $\lim_{p \rightarrow \infty} F_{\bar{\omega}_e}(j_p) = -\infty$

$$(\mathfrak{S}_3) \text{ There is } \eta \in (0, 1) \text{ so that } \lim_{j \rightarrow 0^+} j_\eta F_{\bar{\omega}_e}(j) = 0$$

Now, we introduce our first theorem.

Theorem 9. Let $(\mathcal{T}, \omega_e)$ be a complete modified ω_e -metric-like space and A be an extended F_{ω_e} -Suzuki contraction mapping, then A admits a unique fixed point.

Proof. Let $\mathcal{U}_\circ \in \mathcal{T}$ and $\{\mathcal{U}_i\}_{i=1}^\infty$ defined by $\mathcal{U}_{i+1} = A\mathcal{U}_i = A^{i+1}\mathcal{U}_\circ$. If there is $i \in \mathbb{N}$ so that $\omega_e(\mathcal{U}_i, A\mathcal{U}_i) = 0$. It completes the proof. Otherwise, assume that $0 < \omega_e(A_i, I\mathcal{U}_i) = \omega_e(\mathcal{U}_i, \mathcal{U}_{i+1}) = \omega_e^i$; therefore, for all $i \in \mathbb{N}$,

$$\frac{1}{2} \omega_e(\mathcal{U}_i, A\mathcal{U}_i) < \omega_e(\mathcal{U}_i, A\mathcal{U}_i), \quad (19)$$

it yields or

$$\vartheta + F_{\omega_e}(\omega_e(A\mathcal{U}_i, A^2\mathcal{U}_i)) \leq F_{\omega_e}(\omega_e(\mathcal{U}_i, A\mathcal{U}_i)), \quad (20)$$

$$F_{\omega_e}(\omega_e(A\mathcal{U}_i, A^2\mathcal{U}_i)) \leq F_{\omega_e}(\omega_e(\mathcal{U}_i, A\mathcal{U}_i)) - \vartheta. \quad (21)$$

By the same method, one gets

$$\begin{aligned} F_{\omega_e}(\omega_e^i) &= F_{\omega_e}(\omega_e(\mathcal{U}_i, A\mathcal{U}_i)) = F_{\omega_e}(\omega_e(A\mathcal{U}_{i-1}, A^2\mathcal{U}_{i-1})) \\ &\leq F_{\omega_e}(\omega_e(\mathcal{U}_{i-1}, A\mathcal{U}_{i-1})) - \vartheta \leq F_{\omega_e}(\omega_e(\mathcal{U}_{i-2}, A\mathcal{U}_{i-2})) \\ &\quad - 2\vartheta : \leq F_{\omega_e}(\omega_e(\mathcal{U}_\circ, A\mathcal{U}_\circ)) - i\vartheta \text{ for all } i \geq 1. \end{aligned} \quad (22)$$

Taking $i \rightarrow \infty$ in (22), we have

$$\lim_{i \rightarrow \infty} F_{\omega_e}(\omega_e^i) = -\infty. \quad (23)$$

So, by $(\mathfrak{F}_2)$, we obtain

$$\lim_{i \rightarrow \infty} \omega_e^i = 0. \quad (24)$$

Applying $(\mathfrak{F}_3)$, there is $\eta \in (0, 1)$ so that

$$\lim_{i \rightarrow \infty} (\omega_e^i)^\eta F_{\omega_e}(\omega_e^i) = 0. \quad (25)$$

By (22), one writes for all $i \geq 1$,

$$(\omega_e^i)^\eta (F_{\omega_e}(\omega_e^i) - F_{\omega_e}(\omega_e^\circ)) \leq -i\vartheta (\omega_e^i)^\eta \leq 0. \quad (26)$$

Considering (24) and (25) and passing $i \rightarrow \infty$ in (26), one gets

$$\lim_{i \rightarrow \infty} i(\omega_e^i)^\eta = 0. \quad (27)$$

By (27), there exists $i_1 \in \mathbb{N}$ so that $i(\omega_e^i)^\eta \leq 1$ for all $i \geq i_1$, or

$$\omega_e^i \leq \frac{1}{i^{1/\eta}} \text{ for all } i \geq i_1. \quad (28)$$

Consider the integers $m > i$. Applying (ω_3) and (28), one writes

$$\begin{aligned} \omega_e(\mathcal{U}_i, \mathcal{U}_m) &\leq s(\mathcal{U}_i, \mathcal{U}_m) [\omega_e(\mathcal{U}_i, \mathcal{U}_{i+1}) + \omega_e(\mathcal{U}_{i+1}, \mathcal{U}_m)] \\ &\leq s(\mathcal{U}_i, \mathcal{U}_m) \omega_e(\mathcal{U}_i, \mathcal{U}_{i+1}) + s(\mathcal{U}_i, \mathcal{U}_m) s(\mathcal{U}_{i+1}, \mathcal{U}_m) \\ &\quad \cdot [\omega_e(\mathcal{U}_{i+1}, \mathcal{U}_{i+2}) + \omega_e(\mathcal{U}_{i+2}, \mathcal{U}_m)] \\ &\leq s(\mathcal{U}_i, \mathcal{U}_m) \omega_e(\mathcal{U}_i, \mathcal{U}_{i+1}) + s(\mathcal{U}_i, \mathcal{U}_m) s(\mathcal{U}_{i+1}, \mathcal{U}_m) \\ &\quad \cdot \omega_e(\mathcal{U}_{i+1}, \mathcal{U}_{i+2}) + \cdots + s(\mathcal{U}_i, \mathcal{U}_m) s(\mathcal{U}_{i+1}, \mathcal{U}_m) s \\ &\quad \cdot (\mathcal{U}_{i+2}, \mathcal{U}_m) \cdots s(\mathcal{U}_{m-2}, \mathcal{U}_m) s(\mathcal{U}_{m-1}, \mathcal{U}_m) \\ &\quad \cdot \omega_e(\mathcal{U}_{m-1}, \mathcal{U}_m) \leq s(\mathcal{U}_1, \mathcal{U}_m) s(\mathcal{U}_2, \mathcal{U}_m) \cdots s(\mathcal{U}_i, \mathcal{U}_m) \\ &\quad \cdot \omega_e(\mathcal{U}_i, \mathcal{U}_{i+1}) + s(\mathcal{U}_1, \mathcal{U}_m) s(\mathcal{U}_2, \mathcal{U}_m) \cdots s(\mathcal{U}_{i+1}, \mathcal{U}_m) \\ &\quad \cdot \omega_e(\mathcal{U}_{i+1}, \mathcal{U}_{i+2}) + \cdots + s(\mathcal{U}_1, \mathcal{U}_m) s(\mathcal{U}_2, \mathcal{U}_m) \cdots \\ &\quad \cdot s(\mathcal{U}_{m-1}, \mathcal{U}_m) \omega_e(\mathcal{U}_{m-1}, \mathcal{U}_m) = \sum_{i=1}^\infty \omega_e(\mathcal{U}_i, \mathcal{U}_{i+1}) \\ &\quad \cdot \prod_{j=1}^i s(\mathcal{U}_j, \mathcal{U}_m) \leq \sum_{i=1}^\infty \frac{1}{i^{1/\eta}} \prod_{j=1}^i s(\mathcal{U}_j, \mathcal{U}_m) \\ &\leq \sum_{i=1}^\infty \frac{1}{i^{1/\eta}} \frac{1}{\eta} = \frac{1}{\eta} \sum_{i=1}^\infty \frac{1}{i^{1/\eta}}. \end{aligned} \quad (29)$$

Recall that $\sum_{i=1}^\infty 1/i^{1/\eta}$ converges, so $\omega_e(\mathcal{U}_i, \mathcal{U}_m) \rightarrow 0$. Therefore, $\{\mathcal{U}_i\}$ is a Cauchy sequence in the complete modified ω_e -metric-like space $(\mathcal{T}, \omega_e)$; hence, there is $\mathcal{U}^* \in \mathcal{T}$ such that $\mathcal{U}_i \rightarrow \mathcal{U}^*$, as $i \rightarrow \infty$. That is,

$$\lim_{i, m \rightarrow \infty} \omega_e(\mathcal{U}_i, \mathcal{U}_m) = \lim_{i \rightarrow \infty} \omega_e(\mathcal{U}_i, \mathcal{U}^*) = \omega_e(\mathcal{U}^*, \mathcal{U}^*) = 0. \quad (30)$$

Next, if F_{ω_e} is continuous, then two short cases arise.

Case 1. For each $i \in \mathbb{N}$, there exists $j_i \in \mathbb{N}$ such that $\mathcal{U}_{j_i} = A\mathcal{U}^*$ and $j_i > j_{i-1}$, where $j_0 = 0$. Therefore, one gets $\mathcal{U}^* = \lim_{i \rightarrow \infty} \mathcal{U}_{j_i} = \lim_{i \rightarrow \infty} A\mathcal{U}^* = A\mathcal{U}^*$.

Case 2. There is $i_\circ \in \mathbb{N}$ so that for all $i \geq i_\circ$, $\mathcal{U}_i \neq A\mathcal{U}^*$. It is clear that $1/2\omega_e(\mathcal{U}_i, A\mathcal{U}^*) < \omega_e(\mathcal{U}_i, A\mathcal{U}^*)$ for all $i \geq i_\circ$.

