

## Research Article

# Some Eigenvalues Estimate for the $\phi$ -Laplace Operator on Slant Submanifolds of Sasakian Space Forms

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This paper is aimed at establishing new upper bounds for the first positive eigenvalue of the  $\phi$ -Laplacian operator on Riemannian manifolds in terms of mean curvature and constant sectional curvature. The first eigenvalue for the  $\phi$ -Laplacian operator on closed oriented  $m$ -dimensional slant submanifolds in a Sasakian space form  $\tilde{M}^{2k+1}(\varepsilon)$  is estimated in various ways. Several Reilly-like inequalities are generalized from our findings for Laplacian to the  $\phi$ -Laplacian on slant submanifold in a sphere  $S^{2n+1}$  with  $\varepsilon = 1$  and  $\phi = 2$ .

## 1. Introduction and Statement of Main Results

Finding the bound of the eigenvalue for the Laplacian on a given manifold is a key aspect in Riemannian geometry, and there are different classes of submanifolds such as slant submanifolds, CR-submanifolds, and singular submanifolds, which motivates further exploration and attracts many researchers from different research areas [1–11]. A major objective is to study the eigenvalue that appears as solutions of the Dirichlet or Neumann boundary value problems for curvature functions. Because there are different boundary conditions on a manifold, one can take a philosophical view of the Dirichlet boundary condition, finding the upper bound for the eigenvalue as a method of investigation for the suitable bound of the Laplacian on the given manifold. In recent years, there has been increasing interest to obtain the eigenvalue for the Laplacian operator and the  $\phi$ -Laplacian operators. The linearized operator of the  $(r+1)$ -th anisotropic mean curvature that is an extension of the usual Laplacian operator was also studied in [12]. Let  $\mathcal{N}^m$  be a complete noncompact Riemannian manifold and  $\Sigma$  be the

compact domain in  $\mathcal{N}^m$ . Assume that  $\Lambda_1(\Sigma) > 0$  denotes the first eigenvalue of the Dirichlet boundary value problem:

$$\Delta f + \Lambda f = 0, \text{ in } \Sigma \text{ and } f = 0 \text{ on } \partial\Sigma, \quad (1)$$

where  $\Delta$  denotes the Laplace operator on  $\mathcal{N}^m$ . Then, the first eigenvalue  $\Lambda_1(\mathcal{N})$  is defined by  $\Lambda_1(\mathcal{N}) = \inf_{\Sigma} \Lambda_1(\Sigma)$ . The Reilly formula is solely concerned with the manifold's intrinsic geometry, and most notably with the PDE in question. With the following example, this is easily understood. Let  $(\mathcal{N}^m, g)$  be compact  $m$ -dimensional Riemannian manifold, and let  $\Lambda_1$  denote the first nonzero eigenvalue of the Neumann problem:

$$\Delta f + \Lambda_1 f = 0 \text{ on } \mathcal{N} \text{ and } \frac{\partial f}{\partial N} = 0 \text{ on } \partial\mathcal{N}, \quad (2)$$

where  $N$  is the outward normal on  $\partial\mathcal{N}^m$ . A result of Reilly [13] reads the following.

Let  $\mathcal{N}^m$  be Riemannian manifold, and  $\mathbb{R}^k$  is the Euclidean space having dimensions  $m$  and  $k$ , respectively. The manifold  $\mathcal{N}^m$  is connected, closed, and oriented as well. The  $\mathcal{N}^m$  is isometrically immersed in  $\mathbb{R}^k$  with condition  $\partial\mathcal{N}^m = 0$ . The mean curvature of this isometric immersion is denoted by  $\mathbb{H}$ , and the first nonzero eigenvalue  $\Lambda_1^\nabla$  of the Laplacian on  $\mathcal{N}^m$  can be written as in the sense of Reily [13].

$$\Lambda_1^\nabla \leq \frac{l}{\text{Vol}(\mathcal{N}^m)} \int_{\mathcal{N}^m} |\mathbb{H}|^2 dV. \quad (3)$$

where the volume element of  $\mathcal{N}^m$  is denoted by  $dV$ . It can be seen in literature that many authors prompted to work in such inequalities for different ambient spaces after the breakthrough of inequality (3). In Minkowski spaces, the upper bound for Finsler submanifold is proposed by both Zeng and He [14]. This upper bound relates to the very 1st eigenvalue of the  $\phi$ -Laplacian. For closed manifold, the first eigenvalue of the  $\phi$ -Laplace operator is presented by Seto and Wei [15] by using the condition of integral curvature. In the hyperbolic space, the bottom spectra of the Laplace manifold for complete and no-compact submanifold are calculated by Lin [16], and mean curvature has condition of integral pinching. In addition to this, Xiong [17] contributed his role on closed hyperspace to find the first Hodge-Laplacian eigenvalue. Moreover, Xiong worked for complete Riemannian manifold which includes the Reilly-type sharp upper bounds for the eigenvalues in product manifolds. The generalized Reilly inequality (3) and first nonzero eigenvalue of  $\phi$ -Laplace operator are calculated by Du et al [18]. On compact submanifold, they used the Wentzel-Laplace operator having boundary in Euclidean space. Following the same pattern, for Neumann and Dirichlet boundary restrictions, Blacker and Seto [3] evidenced the Lichnerowicz-type lower bound for the first nonzero eigenvalue of the  $\phi$ -Laplacian. They used the Hessian decomposition on Kaehler manifolds having a positive Ricci curvature. A simply connected space form  $\mathbb{M}^k(\varepsilon)$  having a constant curvature  $\varepsilon$  is obtained, a well-known evaluation for the first nonzero eigenvalue of Laplacian by the immersion of submanifold  $\mathcal{N}^m$  in simply connected space having  $m$ -dimension. This space form includes the Euclidean space  $\mathbb{R}^k$ , the unit sphere  $\mathbb{S}^k(1)$ , and the hyperbolic space  $\mathbb{H}(-1)^k$  with  $\varepsilon = 0, 1$  and  $\varepsilon = -1$ , respectively.

**Theorem 1.** [13, 19] *Let  $\mathcal{N}^m$  be an  $m$ -dimensional closed orientable submanifold in a  $k$ -dimensional space form  $\mathbb{M}^k(\varepsilon)$ . Then, the first nonnull eigenvalue  $\Lambda_1^\nabla$  of Laplacian satisfies*

$$\Lambda_1^\nabla \leq \frac{m}{\text{Vol}(\mathcal{N})} \int_{\mathcal{N}^m} (|\mathbb{H}|^2 + \varepsilon) dV, \quad (4)$$

where  $\mathbb{H}$  is the mean curvature vector of  $\mathcal{N}^m$  in  $\mathbb{M}^k(\varepsilon)$ . The equality holds if and only if  $\mathcal{N}^m$  is minimally immersed in

a geodesic sphere of radius  $r_\varepsilon$  of  $\tilde{\mathbb{M}}^k(\varepsilon)$  with  $r_0 = (m/\Lambda_1^\nabla)^{1/2}$ ,  $r_1 = \arcsin r_0$  and  $r_{-1} = \text{arcsinh } r_0$ .

In [20, 21], the first nonnull eigenvalue of the Laplacian is evidenced which is considered the generalization of the results in Reilly [13]. For various ambient spaces, the outcomes of different classes of Riemannian submanifolds indicate that the result of both 1st nonzero eigenvalues depict alike inequalities and ultimately have identical upper bounds [20, 22]. This result is valid for both Dirichlet and Neumann conditions. For ambient manifold, it is obvious from the literature that Laplace and  $\phi$ -Laplace operators on Riemannian submanifolds helped a lot to acquire different breakthroughs in Riemannian geometry (see [12, 14, 23–29]) through the work of [13]. To define the  $\phi$ -Laplacian which is second order quasilinear elliptic operator on  $\mathcal{N}^m$  (compact Riemannian manifold  $\mathcal{N}^m$  having  $m$ -dimension), we have

$$\Delta_\phi f = \text{div} \left( |\nabla f|^{\phi-2} \nabla f \right), \quad (5)$$

where  $\phi > 1$  to satisfy the above equation. We have the usual Laplacian for  $\phi = 2$ . On the other hand, the eigenvalue of  $\Delta_\phi$  has similarity with Laplacian. For instance, if a function  $f \neq 0$  satisfies the subsequent equation with Dirichlet boundary condition (1) (or Neumann boundary condition (2)), then  $\Lambda$  (any real number) is Dirichlet eigenvalue. Similarly, the above criteria also hold for Neumann boundary condition (2).

$$\Delta_\phi f = -\Lambda |f|^{\phi-2} f. \quad (6)$$

Let us study a Riemannian manifold  $\mathcal{N}^m$  with no boundary. The Rayleigh-type variational characterization is observed in first nonzero eigenvalue of  $\Delta_\phi$  which is given by  $\Lambda_{1,\phi}$ , from (cf. [30])

$$\Lambda_{1,\phi} = \inf \left\{ \frac{\int_{\mathcal{N}} |\nabla f|^\phi}{\int_{\mathcal{N}} \|f\|^\phi} \mid f \in W^{1,\phi}(\mathcal{N}^l) \setminus \{0\}, \int_{\mathcal{N}} |f|^{\phi-2} f = 0 \right\}. \quad (7)$$

This naturally raises the question: Is it possible to generalize the Reilly-type inequalities for submanifolds in spheres through the class almost-contact manifolds which were proved in [1, 20, 21]? In the Sasakian space form, our aim is to derive the 1st eigenvalue for the  $\phi$ -Laplacian on slant submanifold. Following this opinion and motivated by the historical development in the analysis of the first nonnull eigenvalue of the  $\phi$ -Laplacian on submanifold in various space forms, by using the Gauss equation and influenced by studied of [18, 20, 22], our goal is to give general view of the above Reilly's conclusion for  $\phi$ -Laplace operator, and we are going to provide a sharp estimate to the first eigenvalue for the  $\phi$ -Laplacian on slant submanifold of Sasakian space form  $\mathbb{M}^{2k+1}(\varepsilon)$ . In fact, the main finding of this paper will be announced in the following theorem.