By (18), we have

$$\begin{aligned} \vartheta + F_{\omega_e}(\omega_e(\mathcal{U}_{i+1}, A^2\mathcal{U}^*)) &= \vartheta + F_{\omega_e}(\omega_e(A\mathcal{U}_i, A^2\mathcal{U}^*)) \\ &\leq F_{\omega_e}(\omega_e(\mathcal{U}_i, A\mathcal{U}^*)). \end{aligned} \quad (31)$$

Since F_{ω_e} is continuous, we obtain at the limit $i \rightarrow \infty$, or

$$\vartheta + F_{\omega_e}(\omega_e(\mathcal{U}^*, A^2\mathcal{U}^*)) \leq F_{\omega_e}(\omega_e(\mathcal{U}^*, A\mathcal{U}^*)), \quad (32)$$

$$F_{\omega_e}(\omega_e(\mathcal{U}^*, A\mathcal{U}^*)) \leq F_{\omega_e}(\omega_e(\mathcal{U}^*, \mathcal{U}^*)) - \vartheta, \quad (33)$$

which is a contradiction due to $(\mathfrak{F}_1)$. Then, $\omega_e(\mathcal{U}^*, A\mathcal{U}^*) = 0$, which means that $\mathcal{U}^* = A\mathcal{U}^*$.

The two cases above lead to the existence of a fixed point of A , i.e., $\mathcal{U}^* = A\mathcal{U}^*$.

Now, assume that $\mathcal{U}_1^*$ and $\mathcal{U}_2^*$ are so that $\mathcal{U}_1^* = A\mathcal{U}_1^* \neq \mathcal{U}_2^* = A\mathcal{U}_2^*$. We have $1/2\omega_e(\mathcal{U}_1^*, \mathcal{U}_2^*) < \omega_e(\mathcal{U}_1^*, \mathcal{U}_2^*)$, which implies by (18) that

$$\vartheta + F_{\omega_e}(\omega_e(\mathcal{U}_1^*, \mathcal{U}_2^*)) = \vartheta + F_{\omega_e}(\omega_e(A\mathcal{U}_1^*, A\mathcal{U}_2^*)) \leq F_{\omega_e}(\omega_e(\mathcal{U}_1^*, \mathcal{U}_2^*)). \quad (34)$$

It is again a contradiction.

The following examples verify all required hypotheses of Theorem 9.

Example 10. Let $\mathcal{T} = [0, \infty)$. Define $\omega_e : \mathcal{T}^2 \rightarrow \mathbb{R}$ by $\omega_e(\mathcal{U}, \aleph) = (\mathcal{U} + \aleph)^2$ and $s : \mathcal{T} \times \mathcal{T} \rightarrow [1, \infty)$ by $s(\mathcal{U}, \aleph) = 1 + \mathcal{U} + \aleph$, for all $\mathcal{U}, \aleph \in \mathcal{T}$. Here, ω_e is a modified extended ω_e -metric-like space. Define $A : \mathcal{T} \rightarrow \mathcal{T}$ as $A\mathcal{U} = (1/3)\mathcal{U}$, for all $\mathcal{U} \in \mathcal{T}$. It is clear that

$$\begin{aligned} \frac{1}{2}\omega_e(\mathcal{U}, A\mathcal{U}) &= \frac{1}{2}\omega_e\left(\mathcal{U}, \frac{1}{3}\mathcal{U}\right) = \frac{1}{2}\left(\mathcal{U} + \frac{1}{3}\mathcal{U}\right)^2 \\ &= \frac{16}{18}\mathcal{U}^2 \leq \mathcal{U}^2 \leq (\mathcal{U} + \mu)^2 = \omega_e(\mathcal{U}, \mu). \end{aligned} \quad (35)$$

Consider, for all $\mathcal{U}, \mu \in \mathcal{T}$,

$$\begin{aligned} F_{\omega_e}(\omega_e(A\mathcal{U}, A\mu)) &= F_{\omega_e}\left(\omega_e\left(\frac{1}{3}\mathcal{U}, \frac{1}{3}\mu\right)\right) = F_{\omega_e}\left(\left(\frac{1}{3}\mathcal{U} + \frac{1}{3}\mu\right)^2\right) \\ &= F_{\omega_e}\left(\frac{1}{9}(\mathcal{U} + \mu)^2\right). \end{aligned} \quad (36)$$

Also,

$$F_{\omega_e}(\omega_e(\mathcal{U}, \mu)) = F_{\omega_e}((\mathcal{U} + \mu)^2). \quad (37)$$

Let the function $F_{\omega_e} \in \Pi$ be defined by $F_{\omega_e}(\ell) = \ln(\ell)$, for $\ell > 0$. Then,

$$\begin{aligned} F_{\omega_e}(\omega_e(A\mathcal{U}, A\mu)) - F_{\omega_e}(\omega_e(\mathcal{U}, \mu)) &= \ln\left(\frac{1}{9}(\mathcal{U} + \mu)^2\right) - \ln((\mathcal{U} + \mu)^2) = \ln\left(\frac{1/9(\mathcal{U} + \mu)^2}{(\mathcal{U} + \mu)^2}\right) \\ &= \ln\left(\frac{1}{9}\right) = -2.197 \leq -2. \end{aligned} \quad (38)$$

Therefore, A is an extended F_{ω_e} -Suzuki contraction mapping with $\vartheta = 2$. Moreover, if $\mathcal{U}_m = \{1/(m+1)\} \in \mathcal{T}$, we have

$$\lim_{i, m \rightarrow \infty} s(\mathcal{U}_m, \mathcal{U}_i) = \lim_{i, m \rightarrow \infty} \left(1 + \frac{1}{m+1} + \frac{1}{i+1}\right) = 1 < \frac{1}{\eta}, \quad (39)$$

for $\eta \in (0, 1)$. So, all hypotheses of Theorem 9 are satisfied, and A has 0 as a unique fixed point.

Example 11. Let $\mathcal{T} = \{1/3^{2i-1} : i \in \mathbb{N}\} \cup \{0\}$. Suppose that $\omega_e : \mathcal{T} \times \mathcal{T} \rightarrow [0, \infty)$ and $s : \mathcal{T} \times \mathcal{T} \rightarrow [1, \infty)$ are functions defined by $\omega_e(\mathcal{U}, \mu) = (\max\{\mathcal{U}, \mu\})^2$ and $s(\mathcal{U}, \mu) = 1 + \mu + \mathcal{U}$, respectively, for all $\mu, \mathcal{U} \in \mathcal{T}$. Then, the pair $(\mathcal{T}, \omega_e)$ is a

complete modified ω_e -metric-like space. Define a nonlinear mapping $A : \mathcal{T} \rightarrow \mathcal{T}$ by

$$A\mathcal{U} = \begin{cases} \left\{\frac{1}{3^{2i}}\right\}, & \text{if } \mathcal{U} \in \left\{\frac{1}{3^{2i-1}}; i \in \mathbb{N}\right\}, \\ 0, & \text{if } \mathcal{U} = 0. \end{cases} \quad (40)$$

We shall prove that a mapping A is an extended F_{ω_e} -Suzuki contraction with $F_{\omega_e}(\ell) = \ln(\ell)$ for $\ell > 0$ and $\vartheta > 0$, by showing the following cases.

Case 1. Let $\mathcal{U} = 1/3^{2i-1}$ and $\mu = 1/3^{2m-1}$, for $m > i \geq 1$, one can write

$$\begin{aligned} \frac{1}{2}\omega_e(\mathcal{U}, A\mathcal{U}) &= \frac{1}{2}\omega_e\left(\frac{1}{3^{2i-1}}, \frac{1}{3^{2i}}\right) = \frac{1}{2}\left(\max\left\{\frac{1}{3^{2i-1}}, \frac{1}{3^{2i}}\right\}\right)^2 \\ &= \frac{1}{2}\left(\frac{1}{3^{2i-1}}\right)^2 < \left(\frac{1}{3^{2i-1}}\right)^2 \\ &= \left(\max\left\{\frac{1}{3^{2i-1}}, \frac{1}{3^{2m-1}}\right\}\right)^2 = \omega_e(\mathcal{U}, \mu). \end{aligned} \quad (41)$$

Consider

$$\begin{aligned} F_{\omega_e}(\omega_e(A\mathcal{U}, A\mu)) &= F_{\omega_e}\left(\omega_e\left(A\frac{1}{3^{2i-1}}, A\frac{1}{3^{2m-1}}\right)\right) \\ &= F_{\omega_e}\left(\omega_e\left(\frac{1}{3^{2i}}, \frac{1}{3^{2m}}\right)\right) \\ &= F_{\omega_e}\left(\left(\max\left\{\frac{1}{3^{2i}}, \frac{1}{3^{2m}}\right\}\right)^2\right) \\ &= F_{\omega_e}\left(\left(\frac{1}{3^{2i}}\right)^2\right) = \ln\left(\frac{1}{3^{2i}}\right)^2 = 2\ln\left(\frac{1}{3^{2i}}\right). \end{aligned} \quad (42)$$

Also,

$$\begin{aligned} F_{\omega_e}(\omega_e(\mathcal{U}, \mu)) &= F_{\omega_e}\left(\omega_e\left(\frac{1}{3^{2i-1}}, \frac{1}{3^{2m-1}}\right)\right) \\ &= F_{\omega_e}\left(\left(\max\left\{\frac{1}{3^{2i-1}}, \frac{1}{3^{2m-1}}\right\}\right)^2\right) \\ &= F_{\omega_e}\left(\left(\frac{1}{3^{2i-1}}\right)^2\right) = 2\ln\left(\frac{1}{3^{2i-1}}\right). \end{aligned} \quad (43)$$

By subtracting (42) and (43), we find that

$$\begin{aligned} F_{\omega_e}(\omega_e(A\mathcal{U}, A\mu)) - F_{\omega_e}(\omega_e(\mathcal{U}, \mu)) &= 2\left(\ln\left(\frac{1}{3^{2i}}\right) - \ln\left(\frac{1}{3^{2i-1}}\right)\right) = 2\left(\ln\left(\frac{1}{3^{2i}} \times 3^{2i} \cdot 3^{-1}\right)\right) \\ &= -2\ln 3 < -2. \end{aligned} \quad (44)$$

Case 2. Let $\mathfrak{U} = 1/3^{2l-1}$ and $\mu = 0$. We have

$$\begin{aligned} \frac{1}{2} \omega_e(\mathfrak{U}, A\mathfrak{U}) &= \frac{1}{2} \left(\max \left\{ \frac{1}{3^{2l-1}}, \frac{1}{3^{2l}} \right\} \right)^2 = \frac{1}{2} \left(\frac{1}{3^{2l-1}} \right)^2 \\ &< \left(\frac{1}{3^{2n-1}} \right)^2 = \left(\max \left\{ \frac{1}{3^{2l-1}}, 0 \right\} \right)^2 = \omega_e(\mathfrak{U}, \mu). \end{aligned} \quad (45)$$

Suppose that

$$\begin{aligned} F_{\omega_e}(\omega_e(A\mathfrak{U}, A\mu)) &= F_{\omega_e} \left(\omega_e \left(\frac{1}{3^{2l}}, 0 \right) \right) \\ &= F_{\omega_e} \left(\left(\max \left\{ \frac{1}{3^{2l}}, 0 \right\} \right)^2 \right) = F_{\omega_e} \left(\left(\frac{1}{3^{2l}} \right)^2 \right) \\ &= \ln \left(\frac{1}{3^{2l}} \right)^2 = 2 \ln \left(\frac{1}{3^{2l}} \right); \end{aligned} \quad (46)$$

also,

$$\begin{aligned} F_{\omega_e}(\omega_e(\mathfrak{U}, \mu)) &= F_{\omega_e} \left(\omega_e \left(\frac{1}{3^{2l-1}}, 0 \right) \right) \\ &= F_{\omega_e} \left(\left(\max \left\{ \frac{1}{3^{2l-1}}, 0 \right\} \right)^2 \right) \\ &= F_{\omega_e} \left(\left(\frac{1}{3^{2l-1}} \right)^2 \right) = 2 \ln \left(\frac{1}{3^{2l-1}} \right). \end{aligned} \quad (47)$$

By subtracting (46) and (47), we have the same inequality (44).

Case 3. Let $\mathfrak{U} = 0$ and $\mu = 1/3^{2m-1}$. The proof follows immediately as Case 2. Thus, A is an extended F_{ω_e} -Suzuki contraction mapping with $\vartheta = 2$. Here, 0 is the unique fixed point.

3. An Extended Generalized F_{ω_e} -Suzuki Contraction

Definition 12. A self-mapping A on a modified extended b -metric-like space $(\mathbb{T}, \omega_e)$ is called an extended generalized F_{ω_e} -Suzuki contraction if there are $F_{\omega_e} \in \Pi$ and $\vartheta > 0$ such that, if for all $\mathfrak{U}, \mathfrak{N} \in \mathbb{T}$, the following hypothesis is satisfied

$$\begin{aligned} \frac{1}{2} \omega_e(\mathfrak{U}, A\mathfrak{U}) &< \omega_e(\mathfrak{U}, \mathfrak{N}) \Rightarrow \vartheta + F_{\omega_e}(\omega_e(A\mathfrak{U}, A\mathfrak{N})) \\ &\leq F_{\omega_e} \left(\max \left\{ \omega_e(\mathfrak{U}, \mathfrak{N}), \frac{\omega_e(\mathfrak{U}, A\mathfrak{U})}{1 + \omega_e(\mathfrak{U}, A\mathfrak{U})}, \right. \right. \\ &\quad \left. \left. \frac{\omega_e(\mathfrak{N}, A\mathfrak{N})}{1 + \omega_e(\mathfrak{N}, A\mathfrak{N})}, \frac{\omega_e(\mathfrak{U}, A\mathfrak{N}) + \omega_e(\mathfrak{N}, A\mathfrak{U})}{4s(\mathfrak{U}, \mathfrak{N})} \right\} \right). \end{aligned} \quad (48)$$

Remark 13.

- (i) Every extended F_{ω_e} -Suzuki contraction is an extended generalized F_{ω_e} -Suzuki contraction
- (ii) Suppose that A is an extended generalized F_{ω_e} -Suzuki contraction, by Definition 12, for all $\mathfrak{U}, \mathfrak{N} \in \mathbb{T}$, we get $A\mathfrak{U} \neq A\mathfrak{N}$ and $1/2\omega_e(\mathfrak{U}, A\mathfrak{U}) < \omega_e(\mathfrak{U}, \mathfrak{N})$. Thus,

$$\begin{aligned} F_{\omega_e}(\omega_e(A\mathfrak{U}, A\mathfrak{N})) &< \vartheta + F_{\omega_e}(\omega_e(A\mathfrak{U}, A\mathfrak{N})) \\ &\leq F_{\omega_e} \left(\max \left\{ \omega_e(\mathfrak{U}, \mathfrak{N}), \frac{\omega_e(\mathfrak{U}, A\mathfrak{U})}{1 + \omega_e(\mathfrak{U}, A\mathfrak{U})}, \right. \right. \\ &\quad \left. \left. \frac{\omega_e(\mathfrak{N}, A\mathfrak{N})}{1 + \omega_e(\mathfrak{N}, A\mathfrak{N})}, \frac{\omega_e(\mathfrak{U}, A\mathfrak{N}) + \omega_e(\mathfrak{N}, A\mathfrak{U})}{4s(\mathfrak{U}, \mathfrak{N})} \right\} \right) \end{aligned} \quad (49)$$

By condition $(\mathfrak{F}_1)$, for all $\mathfrak{U}, \mathfrak{N} \in \mathbb{T}$ with $A\mathfrak{U} \neq A\mathfrak{N}$, we have

$$\begin{aligned} \omega_e(A\mathfrak{U}, A\mathfrak{N}) &\leq \max \left\{ \omega_e(\mathfrak{U}, \mathfrak{N}), \frac{\omega_e(\mathfrak{U}, A\mathfrak{U})}{1 + \omega_e(\mathfrak{U}, A\mathfrak{U})}, \right. \\ &\quad \left. \frac{\omega_e(\mathfrak{N}, A\mathfrak{N})}{1 + \omega_e(\mathfrak{N}, A\mathfrak{N})}, \frac{\omega_e(\mathfrak{U}, A\mathfrak{N}) + \omega_e(\mathfrak{N}, A\mathfrak{U})}{4s(\mathfrak{U}, \mathfrak{N})} \right\}. \end{aligned} \quad (50)$$

Note that the inverse of the above remark is generally incorrect.

Example 14. Let $\mathbb{T} = [0, \infty)$. Define $\omega_e : \mathbb{T}^2 \rightarrow \mathbb{R}$ by $\omega_e(\mathfrak{U}, \mathfrak{N}) = (\mathfrak{U} + \mathfrak{N})^2$ and $s : \mathbb{T} \times \mathbb{T} \rightarrow [1, \infty)$ by $s(\mathfrak{U}, \mathfrak{N}) = 1 + \mathfrak{U} + \mathfrak{N}$, for all $\mathfrak{U}, \mathfrak{N} \in \mathbb{T}$. Here, ω_e is a modified extended ω_e -metric-like space. Define $A : \mathbb{T} \rightarrow \mathbb{T}$ as

$$A\mathfrak{U} = \begin{cases} 0, & \text{if } 0 \leq \mathfrak{U} < 1, \\ \frac{1}{2}, & \text{if } \mathfrak{U} \geq 1. \end{cases} \quad (51)$$

Note that A is not an extended F_{ω_e} -Suzuki contraction. Indeed, for $0 \leq \mathfrak{U} < 1$ and $\mathfrak{N} = 1$, we can write $1/2\omega_e(\mathfrak{U}, A\mathfrak{U}) = 1/2\omega_e(\mathfrak{U}, 0) = 1/2\mathfrak{U}^2 < (\mathfrak{U} + 1)^2 = \omega_e(\mathfrak{U}, \mathfrak{N})$ and

$$\begin{aligned} &\max \left\{ \omega_e(\mathfrak{U}, 1), \frac{\omega_e(\mathfrak{U}, A\mathfrak{U})}{1 + \omega_e(\mathfrak{U}, A\mathfrak{U})}, \frac{\omega_e(1, A1)}{1 + \omega_e(1, A1)}, \right. \\ &\quad \left. \frac{\omega_e(\mathfrak{U}, A1) + \omega_e(1, A\mathfrak{U})}{4(2 + \mathfrak{U})} \right\} \\ &= \max \left\{ \omega_e(\mathfrak{U}, 1), \frac{\omega_e(\mathfrak{U}, 0)}{1 + \omega_e(\mathfrak{U}, 0)}, \frac{\omega_e(1, A1)}{1 + \omega_e(1, A1)}, \right. \\ &\quad \left. \frac{\omega_e(\mathfrak{U}, A1) + \omega_e(1, 0)}{4(2 + \mathfrak{U})} \right\} \geq \omega_e(1, A1) = \omega_e \left(1, \frac{1}{2} \right) \\ &= \frac{9}{4} > \frac{1}{4} = \omega_e(A\mathfrak{U}, A\mathfrak{N}). \end{aligned} \quad (52)$$