**Theorem 2.** Let  $\mathcal{N}^m$  be an  $m(\geq 2)$ -dimensional closed orientated slant submanifold in a Sasakian space form  $\tilde{\mathbb{M}}^{2k+1}(\varepsilon)$ . Then

- (1) The first nonnull eigenvalue  $\Lambda_{1,\phi}$  of the  $\phi$ -Laplacian satisfies

$$\Lambda_{1,\phi} \leq \left( \frac{2^{(1-\phi/2)}(k+1)^{(1-\phi/2)}m^{\phi/2}}{(Vol(\mathcal{N}))^{\phi/2}} \right) \times \left\{ \int_{\mathcal{N}^m} \left\{ \left( \frac{\varepsilon+3}{4} \right) + \left( \frac{\varepsilon-1}{4} \right) \left( \frac{3 \cos^2\theta - 2}{m} \right) + |\mathbb{H}|^2 \right\} dV \right\}^{\frac{\phi}{2}} \text{ for } 1 < \phi \leq 2, \tag{8}$$

$$\Lambda_{1,\phi} \leq \left( \frac{2^{(1-\phi/2)}(k+1)^{(\phi/2-1)}m^{\phi/2}}{Vol(\mathcal{N})} \right) \times \int_{\mathcal{N}^m} \left\{ \left( \frac{\varepsilon+3}{4} \right) + \left( \frac{\varepsilon-1}{4} \right) \left( \frac{3 \cos^2\theta - 2}{m} \right) + |\mathbb{H}|^2 \right\}^{\phi/2} dV \text{ for } 2 < \phi \leq \frac{m}{2} + 1 \tag{9}$$

- (2) The equality carries in (8) and (9) if and only if  $\phi = 2$  and  $\mathcal{N}^m$  is minimally immersed in a geodesic sphere of radius  $r_\varepsilon$  of  $\tilde{\mathbb{M}}^{2k+1}(\varepsilon)$  with the following equalities:

$$\begin{aligned} r_0 &= \left( \frac{m}{\Lambda_1^\Delta} \right)^{1/2}, \\ r_1 &= \sin^{-1} r_0, \\ r_{-1} &= \sinh^{-1} r_0. \end{aligned} \tag{10}$$

*Remark 3.* For  $\phi = 2$ , our estimate finds the corollary.

**Corollary 4.** Let  $\mathcal{N}^m$  be an  $m$ -dimensional closed orientated slant submanifold in Sasakian space form  $\tilde{\mathbb{M}}^{2k+1}(\varepsilon)$ . Then,  $\Lambda_1^\Delta$  satisfies the following inequality for the Laplacian:

$$\Lambda_1^\Delta \leq \frac{m}{Vol(\mathcal{N})} \int_{\mathcal{N}} \left\{ |\mathbb{H}|^2 + \left( \frac{\varepsilon+3}{4} \right) + \left( \frac{\varepsilon-1}{4} \right) \left( \frac{3 \cos^2\theta - 2}{m} \right) \right\} dV. \tag{11}$$

The equality's cases are same as that in Theorem 2 (2). This is an immediate application of Theorem 2 by using  $1 < \phi \leq 2$ , as Sasakian space form.

**Theorem 5.** Let  $\mathcal{N}^m$  be an  $m(\geq 2)$ -dimensional closed orientated slant submanifold in Sasakian space form  $\tilde{\mathbb{M}}^{2k+1}(\varepsilon)$ . Then,  $\Lambda_{1,\phi}$  satisfies the following inequality for the  $\phi$ -

Laplacian:

$$\Lambda_{1,\phi} \leq \left( \frac{2^{1-\phi/2}(m+1)^{(1-\phi/2)}m^{\phi/2}}{(Vol(\mathcal{N}))^{(\phi-1)}} \right) \times \left\{ \int_{\mathcal{N}^m} \left( \left( \frac{\varepsilon+3}{4} \right) + \left( \frac{\varepsilon-1}{4} \right) \left( \frac{3 \cos^2\theta - 2}{m} \right) + |\mathbb{H}|^2 \right)^{\frac{\phi}{2(\phi-1)}} dV \right\}^{(\phi-1)}, \tag{12}$$

for  $1 < \phi \leq 2$ .

*Remark 6.* Consider the inequality (12) and give value  $\phi = 2$ , and then inequality (12) reduces to the Reilly-type inequality (11). This shows that Reilly-type calculates the first eigenvalue for the Laplace operator on slant submanifold in Euclidean sphere  $\mathbb{S}^{2k+1}$  (see Theorem 2 in [20] and Theorem 1.5 in [21]), which are the same on the case of our Theorem 2 for  $\varepsilon = 1$  and  $\phi = 2$ .