Let the function $F_{\omega_e} \in \Pi$ be defined by $F_{\omega_e}(\ell) = \ln(\ell)$, for $\ell > 0$. Then,

$$\begin{aligned}
& F_{\omega_e}(\omega_e(A\mathcal{U}, A\mathcal{N})) - F_{\omega_e}(\omega_e(\mathcal{U}, \mathcal{N})) \\
& \leq F_{\omega_e}(\omega_e(A\mathcal{U}, A1)) - F_{\omega_e}(\omega_e(1, A1)) \\
& = F_{\omega_e}\left(\frac{1}{4}\right) - F_{\omega_e}\left(\frac{9}{4}\right) = \ln\left(\frac{1}{4} \times \frac{4}{9}\right) = -\ln(9) < -2.
\end{aligned} \tag{53}$$

Therefore, A is an extended generalized F_{ω_e} -Suzuki contraction (for $\vartheta = 2$).

The following theorem is the main consequence of this part.

Theorem 15. Let $(\mathcal{T}, \omega_e)$ be an extended b -metric-like space and A be an extended generalized F_{ω_e} -Suzuki contraction self-mapping, then A has a unique fixed point, provided that $\lim_{i,m} s(\mathcal{U}_i, \mathcal{U}_m) \leq 1/\eta$, for $0 < \eta < 1$.

Proof. By the first lines of proof of Theorem 9, we build a sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ as $\mathcal{U}_{i+1} = A\mathcal{U}_i = A^{i+1}\mathcal{U}_0$. Here, we consider $i \in \mathbb{N} \cup \{0\}$, $0 < \omega_e(\mathcal{U}_i, A\mathcal{U}_i) = \omega_e(\mathcal{U}_i, \mathcal{U}_{i+1})$, so

$$\frac{1}{2}\omega_e(\mathcal{U}_i, A\mathcal{U}_i) < \omega_e(\mathcal{U}_i, A\mathcal{U}_i). \tag{54}$$

Applying conditions (48) and (ω_{e3}) , we get

$$\begin{aligned}
& \vartheta + F_{\omega_e}(\omega_e(\mathcal{U}_{i+1}, \mathcal{U}_{i+2})) = \vartheta + F_{\omega_e}(\omega_e(A\mathcal{U}_i, A(A\mathcal{U}_i))) \\
& \leq F_{\omega_e}\left(\max\left\{\omega_e(\mathcal{U}_i, A\mathcal{U}_i), \frac{\omega_e(\mathcal{U}_i, A\mathcal{U}_i)}{1 + \omega_e(\mathcal{U}_i, A\mathcal{U}_i)}, \frac{\omega_e(A\mathcal{U}_i, A^2\mathcal{U}_i)}{1 + \omega_e(A\mathcal{U}_i, A^2\mathcal{U}_i)}, \frac{\omega_e(\mathcal{U}_i, A^2\mathcal{U}_i) + \omega_e(A\mathcal{U}_i, A\mathcal{U}_i)}{4s(\mathcal{U}_i, A\mathcal{U}_i)}\right\}\right) \\
& = F_{\omega_e}\left(\max\left\{\omega_e(\mathcal{U}_i, \mathcal{U}_{i+1}), \frac{\omega_e(\mathcal{U}_i, \mathcal{U}_{i+1})}{1 + \omega_e(\mathcal{U}_i, \mathcal{U}_{i+1})}, \frac{\omega_e(\mathcal{U}_{i+1}, \mathcal{U}_{i+2})}{1 + \omega_e(\mathcal{U}_{i+1}, \mathcal{U}_{i+2})}, \frac{\omega_e(\mathcal{U}_i, \mathcal{U}_{i+2}) + \omega_e(\mathcal{U}_{i+1}, \mathcal{U}_{i+1})}{4s(\mathcal{U}_i, \mathcal{U}_{i+1})}\right\}\right) \\
& \leq F_{\omega_e}\left(\max\left\{\omega_e(\mathcal{U}_i, \mathcal{U}_{i+1}), \frac{\omega_e(\mathcal{U}_i, \mathcal{U}_{i+1})}{1 + \omega_e(\mathcal{U}_i, \mathcal{U}_{i+1})}, \frac{\omega_e(\mathcal{U}_{i+1}, \mathcal{U}_{i+2})}{1 + \omega_e(\mathcal{U}_{i+1}, \mathcal{U}_{i+2})}, \frac{s(\mathcal{U}_i, \mathcal{U}_{i+1})[\omega_e(\mathcal{U}_i, \mathcal{U}_{i+1}) + \omega_e(\mathcal{U}_{i+1}, \mathcal{U}_{i+2})] + 2s(\mathcal{U}_i, \mathcal{U}_{i+2})\omega_e(\mathcal{U}_i, \mathcal{U}_{i+1})}{4s(\mathcal{U}_i, \mathcal{U}_{i+2})}\right\}\right) \\
& = F_{\omega_e}\left(\max\left\{\omega_e(\mathcal{U}_i, \mathcal{U}_{i+1}), \frac{\omega_e(\mathcal{U}_i, \mathcal{U}_{i+1})}{1 + \omega_e(\mathcal{U}_i, \mathcal{U}_{i+1})}, \frac{\omega_e(\mathcal{U}_{i+1}, \mathcal{U}_{i+2})}{1 + \omega_e(\mathcal{U}_{i+1}, \mathcal{U}_{i+2})}, \frac{3\omega_e(\mathcal{U}_i, \mathcal{U}_{i+1}) + \omega_e(\mathcal{U}_{i+1}, \mathcal{U}_{i+2})}{4}\right\}\right) \\
& \leq F_{\omega_e}(\max\{\omega_e(\mathcal{U}_i, \mathcal{U}_{i+1}), \omega_e(\mathcal{U}_{i+1}, \mathcal{U}_{i+2})\}).
\end{aligned} \tag{55}$$

Now, if $\omega_e(\mathcal{U}_i, \mathcal{U}_{i+1}) < \omega_e(\mathcal{U}_{i+1}, \mathcal{U}_{i+2})$, then

$$F_{\omega_e}(\omega_e(\mathcal{U}_{i+1}, \mathcal{U}_{i+2})) \leq F_{\omega_e}(\omega_e(\mathcal{U}_{i+1}, \mathcal{U}_{i+2})) - \vartheta, \tag{56}$$

which is a contradiction due to $(\mathfrak{F}_1)$, so we should write

$$F_{\omega_e}(\omega_e(\mathcal{U}_{i+1}, \mathcal{U}_{i+2})) \leq F_{\omega_e}(\omega_e(\mathcal{U}_i, \mathcal{U}_{i+1})) - \vartheta, \quad \forall i \in \mathbb{N} \cup \{0\}. \tag{57}$$

By the same manner,

$$F_{\omega_e}(\omega_e(\mathcal{U}_i, \mathcal{U}_{i+1})) \leq F_{\omega_e}(\omega_e(\mathcal{U}_{i-1}, \mathcal{U}_i)) - \vartheta, \quad \forall i \in \mathbb{N} \cup \{0\}. \tag{58}$$

From (57) and (58), one can write

$$F_{\omega_e}(\omega_e(\mathcal{U}_{i+1}, \mathcal{U}_{i+2})) \leq F_{\omega_e}(\omega_e(\mathcal{U}_{i-1}, \mathcal{U}_i)) - 2\vartheta, \quad \forall i \in \mathbb{N} \cup \{0\}. \tag{59}$$

Repeating the same scenario, we have

$$F_{\omega_e}(\omega_e(\mathcal{U}_i, \mathcal{U}_{i+1})) \leq F_{\omega_e}(\omega_e(\mathcal{U}_0, \mathcal{U}_1)) - i\vartheta, \quad \forall i \in \mathbb{N} \cup \{0\}. \tag{60}$$

The proof of Theorem 9, namely, relations (22)–(29), yields that $\{\mathcal{U}_i\}$ is Cauchy sequence in $(\mathcal{T}, \omega_e)$, which is complete; hence, there is $\mathcal{U}^* \in \mathcal{T}$ so that $\mathcal{U}_i \rightarrow \mathcal{U}^*$ as $i \rightarrow \infty$. That is,

$$\lim_{i,m \rightarrow \infty} \omega_e(\mathcal{U}_i, \mathcal{U}_m) = \lim_{i \rightarrow \infty} \omega_e(\mathcal{U}_i, \mathcal{U}^*) = \omega_e(\mathcal{U}^*, \mathcal{U}^*) = 0. \tag{61}$$

Now, if A is continuous, by (24) we get

$$\omega_e(A\mathcal{U}^*, \mathcal{U}^*) = \lim_{i \rightarrow \infty} \omega_e(A\mathcal{U}_i, \mathcal{U}_i) = \lim_{i \rightarrow \infty} \omega_e(\mathcal{U}_{i+1}, \mathcal{U}_i) = 0. \tag{62}$$

Thus, $A\mathcal{U}^* = \mathcal{U}^*$; that is, $\mathcal{U}^*$ is a fixed point of A .