## 2. Preliminaries and Notations

An almost-contact manifold is odd-dimensional  $C^\infty$ -manifold  $(\tilde{\mathbb{M}}^{2k+1}, g)$  with almost-contact structure  $(\psi, \xi, \eta)$  that satisfies the succeeding properties, i.e.

$$\begin{aligned} \psi^2 &= -I + \eta \otimes \xi, \\ \eta(\xi) &= 1, \\ \psi(\xi) &= 0, \\ \eta \circ \psi &= 0, \\ g(\psi U_2, \psi V_2) &= g(U_2, V_2) - \eta(U_2)\eta(V_2), \\ \eta(U_2) &= g(U_2, \xi), \end{aligned} \tag{13}$$

for any  $U_2, V_2$  belonging to  $\tilde{\mathbb{M}}^{2k+1}$ .

The three parameters of almost-contact structure can be developed on its own as  $\psi$  is a  $(1, 1)$ -type tensor field, whereas  $\xi$  is the structure vector field and  $\eta$  is dual 1-form. In the perspective of the Riemannian connection, an almost-contact manifold can be a Sasakian manifold [2, 31] if

$$(\nabla_{U_2}^\sim \psi)V_2 = g(U_2, V_2)\xi - \eta(V_2)U_2. \tag{14}$$

It indicates that

$$\nabla_{U_2}^\sim \xi = -\psi U_2, \tag{15}$$

where  $\nabla$  indicates the Riemannian connection in regard to  $g$  and  $U_2, V_2$  is any vector fields on  $\tilde{\mathbb{M}}^{2k+1}$ . We consider that  $\tilde{\mathbb{M}}^{2k+1}$  converts into a Sasakian space form if it has  $\psi$ -sectional constant curvature  $\varepsilon$  and is represented by  $\tilde{\mathbb{M}}^{2k+1}(\varepsilon)$ . With all this, we can represent the curvature tensor  $\tilde{R}$  of

$\tilde{\mathbb{M}}^{2m+1}(\varepsilon)$  as

$$\begin{aligned} \tilde{R}(X_2, Y_2, Z_2, W_2) = & \frac{\varepsilon + 3}{4} \{g(Y_2, Z_2)g(X_2, W_2) \\ & - g(X_2, Z_2)g(Y_2, W_2)\} \\ & + \frac{\varepsilon - 1}{4} \{\eta(X_2)\eta(Z_2)g(Y_2, W_2) \\ & + \eta(W_2)\eta(Y_2)g(X_2, Z_2) \\ & - \eta(Y_2)\eta(Z_2)g(X_2, W_2) \\ & - \eta(X_2)g(Y_2, Z_2)\eta(W_2) \\ & + g(\psi Y_2, Z_2)g(\psi X_2, W_2) \\ & - g(\psi X_2, Z_2)g(\psi Y_2, W_2) \\ & + 2g(X_2, \psi Y_2)g(\psi Z_2, W_2)\}, \end{aligned} \quad (16)$$

for any arbitrary  $X_2, Y_2, Z_2, W_2$  that belong to  $\tilde{\mathbb{M}}^{2k+1}$  (for more details, see [2, 31, 32]).

Assuming that  $\mathcal{N}^m$  is an  $m$ -dimensional submanifold isometrically immersed in a Sasakian space form  $\tilde{\mathbb{M}}^{2k+1}(\varepsilon)$ . If  $\nabla$  and  $\nabla^\perp$  are generated connections on the tangent bundle  $T\mathcal{N}$  and normal bundle  $T^\perp\mathcal{N}$  of  $\mathcal{N}$ , respectively, then the Gauss and Weingarten formulas are given by

$$(i) \nabla_{U_2}^\perp V_2 = \nabla_{U_2} V_2 + h(U_2, V_2), (ii) \nabla_{U_2}^\perp \zeta = -A_\zeta U_2 + \nabla_{U_2}^\perp \zeta, \quad (17)$$

for each  $U_2, V_2 \in \Gamma(T\mathcal{N})$  and  $\zeta \in \Gamma(T^\perp\mathcal{N})$ , where  $h$  and  $A_\zeta$  are the second fundamental form and shape operator (analogous to the normal vector field  $\zeta$ ), respectively, for the immersion of  $\mathcal{N}^m$  into  $\tilde{\mathbb{M}}^{2k+1}(\varepsilon)$ . They are linked as  $g(h(U_2, V_2), \zeta) = g(A_\zeta U_2, V_2)$ . In the whole article,  $\zeta$  is assumed to be tangential to  $\mathcal{N}$ ; otherwise  $\mathcal{N}$  is simply anti-invariant. Now for any  $U \in \Gamma(T\mathcal{N})$  and  $N \in \Gamma(T^\perp\mathcal{N})$ , we have