Next, in the case that F_{ω_e} is continuous, we claim that

$$\omega_e(\mathcal{U}_m, \mathcal{U}^*) \leq \omega_e(\mathcal{U}^*, A\mathcal{U}_m), \quad \forall m \in \mathbb{N} \cup \{0\}. \tag{63}$$

By the fact $1/2\omega(\mathcal{U}_m, A\mathcal{U}_m) < \omega(\mathcal{U}_m, A\mathcal{U}_m)$ and using (48), we obtain that

$$\begin{aligned}
& \vartheta + F_{\omega_e}(\omega_e(A\mathcal{U}_m, A^2\mathcal{U}_m)) \leq F_{\omega_e}\left(\max\left\{\omega_e(\mathcal{U}_m, A\mathcal{U}_m), \frac{\omega_e(\mathcal{U}_m, A\mathcal{U}_m)}{1 + \omega_e(\mathcal{U}_m, A\mathcal{U}_m)}, \frac{\omega_e(A\mathcal{U}_m, A^2\mathcal{U}_m)}{1 + \omega_e(A\mathcal{U}_m, A^2\mathcal{U}_m)}, \frac{\omega_e(\mathcal{U}_m, A^2\mathcal{U}_m) + \omega_e(A\mathcal{U}_m, A\mathcal{U}_m)}{4s(\mathcal{U}_m, A\mathcal{U}_m)}\right\}\right) \\
& \leq F\left(\max\left\{\omega_e(\mathcal{U}_m, A\mathcal{U}_m), \frac{\omega_e(\mathcal{U}_m, A\mathcal{U}_m)}{1 + \omega_e(\mathcal{U}_m, A\mathcal{U}_m)}, \frac{\omega_e(A\mathcal{U}_m, A^2\mathcal{U}_m)}{1 + \omega_e(A\mathcal{U}_m, A^2\mathcal{U}_m)}, \frac{s(\mathcal{U}_m, A\mathcal{U}_m)[\omega_e(\mathcal{U}_m, A\mathcal{U}_m) + \omega_e(A\mathcal{U}_m, A^2\mathcal{U}_m)] + 2s(\mathcal{U}_m, A\mathcal{U}_m)\omega_e(\mathcal{U}_m, A\mathcal{U}_m)}{4s(\mathcal{U}_m, A\mathcal{U}_m)}\right\}\right) \\
& \leq F\left(\max\left\{\omega_e(\mathcal{U}_m, A\mathcal{U}_m), \frac{\omega_e(\mathcal{U}_m, A\mathcal{U}_m)}{1 + \omega_e(\mathcal{U}_m, A\mathcal{U}_m)}, \frac{\omega_e(A\mathcal{U}_m, A^2\mathcal{U}_m)}{1 + \omega_e(A\mathcal{U}_m, A^2\mathcal{U}_m)}, \frac{3\omega_e(\mathcal{U}_m, A\mathcal{U}_m) + \omega_e(A\mathcal{U}_m, A^2\mathcal{U}_m)}{4}\right\}\right) \\
& \leq F(\max\{\omega_e(\mathcal{U}_m, A\mathcal{U}_m), \omega_e(A\mathcal{U}_m, A^2\mathcal{U}_m)\}).
\end{aligned} \tag{64}$$

If $\varpi_e(\mathcal{U}_m, A\mathcal{U}_m) < \varpi_e(A\mathcal{U}_m, A^2\mathcal{U}_m)$, then we have

$$F_{\varpi_e}(\varpi_e(A\mathcal{U}_m, A^2\mathcal{U}_m)) \leq F_{\varpi_e}(\varpi_e(A\mathcal{U}_m, A^2\mathcal{U}_m)) - \vartheta, \quad (65)$$

a contradiction due to $(\mathfrak{F}_1)$. So, we should write

$$F_{\varpi_e}(\varpi_e(A\mathcal{U}_m, A^2\mathcal{U}_m)) \leq F_{\varpi_e}(\varpi_e(\mathcal{U}_m, A\mathcal{U}_m)) - \vartheta. \quad (66)$$

Since F_{ϖ_e} is continuous and strictly increasing, it follows that

$$\varpi_e(A\mathcal{U}_m, A^2\mathcal{U}_m) < \varpi_e(\mathcal{U}_m, A\mathcal{U}_m). \quad (67)$$

Now, to ensure the existence of a fixed point, two cases arise as follows.

Case 1. For each $\iota \in \mathbb{N}$, there is $j_\iota \in \mathbb{N}$ so that $\mathcal{U}_{j_\iota} = A\mathcal{U}^*$ and $j_\iota > j_{\iota-1}$, where $j_0 = 1$. Then, we have $\mathcal{U}^* = \lim_{\iota \rightarrow \infty} \mathcal{U}_{j_\iota} = \lim_{\iota \rightarrow \infty} A\mathcal{U}^* = A\mathcal{U}^*$, i.e., $\mathcal{U}^*$ is a fixed point of A .

Case 2. There is $\iota_0 \in \mathbb{N}$ such that for all $\iota \geq \iota_0$, $\mathcal{U}_{\iota+1} \neq A\mathcal{U}^*$. It is clear that $1/2\varpi_e(\mathcal{U}_\iota, A\mathcal{U}^*) < \varpi_e(\mathcal{U}_\iota, A\mathcal{U}^*)$ for all $\iota \geq \iota_0$.

By (48) and (63), we get

$$\begin{aligned} \vartheta + F_{\varpi_e}(\varpi_e(A\mathcal{U}_\iota, A^2\mathcal{U}^*)) &\leq F_{\varpi_e}\left(\max\left\{\varpi_e(\mathcal{U}_\iota, A\mathcal{U}^*), \frac{\varpi_e(\mathcal{U}_\iota, A\mathcal{U}_\iota)}{1 + \varpi_e(\mathcal{U}_\iota, A\mathcal{U}_\iota)}, \frac{\varpi_e(A\mathcal{U}^*, A^2\mathcal{U}^*)}{1 + \varpi_e(A\mathcal{U}^*, A^2\mathcal{U}^*)}, \frac{\varpi_e(\mathcal{U}_\iota, A^2\mathcal{U}^*) + \varpi_e(A\mathcal{U}^*, A\mathcal{U}_\iota)}{4s(\mathcal{U}_\iota, A\mathcal{U}^*)}\right\}\right) \\ &\leq F_{\varpi_e}\left(\max\left\{\varpi_e(\mathcal{U}_\iota, A\mathcal{U}^*), \frac{\varpi_e(\mathcal{U}_\iota, A\mathcal{U}_\iota)}{1 + \varpi_e(\mathcal{U}_\iota, A\mathcal{U}_\iota)}, \frac{\varpi_e(A\mathcal{U}^*, A^2\mathcal{U}^*)}{1 + \varpi_e(A\mathcal{U}^*, A^2\mathcal{U}^*)}, \frac{2\varpi_e(\mathcal{U}_\iota, A\mathcal{U}^*) + \varpi_e(A\mathcal{U}^*, A^2\mathcal{U}^*) + \varpi_e(\mathcal{U}_\iota, A\mathcal{U}_\iota)}{4}\right\}\right) \\ &\leq F_{\varpi_e}(\max\{\varpi_e(\mathcal{U}_\iota, A\mathcal{U}^*), \varpi_e(\mathcal{U}_\iota, A\mathcal{U}_\iota), \varpi_e(A\mathcal{U}^*, A^2\mathcal{U}^*)\}) \\ &< F_{\varpi_e}(\max\{\varpi_e(\mathcal{U}_\iota, A\mathcal{U}^*), \varpi_e(\mathcal{U}_\iota, A\mathcal{U}_\iota), \varpi_e(\mathcal{U}^*, A\mathcal{U}^*)\}). \end{aligned} \quad (68)$$

Since F_{ϖ_e} is continuous, we find at the limit $\iota \rightarrow \infty$, or

$$\vartheta + F_{\varpi_e}(\varpi_e(\mathcal{U}^*, A^2\mathcal{U}^*)) \leq F_{\varpi_e}(\varpi_e(\mathcal{U}^*, A\mathcal{U}^*)), \quad (69)$$

$$F_{\varpi_e}(\varpi_e(\mathcal{U}^*, A\mathcal{U}^*)) \leq F_{\varpi_e}(\varpi_e(\mathcal{U}^*, \mathcal{U}^*)) - \vartheta, \quad (70)$$

which is a contradiction. So, $\varpi_e(\mathcal{U}^*, A\mathcal{U}^*) = 0$, which leads to $\mathcal{U}^* = A\mathcal{U}^*$.

The two cases above ensure the existence of a fixed point of A .