$$(i) \psi U_2 = TU_2 + FU_2, \quad (18)$$

$$(ii) \psi \zeta = t\zeta + f\zeta, \quad (19)$$

where  $TU_2(t\zeta)$  and  $FU_2(f\zeta)$  are the tangential and normal components of  $\psi U_2(\psi \zeta)$ , respectively. From (18), it is not difficult to check that for each  $U_2, V_2 \in \Gamma(T\mathcal{N})$ :

$$g(TU_2, V_2) = -g(U_2, TV_2). \quad (20)$$

A submanifold  $\mathcal{N}^m$  is defined to be slant submanifold if for any  $x \in \mathcal{N}$  and for any vector field  $U_2 \in \Gamma(T\mathcal{N}^m)$ , linearly independent on  $\xi$ , the angle between  $\psi U_2$  and  $T\mathcal{N}$  is a constant angle  $\theta(U_2)$  that lies between zero and  $\pi/2$ .

It follows the definition of slant immersions by Cabrerizo et al. [33] who obtained the necessary and sufficient condition that a submanifold  $\mathcal{N}^m$  is said to be a slant submanifold if and only if there exists a constant  $C \in [0, \pi/2]$  and one one tensor field  $T$  is satisfied by the following:

$$T^2 = -C(I - \eta \otimes \xi), \quad (21)$$

such that  $C = \cos^2\theta$ . Also, we have consequence of above formula:

$$g(TU_2, TV_2) = \cos^2\theta \{g(U_2, V_2) - \eta(U_2)\eta(V_2)\}. \quad (22)$$

*Remark 7.* It is clear that slant submanifold is generalized to invariant submanifold with slant angle  $\theta = 0$ .

*Remark 8.* Totally real submanifold is a particular case of slant submanifold with slant angle  $\theta = \pi/2$ .

With the help of moving frame method, we explore some of the interesting features of conformal geometry and slant submanifolds. The specific convection has been applied on indices range. Though we exclude in a way the following:

$$1 \leq i, j, s, \dots \leq m; m+1 \leq \alpha, \beta, \gamma, \dots \leq 2k+1; 1 \leq a, b, c, \dots \leq 2k+1. \quad (23)$$

The mean curvature and squared norm of the mean curvature vector  $H_{\mathcal{N}}$  of a Riemannian submanifold  $\mathcal{N}^m$  is defined by

$$\mathbb{H} = \frac{1}{m} \sum_{i=1}^n h(e_i, e_i) \text{ and } \|\mathbb{H}\|^2 = \frac{1}{m^2} \sum_{r=m+1}^k \left( \sum_{i=1}^m h_{ii}^r \right)^2. \quad (24)$$

Similarly, the length of the second fundamental form  $h$  is given:

$$h_{ij}^r = g(h(e_i, e_j), e_r) \text{ and } S = \|h\|^2 = \sum_{r=m+1}^k \sum_{i,j=1}^n \left( h_{ij}^r \right)^2. \quad (25)$$

In addition, we denoted the following:

$$\|T\|^2 = \sum_{i,j=1}^m g^2(Te_i, e_j). \quad (26)$$

Our main motivation comes from the following example:

*Example 9.* (see [33]). Let  $(\mathbb{R}^{2k+1}, \psi, \xi, \eta, g)$  denotes the Sasakian manifold with Sasakian structure:

$$\begin{aligned} \eta &= \frac{1}{2} \left( dz_1 - \sum_{i=1}^k y_1^i dx_1^i \right), \xi = 2 \frac{\partial}{\partial z_1}, \\ g &= \eta \otimes \eta + \frac{1}{4} \left( \sum_{i=1}^k (dx_1^i \otimes dx_1^i \otimes dy_1^i \otimes dy_1^i) \right), \end{aligned} \quad (27)$$

$$\begin{aligned} \psi & \left( \sum_{i=1}^k \left( X_i \frac{\partial}{\partial x_1^i} + Y_i \frac{\partial}{\partial y_1^i} \right) + Z \frac{\partial}{\partial z_1} \right) \\ &= \sum_{i=1}^k \left( Y_i \frac{\partial}{\partial y_1^i} - X_i \frac{\partial}{\partial x_1^i} \right) + \sum_{i=1}^k Y_i y_1^i \frac{\partial}{\partial z_1}, \end{aligned}$$

where  $(x_1^i, y_1^i, z_1), i = 1 \dots k$  are the coordinates system. It is easy to explain that  $(\mathbb{R}^{2k+1}, \psi, \xi, \eta, g)$  is an almost-contact metric manifold. Now consider the 3-dimension submanifold in  $\mathbb{R}^5$  with Sasakian structure, for any  $\theta \in [0, (\pi/2)]$  such that

$$\psi(u_1, v_1, t) = 2(u_1 \cos \theta, u_1 \sin \theta, v_1, 0, t). \tag{28}$$

Under above immersion,  $\mathcal{N}^3$  is a three-dimension mini-

$$\psi(u_1, v_1, t) = 2\left(e^{\lambda u_1} \cos u_1 \cos v_1, e^{\lambda u_1} \sin u_1 \cos v_1, e^{\lambda u_1} \cos u_1 \sin v_1, e^{\lambda u_1} \sin u_1 \sin v_1, t\right). \tag{29}$$

It is easy see that above immersion is a three-dimension slant submanifold with slant angle  $\theta = \cos^{-1}(|\lambda|/(\sqrt{1 + \lambda^2}))$ . Moreover, scalar curvature  $\tau = -\lambda^2/(3(1 + \lambda^2))$  and mean curvature  $|\mathbb{H}| = 2e^{-\lambda u_1}/3\sqrt{1 + \lambda^2}$ .

It is necessary to clarify the definition of the curvature tensor  $\tilde{R}$  for slant submanifold in Sasakian space form  $\tilde{\mathbb{M}}^{2k+1}(\varepsilon)$  and is given by

$$\begin{aligned} \psi \tilde{R}(e_i, e_j, e_i, e_j) &= \left(\frac{\varepsilon + 3}{4}\right)(m^2 - m) \\ &+ \left(\frac{\varepsilon - 1}{4}\right) \left\{ 3 \sum_{i,j=1}^m g^2(\psi e_i, e_j) - 2(m - 1) \right\}. \end{aligned} \tag{30}$$

On the other hand, let  $\{e_1, \dots, e_q, \dots, e_m = \xi\}$  be an orthonormal basis of  $T_x \mathcal{N}$  such that  $e_1, e_2 = \sec \theta T e_1, \dots, e_{2q} = \sec \theta T e_{2q-1}, \dots, e_{2q+1} = \xi$ , since we define

$$g(\psi e_1, e_2) = g(\psi e_1, \sec \theta T e_1) = \sec \theta g(\psi e_1, T e_1) = \sec \theta g(T e_1, T e_1). \tag{31}$$

It is clear that the dimension of  $\mathcal{N}^m$  can be decomposed as  $m = 2q + 1$ . Then, from (22), we derive that

$$g(\psi e_1, e_2) = \cos \theta. \tag{32}$$

In similar way, we repeat that

$$g^2(\psi e_i, e_{i+1}) = \cos^2 \theta \sum_{i,j=1}^m g^2(\psi e_i, e_j) = (m - 1) \cos^2 \theta. \tag{33}$$

Merging (30) and (33) implies that

$$\tilde{R}(e_i, e_j, e_i, e_j) = \left(\frac{\varepsilon + 3}{4}\right)(m^2 - m) + \left(\frac{\varepsilon - 1}{4}\right)(m - 1)\{3 \cos^2 \theta - 2\}. \tag{34}$$

**2.1. Structure Equations for Slant Submanifolds.** Let  $x$  be a

mal slant submanifold containing slant angle  $\theta$  and scalar curvature  $\tau = -(\cos^2 \theta/3)$ .

Similarly, we give more examples for nonminimal submanifold.

*Example 10* (see [33]). For any constant  $\lambda$ , we define an immersion:

totally real embedding from  $\mathcal{N}^m$  to an  $(2k + 1)$ -dimensional Riemannian manifold  $(\tilde{\mathbb{M}}, \tilde{g})$ . Then  $\mathcal{N}^m$  has a generated metric  $g_{\mathcal{N}} = x^* \tilde{g}$ . Let us consider  $\tilde{\mathbb{M}}^{2k+1} = \tilde{\mathbb{M}}^{2k+1}(\varepsilon)$ , then pulling back in [[1] Eq. (12)] by  $x$  and using [[1] Eqs (13), (14)], we obtain the Gauss equations for slant submanifold in Sasakian space form  $\tilde{\mathbb{M}}^{2k+1}(\varepsilon)$  and taking into account (30).

$$\begin{aligned} R_{ijtl} &= \left(\frac{\varepsilon + 3}{4}\right)(\delta_{it}\delta_{jl} - \delta_{il}\delta_{jt}) + \left(\frac{\varepsilon - 1}{4}\right)\{3(\psi e_i, e_j) \\ &- 2(m - 1)\} + \sum_{\alpha} \left(h_{it}^{\alpha} h_{jl}^{\alpha} - h_{il}^{\alpha} h_{jt}^{\alpha}\right). \end{aligned} \tag{35}$$

Taking trace of the above equation and using (34), we get:

$$\begin{aligned} R &= m^2 |\mathbb{H}|^2 - S + \left(\frac{\varepsilon + 3}{4}\right)m(m - 1) \\ &+ \left(\frac{\varepsilon - 1}{4}\right)(m - 1)\{3 \cos^2 \theta - 2\}, \end{aligned} \tag{36}$$

where  $R$  is the scalar curvature of  $\mathcal{N}^m$  and  $S$  is the length of the second fundamental form  $h$ .

**2.2. Conformal Relations.** In this section, we'll look at how the conformal transformation affects curvature and the second fundamental form. Although these relationships are well-known (cf. [1]), we use the moving frame method to provide a quick proof for readers' convenience.