To ensure the uniqueness, suppose that $\mathcal{U}^*, v^*$ are distinct fixed points of A . Hence, $1/2\varpi_e(\mathcal{U}^*, v^*) < \varpi_e(\mathcal{U}^*, v^*)$, which implies that

$$\begin{aligned} \vartheta + F_{\varpi_e}(\varpi_e(\mathcal{U}^*, v^*)) &= \vartheta + F_{\varpi_e}(\varpi_e(A\mathcal{U}^*, Av^*)) \\ &\leq F_{\varpi_e}\left(\max\left\{\varpi_e(\mathcal{U}^*, v^*), \frac{\varpi_e(\mathcal{U}^*, A\mathcal{U}^*)}{1 + \varpi_e(\mathcal{U}^*, A\mathcal{U}^*)}, \frac{\varpi_e(v^*, Av^*)}{1 + \varpi_e(v^*, Av^*)}, \frac{\varpi_e(\mathcal{U}^*, Av^*) + \varpi_e(v^*, A\mathcal{U}^*)}{4s(\mathcal{U}^*, v^*)}\right\}\right) \\ &\leq F_{\varpi_e}(\varpi_e(\mathcal{U}^*, v^*)), \end{aligned} \quad (71)$$

which is a contradiction again. Hence, the fixed point is unique.

In the following, we justify all required hypotheses of Theorem 15.

Example 16. Suppose that $\mathcal{T} = [0, \infty)$. Define functions $\varpi_e : \mathcal{T}^2 \rightarrow \mathbb{R}$ and $s : \mathcal{T} \times \mathcal{T} \rightarrow [1, \infty)$ by $\varpi_e(\mathcal{U}, \aleph) = (\mathcal{U} + \aleph)^2$ and $s(\mathcal{U}, \aleph) = 1 + \mathcal{U} + \aleph$, respectively. Then, $(\mathcal{T}, \varpi_e)$ is a complete modified ϖ_e -metric-like space. Define $A : \mathcal{T} \rightarrow \mathcal{T}$ by

$$A\mathcal{U} = \begin{cases} 0, & \text{if } \mathcal{U} \in [0, \frac{1}{4}), \\ \left\{\frac{1}{4^\iota}\right\}, & \text{if } \mathcal{U} \in \left[\frac{1}{4}, \infty\right), \iota \in \mathcal{T}. \end{cases} \quad (72)$$

Define the function $F_{\varpi_e} \in \Pi$ by $F_{\varpi_e}(\ell) = \ln(\ell)$ for $\ell > 0$ and $\vartheta > 0$. We state the following.

Case 1. Let $\mathcal{U} = 1/4^{\iota-1}$ and $\aleph = 1/4^{m-1}$, for $m > \iota \geq 2$. Now, for $\iota = 1$ and $m = 2$, we have $\mathcal{U} = 1$ and $\aleph = 1/4$. Therefore,

$$\begin{aligned} \frac{1}{2}\varpi_e(\mathcal{U}, A\mathcal{U}) &= \frac{1}{2}\varpi_e(1, A1) = \frac{1}{2}\left(1 + \frac{1}{4}\right)^2 = \frac{25}{32} < \frac{25}{16} \\ &= \left(1 + \frac{1}{4}\right)^2 = \varpi_e(\mathcal{U}, \aleph). \end{aligned} \quad (73)$$

Let

$$\begin{aligned} F_{\varpi_e}(\varpi_e(A\mathcal{U}, A\aleph)) &= F_{\varpi_e}\left(\varpi_e\left(A1, A\frac{1}{4}\right)\right) = F_{\varpi_e}\left(\varpi_e\left(\frac{1}{4}, \frac{1}{16}\right)\right) \\ &= F_{\varpi_e}\left(\frac{5}{16}\right)^2 = 2 \ln\left(\frac{5}{16}\right) = -2.326, \end{aligned} \quad (74)$$

as well as,

$$\begin{aligned} F_{\varpi_e}\left(\max\left\{\varpi_e(\mathcal{U}, \aleph), \frac{\varpi_e(\mathcal{U}, A\mathcal{U})}{1 + \varpi_e(\mathcal{U}, A\mathcal{U})}, \frac{\varpi_e(\aleph, A\aleph)}{1 + \varpi_e(\aleph, A\aleph)}, \frac{\varpi_e(\mathcal{U}, A\aleph) + \varpi_e(\aleph, A\mathcal{U})}{4s(\mathcal{U}, \aleph)}\right\}\right) \\ = F_{\varpi_e}\left(\max\left\{\varpi_e\left(1, \frac{1}{4}\right), \frac{\varpi_e(1, A1)}{1 + \varpi_e(1, A1)}, \frac{\varpi_e(1/4, A(1/4))}{1 + \varpi_e(1/4, A(1/4))}, \frac{\varpi_e(1, A(1/4)) + \varpi_e(1/4, A1)}{4s(1, 1/4)}\right\}\right) \\ = F_{\varpi_e}\left(\max\left\{\frac{25}{16}, \frac{25}{41}, \frac{25}{281}, \frac{353}{2304}\right\}\right) \\ = F_{\varpi_e}\left(\frac{25}{16}\right) = \ln\left(\frac{25}{16}\right) = 0.0446. \end{aligned} \quad (75)$$

So, we get

$$\begin{aligned} F_{\omega_e}(\omega_e(A\mathfrak{U}, A\aleph)) - F_{\omega_e}\left(\max\left\{\omega_e(\mathfrak{U}, \aleph), \frac{\omega_e(\mathfrak{U}, A\mathfrak{U})}{1 + \omega_e(\mathfrak{U}, A\mathfrak{U})}, \frac{\omega_e(\aleph, A\aleph)}{1 + \omega_e(\aleph, A\aleph)}, \frac{\omega_e(\mathfrak{U}, A\aleph) + \omega_e(\aleph, A\mathfrak{U})}{4s(\mathfrak{U}, \aleph)}\right\}\right) \\ = -2.326 - 0.044 = -2.37 < -2. \end{aligned} \quad (76)$$

Case 2. Let $\mathfrak{U} = 1/4$ and $\aleph = 0$. So, we get

$$\frac{1}{2}\omega_e(\mathfrak{U}, A\mathfrak{U}) = \frac{1}{2}\omega_e\left(\frac{1}{4}, \frac{1}{16}\right) = \frac{25}{512} < \frac{1}{16} = \left(\frac{1}{4} + 0\right)^2 = \omega_e(\mathfrak{U}, \aleph). \quad (77)$$

Consider

$$\begin{aligned} F_{\omega_e}(\omega_e(A\mathfrak{U}, A\aleph)) &= F_{\omega_e}\left(\omega_e\left(A\frac{1}{4}, A0\right)\right) = F_{\omega_e}\left(\omega_e\left(\frac{1}{16}, 0\right)\right) \\ &= F_{\omega_e}\left(\frac{1}{16}\right)^2 = 2 \ln\left(\frac{1}{16}\right) = -5.545. \end{aligned} \quad (78)$$

Additionally,

$$\begin{aligned} F_{\omega_e}\left(\max\left\{\omega_e(\mathfrak{U}, \aleph), \frac{\omega_e(\mathfrak{U}, A\mathfrak{U})}{1 + \omega_e(\mathfrak{U}, A\mathfrak{U})}, \frac{\omega_e(\aleph, A\aleph)}{1 + \omega_e(\aleph, A\aleph)}, \frac{\omega_e(\mathfrak{U}, A\aleph) + \omega_e(\aleph, A\mathfrak{U})}{4s(\mathfrak{U}, \aleph)}\right\}\right) \\ = F_{\omega_e}\left(\max\left\{\omega_e\left(\frac{1}{4}, 0\right), \frac{\omega_e(1/4, A(1/4))}{1 + \omega_e(1/4, A(1/4))}, \frac{\omega_e(0, A0)}{1 + \omega_e(0, A0)}, \frac{\omega_e(1/4, A0) + \omega_e(0, A(1/4))}{4s(1/4, 0)}\right\}\right) \\ = F_{\omega_e}\left(\max\left\{\frac{1}{16}, \frac{25}{281}, 0, \frac{17}{1280}\right\}\right) = F_{\omega_e}\left(\frac{25}{281}\right) \\ = \ln\left(\frac{25}{281}\right) = -2.419. \end{aligned} \quad (79)$$

Subtracting the two relations, we have

$$\begin{aligned} F_{\omega_e}(\omega_e(A\mathfrak{U}, A\aleph)) - F_{\omega_e}\left(\max\left\{\omega_e(\mathfrak{U}, \aleph), \frac{\omega_e(\mathfrak{U}, A\mathfrak{U})}{1 + \omega_e(\mathfrak{U}, A\mathfrak{U})}, \frac{\omega_e(\aleph, A\aleph)}{1 + \omega_e(\aleph, A\aleph)}, \frac{\omega_e(\mathfrak{U}, A\aleph) + \omega_e(\aleph, A\mathfrak{U})}{4s(\mathfrak{U}, \aleph)}\right\}\right) \\ = -5.545 + 2.419 = -3.126 < -2. \end{aligned} \quad (80)$$

From the above, we deduce that the mapping A is an extended generalized F_{ω_e} -Suzuki contraction with $\vartheta = 2$. Moreover, if $\mathfrak{U}_m = \{1/4^m\} \in \mathfrak{T}$, we have

$$\lim_{i,m \rightarrow \infty} s(\mathfrak{U}_m, \mathfrak{U}_i) = \lim_{i,m \rightarrow \infty} \left(1 + \frac{1}{4^m} + \frac{1}{4^i}\right) = 1 < \frac{1}{\eta}, \quad (81)$$

for $\eta \in (0, 1)$. Hence, the requirements of Theorem 15 hold; therefore, A has a unique fixed point. Here, it is 0.

4. Supportive Applications

This part is considered as the strength of the paper, where we use the results presented in Theorems 9 and 15 to get the analytical solutions both of the Fredholm integral equation and the second-order differential equation, respectively. For this purpose, we will divide this section into two parts as follows.