Assume that  $\tilde{\mathbb{M}}^{2k+1}$  consist a conformal metric  $\tilde{g} = e^{2\rho} \tilde{g}$ , where  $\rho \in C^{\infty}(\tilde{\mathbb{M}})$ . Then  $\tilde{\Omega}_a = e^{\rho} \Omega_a$  stands for the dual coframe of  $(\tilde{\mathbb{M}}, \tilde{g})$ , and  $\tilde{e}_a = e^{\rho} e_a$  stands for the orthogonal frame of  $(\tilde{\mathbb{M}}, \tilde{g})$ . The equality's equations of  $(\tilde{\mathbb{M}}, \tilde{g})$  are given in [[1], Eqs. (20), (21), (22) (23)] by:

$$\tilde{\Omega}_{ab} = \Omega_{ab} + \rho_a \Omega_b - \rho_b \Omega_a, \tag{37}$$

where  $\rho_a$  is the covariant derivative of  $\rho$  with along to  $e_a$ ,



that is,  $d\rho = \sum_a \rho_a e_a$ .

$$\begin{aligned} e^{2\rho} \tilde{R}_{ijil} &= R_{ijil} - \left( \rho_{it} \delta_{jl} + \rho_{jl} \delta_{it} - \rho_{il} \delta_{jt} - \rho_{jt} \delta_{il} \right) \\ &\quad + \left( \rho_i \rho_t \delta_{jl} + \rho_j \rho_l \delta_{it} - \rho_j \rho_t \delta_{il} - \rho_i \rho_l \delta_{jt} \right) \\ &\quad - |\nabla_\alpha|^2 (\delta_{it} \delta_{jl} - \delta_{il} \delta_{jt}). \end{aligned} \quad (38)$$

By pulling back (33) to  $\mathcal{N}^m$  by  $x$ , we have:

$$\tilde{h}_{ij}^\alpha = e^{-\rho} \left( h_{ij}^\alpha - \rho_\alpha \delta_{ji} \right), \quad (39)$$

$$\tilde{\mathbb{H}}^\alpha = e^{-\alpha} (\mathbb{H}^\alpha - \rho_\alpha), \quad (40)$$

from there, it is easy to get the meaningful relationship:

$$e^{2\rho} \left( \tilde{S} - m |\tilde{\mathbb{H}}|^2 \right) + m |\mathbb{H}|^2 = S. \quad (41)$$

### 3. Proof of Main Results

This section is about proving Theorem 2 announced in the previous section. First of all, some fundamental formulas will be presented and some useful lemmas from [27] will be recalled to our setting. For the purpose of this paper, we will provide a significant lemma that is motivated by the review in [1, 27].

*Remark 11.* A simply connected Sasakian space form  $\mathbb{M}^{2k+1}$  is a  $(2k+1)$ -sphere  $\mathbb{S}^{2k+1}$  and Euclidean space  $\mathbb{R}^{2k+1}$  with constant  $\psi$ -sectional curvature  $\varepsilon = 1$  and  $\varepsilon = -3$ , respectively.

Based on the above arguments, we have the following lemma:

**Lemma 12** (see [1]). *Let  $\mathcal{N}^m$  be a slant submanifold of Sasakian space form  $\tilde{\mathbb{M}}^{2k+1}(\varepsilon)$  that is close and oriented with dimension  $m \geq 2$ . If  $x : \mathcal{N}^m \rightarrow \tilde{\mathbb{M}}^{2k+1}(\varepsilon)$  is embedding from  $\mathcal{N}^m$  into  $\tilde{\mathbb{M}}^{2k+1}(\varepsilon)$ , then there is a standard conformal map  $\Gamma : \tilde{\mathbb{M}}^{2k+1}(\varepsilon) \rightarrow \mathbb{S}^{2k+1}(1) \subset \mathbb{R}^{2k+2}$  such that the embedding  $\omega = \Gamma \circ x = (\omega^1, \dots, \omega^{2k+2})$  satisfies*

$$\int_{\mathcal{N}^m} |\omega^a|^{\phi-2} \omega^a dV_{\mathcal{N}} = 0, \quad a = 1, \dots, 2(k+1), \quad (42)$$

for  $\phi > 1$ .

In the above Lemma 12 by the constructed test function, we produce a higher bound for  $\Lambda_{1,\phi}$  in the form of the conformal function and in comparability with Lemma 2.7 in [27].

**Proposition 13.** *Let  $\mathcal{N}^m$  be an  $m \geq 2$ -dimensional closed oriented slant submanifold into Sasakian space form  $\tilde{\mathbb{M}}^{2k+1}(\varepsilon)$ .*

Then, we have

$$\Lambda_{1,\phi} \text{Vol}(\mathcal{N}^m) \leq 2^{|1-\frac{\phi}{2}|} (k+1)^{|1-\frac{\phi}{2}|} m^{\frac{\phi}{2}} \int_{\mathcal{N}^m} (e^{2\rho})^{\frac{\phi}{2}} dV, \quad (43)$$

where  $\Gamma$  stands for conformal map in Lemma 12 and for all  $\phi > 1$ . Identified by  $Y_\varepsilon$ , the standard metric on  $\tilde{\mathbb{M}}^{2k+1}(\varepsilon)$  and considered  $\Gamma^* Y_1 = e^{2\rho} Y_\varepsilon$ .