4.1. Analytical Solution of Fredholm Integral Equation. Let the Fredholm integral equation given by

$$\mathfrak{U}(\eta) = \int_u^v \Phi(\eta, \zeta, \mathfrak{U}(\zeta)) d\zeta, \quad (82)$$

for all $\eta, \zeta \in [u, v]$, where $F_{\omega_e} : [u, v] \rightarrow \mathbb{R}$ and $\Phi : [u, v] \times u, v] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions.

Let $\mathfrak{T} = C([u, v], \mathbb{R})$ be the set of all real continuous functions defined on $[u, v]$, endowed with

$$\omega_e(\mathfrak{U}, \aleph) = (\|\mathfrak{U} + \aleph\|_{\infty})^2 \text{ for all } \mathfrak{U}, \aleph \in \mathfrak{T}, \quad (83)$$

where $\|\mathfrak{U}\|_{\infty} = \sup_{\eta \in [u, v]} \{|\mathfrak{U}(\eta)|e^{-\eta\vartheta}\}$ with $s(\mathfrak{U}, \aleph) = 1 + |\mathfrak{U}| + |\aleph|$, where $s : \mathfrak{T} \times \mathfrak{T} \rightarrow [1, \infty)$. Note that $(\mathfrak{T}, \omega_e)$ is a complete modified ω_e -metric-like space.

Now, the following is the main result of this part.

Theorem 17. Let A be self-mapping on the complete modified ω_e -metric-like space $(\mathfrak{T}, \omega_e)$. Assume that

(i) for each $\eta, \zeta \in [u, v]$ and $\mathfrak{U}, \aleph \in \mathfrak{T}$,

$$\frac{1}{2} \left(\left\| \mathfrak{U}(\eta) + \int_u^v \Phi(\eta, \zeta, \mathfrak{U}(\zeta)) d\zeta \right\|_{\infty} \right)^2 \leq (\|\mathfrak{U}(\eta)\|_{\infty} + \|\aleph(\eta)\|_{\infty})^2 \quad (84)$$

(ii) for all $\eta, \zeta \in [u, v]$, there is a constant $\vartheta \in \mathbb{R}^+$ such that

$$|\Phi(\eta, \zeta, \mathfrak{U}(\zeta)) + \Phi(\eta, \zeta, \aleph(\zeta))| \leq \frac{e^{-\vartheta/2}}{(v-u)} (|\mathfrak{U}(\zeta) + \aleph(\zeta)|) \quad (85)$$

Then, there exists a solution of the problem (82).

Proof. Consider the nonlinear self-mapping $A : \mathcal{T} \rightarrow \mathcal{T}$ given as

$$A\mathcal{U}(\eta) = \int_u^v \Phi(\eta, \zeta, \mathcal{U}(\zeta)) d\zeta. \quad (86)$$

Clearly, if $\mathcal{U}^* = A\mathcal{U}^*$, then it is a solution of the problem (82).

Let $\mathcal{U}, \mathcal{N} \in \mathcal{T}$, so, by condition (i), we deduce that $1/2\bar{\omega}_e(\mathcal{U}(\eta), A\mathcal{U}(\eta)) < \bar{\omega}_e(\mathcal{U}(\eta), \mathcal{N}(\eta))$. After applying the condition (ii), for any $\mathcal{U}(\eta), \mathcal{N}(\eta) \in \mathcal{T}$, we can write

$$\begin{aligned} |A\mathcal{U}(\eta) + A\mathcal{N}(\eta)|^2 &\leq \left(\int_u^v |\Phi(\eta, \zeta, \mathcal{U}(\zeta)) + \Phi(\eta, \zeta, \mathcal{N}(\zeta))| d\zeta \right)^2 \\ &\cdot \left(\int_u^v \frac{e^{-\vartheta/2}}{(v-u)} (|\mathcal{U}(\zeta) + \mathcal{N}(\zeta)|) d\zeta \right)^2 \\ &\leq \left(\int_u^v \frac{e^{-\vartheta/2}}{v-u} \times \sqrt{e^{-2\vartheta\eta} \times e^{2\vartheta\eta}} \times (|\mathcal{U}(\zeta) + \mathcal{N}(\zeta)|) d\zeta \right)^2 \\ &\leq \frac{e^{-\vartheta}}{(v-u)^2} \times \bar{\omega}_e(\mathcal{U}, \mathcal{N}) \times e^{2\vartheta\eta} \left(\int_u^v d\zeta \right)^2 \leq e^{-\vartheta} \bar{\omega}_e(\mathcal{U}, \mathcal{N}) \times e^{2\vartheta\eta}, \end{aligned} \quad (87)$$

so we have

$$\left(|A\mathcal{U}(\eta) + A\mathcal{N}(\eta)| \times e^{-\vartheta\eta} \right)^2 \leq e^{-\vartheta} \bar{\omega}_e(\mathcal{U}, \mathcal{N}), \quad (88)$$

which leads to

$$(\|A\mathcal{U}(\eta) + A\mathcal{N}(\eta)\|_\infty)^2 \leq e^{-\vartheta} \bar{\omega}_e(\mathcal{U}, \mathcal{N}). \quad (89)$$

It yields that

$$\bar{\omega}_e(A\mathcal{U}(\eta), A\mathcal{N}(\eta)) \leq e^{-\vartheta} \bar{\omega}_e(\mathcal{U}, \mathcal{N}). \quad (90)$$

Taking $F_{\bar{\omega}_e}(\ell) = \ln(\ell)$ for $\ell > 0$, one gets or

$$\ln(\bar{\omega}_e(A\mathcal{U}(\eta), A\mathcal{N}(\eta))) \leq \ln(e^{-\vartheta} \bar{\omega}_e(\mathcal{U}, \mathcal{N})), \quad (91)$$

$$\vartheta + \ln(\bar{\omega}_e(A\mathcal{U}(\eta), A\mathcal{N}(\eta))) \leq \ln(\bar{\omega}_e(\mathcal{U}, \mathcal{N})). \quad (92)$$

Equivalently,

$$\vartheta + F_{\bar{\omega}_e}(\bar{\omega}_e(A\mathcal{U}(\eta), A\mathcal{N}(\eta))) \leq F_{\bar{\omega}_e}(\bar{\omega}_e(\mathcal{U}, \mathcal{N})). \quad (93)$$

By Theorem 9, A admits a fixed point, which is a solution of the problem (82).

4.2. Analytical Solution of Second-Order Differential Equation. Consider the second-order differential equation given as follows:

$$\begin{cases} \mathcal{U}'(\eta) = -\Phi(\eta, \mathcal{U}(\eta)), & \eta \in [0, \gamma], \\ \mathcal{U}(0) = \mathcal{U}(\eta) = 0, \end{cases} \quad (94)$$

where $\Phi : [0, \gamma] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

Resolving the problem (94) is equivalent to resolving the following integral equation:

$$\mathcal{U}(\eta) = \int_0^\eta \phi(\eta, \zeta) \Phi(\zeta, \mathcal{U}(\zeta)) d\zeta, \quad \forall \eta \in [0, \gamma], \quad (95)$$

where ϕ is Green's function defined by

$$\phi(\eta, \zeta) = \begin{cases} \eta(1-\zeta), & \text{if } 0 \leq \eta \leq \zeta \leq \gamma, \\ \zeta(1-\eta), & \text{if } 0 \leq \zeta \leq \eta \leq \gamma, \end{cases} \quad (96)$$

and Φ is a function as in Theorem 17. Hence, if $\mathcal{U} \in C([0, \gamma])$, then $\mathcal{U}$ is a solution of the problem (94) if and only if $\mathcal{U}$ is a solution of the problem (95).

Let $\mathcal{T} = C([0, \gamma], \mathbb{R})$ be the set of all continuous functions defined on $[0, \gamma]$, and define a norm $\|\mathcal{U}\|_\vartheta = \max_{\eta \in [0, \gamma]} \{|\mathcal{U}(\eta)| e^{-1/2\eta\vartheta}\}$, for arbitrary $\eta \geq 1$. Obviously, $\|\cdot\|_\vartheta$ is equivalent to the maximum norm $\|\cdot\|$ on $\mathcal{T}$, and $\mathcal{T}$ is endowed with the extended generalized $\bar{\omega}_{e_\vartheta}$ -metric-like as

$$\begin{aligned} \bar{\omega}_{e_\vartheta}(\mathcal{U}, \mathcal{N}) &= (\|\mathcal{U} + \mathcal{N}\|_\vartheta)^2 = \max_{\eta \in [0, \gamma]} \left\{ |\mathcal{U}(\eta) + \mathcal{N}(\eta)|^2 e^{-\eta\vartheta} \right\} \text{ for all } \mathcal{U}, \\ &\quad \mathcal{N} \in \mathcal{T} \text{ and } e^{\eta\vartheta} \geq 1. \end{aligned} \quad (97)$$

Then, $(\mathcal{T}, \bar{\omega}_e)$ is a complete modified $\bar{\omega}_e$ -metric-like space with $s(\mathcal{U}, \mathcal{N}) = 1 + |\mathcal{U}| + |\mathcal{N}|$. Our main theorem is as follows.

Theorem 18. Suppose that $(\mathcal{T}, \bar{\omega}_e)$ is a complete modified b -metric-like space and A is a nonlinear self-mapping on $\mathcal{T}$, then (95) possesses a unique solution $\mathcal{U} \in C([0, \gamma], \mathbb{R})$, if

(a1) for each $\eta, \zeta \in [0, \gamma]$ and $\mathcal{U}, \mathcal{N} \in \mathcal{T}$,

$$\frac{1}{2} \left\| \mathcal{U}(\eta) + \int_0^\eta \phi(\eta, \zeta) \Phi(\zeta, \mathcal{U}(\zeta)) d\zeta \right\|_\vartheta \leq \|\mathcal{U} + \mathcal{N}\|_\vartheta \quad (98)$$

(a2) $\Phi \in C([0, \gamma] \times \mathbb{R})$ and $\phi \in C([0, \gamma] \times [0, \gamma])$

(a3) Φ satisfies

$$|\Phi(\zeta, \mathcal{U}(\zeta)) + \Phi(\zeta, \mathcal{N}(\zeta))|^2 \leq \vartheta e^{-\vartheta} \psi(\mathcal{U}, \mathcal{N}), \quad (99)$$

for all $\zeta \in [0, \gamma]$ and $\mathcal{U}, \mathcal{N} \in \mathbb{R}$, where

$$\psi(\mathcal{U}, \mathcal{N}) = \max \left\{ |\mathcal{U} + \mathcal{N}|^2, \frac{|\mathcal{U} + A\mathcal{U}|^2}{1 + |\mathcal{U} + A\mathcal{U}|^2}, \frac{|\mathcal{N} + A\mathcal{N}|^2}{1 + |\mathcal{N} + A\mathcal{N}|^2}, \frac{|\mathcal{U} + A\mathcal{N}|^2 + |\mathcal{N} + A\mathcal{U}|^2}{4(1 + |\mathcal{U}| + |\mathcal{N}|)} \right\} \quad (100)$$

$$(a4) \max \int_0^\eta \phi(\eta, \zeta) d\zeta \leq 1, \text{ for all } \eta \in [0, \gamma]$$

Proof. Consider on the set $\mathcal{T}$, the mapping A as

$$A\mathcal{U}(\eta) = \int_0^\eta \phi(\eta, \zeta) \Phi(\zeta, \mathcal{U}(\zeta)) d\zeta, \quad (101)$$

for all $\eta \in [0, \gamma]$ and $\mathcal{U} \in \mathcal{T}$. The solution of (95) is also a fixed point of A on $\mathcal{T}$. By condition (a1) and the definition of A , we can write $1/2\bar{\omega}_{e_9}(\mathcal{U}(\eta), A\mathcal{U}(\eta)) < \bar{\omega}_{e_9}(\mathcal{U}(\eta), \mathcal{N}(\eta))$.

Let $\mathcal{U}, \mathcal{N} \in \mathcal{T}$. By the hypotheses (a2)-(a4), we have

$$\begin{aligned} (|A\mathcal{U}(\eta) + A\mathcal{N}(\eta)|)^2 &= \left| \int_0^\eta \phi(\eta, \zeta) [\Phi(\zeta, \mathcal{U}(\zeta)) + \Phi(\zeta, \mathcal{N}(\zeta))] d\zeta \right|^2 \\ &\leq \int_0^\eta |\phi(\eta, \zeta)|^2 |\Phi(\zeta, \mathcal{U}(\zeta)) + \Phi(\zeta, \mathcal{N}(\zeta))|^2 d\zeta \\ &\leq \int_0^\eta |\Phi(\zeta, \mathcal{U}(\zeta)) + \Phi(\zeta, \mathcal{N}(\zeta))|^2 d\zeta \\ &\leq \int_0^\eta \vartheta e^{-\vartheta} \psi(\mathcal{U}(\zeta), \mathcal{N}(\zeta)) d\zeta \\ &\leq \vartheta e^{-\vartheta} \int_0^\eta e^{\zeta\vartheta} \max \left\{ |\mathcal{U} + \mathcal{N}|^2 e^{-\zeta\vartheta}, \frac{|\mathcal{U} + A\mathcal{U}|^2 e^{-\zeta\vartheta}}{1 + |\mathcal{U} + A\mathcal{U}|^2 e^{-\zeta\vartheta}}, \frac{|\mathcal{N} + A\mathcal{N}|^2 e^{-\zeta\vartheta}}{1 + |\mathcal{N} + A\mathcal{N}|^2 e^{-\zeta\vartheta}}, \frac{|\mathcal{U} + A\mathcal{N}|^2 + |\mathcal{N} + A\mathcal{U}|^2}{4(1 + |\mathcal{U}| + |\mathcal{N}|)} e^{-\zeta\vartheta} \right\} d\zeta \\ &\leq \vartheta e^{-\vartheta} \int_0^\eta e^{\zeta\vartheta} \max \left\{ \bar{\omega}_{e_9}(\mathcal{U}, \mathcal{N}), \frac{\bar{\omega}_{e_9}(\mathcal{U}, A\mathcal{U})}{1 + \bar{\omega}_{e_9}(\mathcal{U}, A\mathcal{U})}, \frac{\bar{\omega}_{e_9}(\mathcal{N}, A\mathcal{N})}{1 + \bar{\omega}_{e_9}(\mathcal{N}, A\mathcal{N})}, \frac{\bar{\omega}_{e_9}(\mathcal{U}, A\mathcal{N}) + \bar{\omega}_{e_9}(\mathcal{N}, A\mathcal{U})}{4s(\mathcal{U}, \mathcal{N})} \right\} d\zeta \\ &= \vartheta e^{-\vartheta} \psi(\mathcal{U}, \mathcal{N}) \int_0^\eta e^{\zeta\vartheta} d\zeta = \vartheta e^{-\vartheta} \psi(\mathcal{U}, \mathcal{N}) \times \left(\frac{e^{\eta\vartheta}}{\vartheta} - 1 \right) \\ &\leq \vartheta e^{-\vartheta} \psi(\mathcal{U}, \mathcal{N}) \times \frac{e^{\eta\vartheta}}{\vartheta} \leq e^{-\vartheta} \psi(\mathcal{U}, \mathcal{N}) e^{\eta\vartheta}. \end{aligned} \quad (102)$$

Hence, for all $\mathcal{U}, \mathcal{N} \in \mathcal{T}$,

$$(|A\mathcal{U}(\eta) + A\mathcal{N}(\eta)|)^2 \times e^{-\eta\vartheta} \leq e^{-\vartheta} \psi(\mathcal{U}, \mathcal{N}), \quad (103)$$

which yields

$$\bar{\omega}_{e_9}(A\mathcal{U}, A\mathcal{N}) = \max_{\eta \in [0, \gamma]} \left\{ (|A\mathcal{U}(\eta) + A\mathcal{N}(\eta)|)^2 \times e^{-\eta\vartheta} \right\} \leq e^{-\vartheta} \psi(\mathcal{U}, \mathcal{N}). \quad (104)$$

That is,

$$\vartheta + \ln \bar{\omega}_{e_9}(A\mathcal{U}, A\mathcal{N}) \leq \ln \psi(\mathcal{U}, \mathcal{N}). \quad (105)$$

Defining the function $F_{\bar{\omega}_e}(\alpha) = \ln(\alpha), \alpha > 0$ in (105), such that $F_{\bar{\omega}_e} \in \Pi$, we have

$$\vartheta + F_{\bar{\omega}_e}(\bar{\omega}_e(A\mathcal{U}, A\mathcal{N})) \leq F_{\bar{\omega}_e} \left(\max \left\{ \bar{\omega}_e(\mathcal{U}, \mathcal{N}), \frac{\bar{\omega}_e(\mathcal{U}, A\mathcal{U})}{1 + \bar{\omega}_e(\mathcal{U}, A\mathcal{U})}, \frac{\bar{\omega}_e(\mathcal{N}, A\mathcal{N})}{1 + \bar{\omega}_e(\mathcal{N}, A\mathcal{N})}, \frac{\bar{\omega}_e(\mathcal{U}, A\mathcal{N}) + \bar{\omega}_e(\mathcal{N}, A\mathcal{U})}{4s(\mathcal{U}, \mathcal{N})} \right\} \right). \quad (106)$$

Hence, all requirements of Theorem 15 hold and A is an extended generalized F -Suzuki contraction; hence, F possesses a fixed point $\mathcal{U} \in \mathcal{T}$, which is a solution of the problem (95).

5. Conclusion

A modified $\bar{\omega}_e$ -metric-like space is presented, and related fixed point results via it are discussed. Nontrivial examples are conducted for supporting the mentioned space and theorems. Thereafter, by using a fixed point technique, a simple and efficient solution for the integral and differential equations is found in the setting of a modified $\bar{\omega}_e$ -metric-like space. A lot of authors connected fixed point techniques and classical integral equations in various abstract spaces such as metric spaces, b -metric spaces, and partial metric spaces. We also follow the same method in the new space. In the literature, our obtained applications are an extension and/or a generalization of many existing classical integral and differential equations. The observed results of this paper open new framework research avenues for

- (i) fixed point techniques for solving Volterra-Fredholm integral equation in a modified $\bar{\omega}_e$ -metric-like space
- (ii) collocation-type methods for Volterra-Hammerstein integral equations in modified $\bar{\omega}_e$ -metric-like spaces

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests concerning the publication of this article.

Authors' Contributions

All authors contributed equally and significantly in writing this article.

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