*Proof.* Considering  $\omega^a$  as a test function along with Lemma 12, we derive

$$\Lambda_{1,\phi} \int_{\mathcal{N}^m} |\omega^a|^\phi \leq |\nabla \omega^a|^\phi dV, \quad 1 \leq a \leq 2(k+1). \quad (44)$$

Observe  $\sum_{a=1}^{2k+2} |\omega^a|^2 = 1$ ; then  $|\omega^a| \leq 1$ . We accomplish

$$\sum_{a=1}^{2k+2} |\nabla \omega^a|^2 = \sum_{i=1}^m |\nabla_{e_i} \omega|^2 = m e^{2\rho}. \quad (45)$$

By using  $1 < \phi \leq 2$ , then we derive

$$|\omega^a|^2 \leq |\omega^a|^\phi. \quad (46)$$

Using the Hölder inequality along with (44)–(46), we are able to get

$$\begin{aligned} \Lambda_{1,\phi} \text{Vol}(\mathcal{N}) &= \Lambda_{1,\phi} \sum_{a=1}^{2k+2} \int_{\mathcal{N}^m} |\omega^a|^2 dV \leq \Lambda_{1,\phi} \sum_{a=1}^{2k+2} \int_{\mathcal{N}^m} |\omega^a|^\phi dV \\ &\leq \Lambda_{1,\phi} \int_{\mathcal{N}^m} \sum_{a=1}^{2k+2} |\nabla \omega^a|^\phi dV \\ &\leq (2k+2)^{1-\phi/2} \int_{\mathcal{N}^m} \left( \sum_{a=1}^m |\nabla \omega^a|^2 \right)^{\frac{\phi}{2}} dV \\ &= 2^{1-\frac{\phi}{2}} (k+1)^{1-\frac{\phi}{2}} \int_{\mathcal{N}^m} (m e^{2\rho})^{\frac{\phi}{2}} dV. \end{aligned} \quad (47)$$

This gives us the desired outcome (43). On the contrary, if we assume  $\phi \geq 2$ , then by applying Hölder inequality, we have

$$1 = \sum_{a=1}^{2k+2} |\omega^a|^2 \leq (2k+2)^{1-\frac{2}{\phi}} \left( \sum_{a=1}^{2k+2} |\omega^a|^\phi \right)^{\frac{2}{\phi}}. \quad (48)$$

And the outcome we get is

$$\Lambda_{1,\phi} \text{Vol}(\mathcal{N}^m) \leq (2k+2)^{\frac{\phi}{2}-1} \left( \sum_{a=1}^{2k+2} \Lambda_{1,\phi} \int_{\mathcal{N}^m} |\omega^a|^\phi dV \right). \quad (49)$$

The Minkowski's inequality gives

$$\sum_{a=1}^{2k+2} |\nabla \omega^a|^\phi \leq \left( \sum_{a=1}^{2k+2} |\nabla \omega^a|^2 \right)^{\frac{\phi}{2}} = (m e^{2\rho})^{\frac{\phi}{2}}. \quad (50)$$

Hence, (43) follows from (44), (49), and (50). This completes the proof of proposition.  $\square$

We are now in the position to prove Theorem 2.

3.1. Proof of Theorem 2. To begin with  $1 < \phi \leq 2$ , then  $\phi/2 \leq 1$ . Taking help from Proposition 13 and implementing the Hölder inequality, we have

$$\begin{aligned} \Lambda_{1,\phi} \text{Vol}(\mathcal{N}^m) &\leq 2^{1-\frac{\phi}{2}}(k+1)^{1-\frac{\phi}{2}} m^{\frac{\phi}{2}} \int_{\mathcal{N}^m} (e^{2\rho})^{\frac{\phi}{2}} dV \\ &\leq 2^{1-\frac{\phi}{2}}(k+1)^{1-\frac{\phi}{2}} m^{\frac{\phi}{2}} (\text{Vol}(\mathcal{N}))^{1-\frac{\phi}{2}} \left( \int_{\mathcal{N}^m} e^{2\rho} dV \right)^{\frac{\phi}{2}}. \end{aligned} \quad (51)$$

By using both conformal relations and Gauss equations, it is possible to calculate  $e^{2\rho}$ . Let  $\tilde{\mathbb{M}}^{2k+1} = \tilde{\mathbb{M}}^{2k+1}(\varepsilon)$ , and  $\tilde{g} = e^{-2\rho} Y_\varepsilon$ ,  $\tilde{g} = \Gamma^* Y_1$  in previous. From (36), the Gauss equations for the embedding  $x$  and the slant embedding  $\omega = \Gamma \circ x$  are respectively

$$\begin{aligned} R &= \left( \frac{\varepsilon+3}{4} \right) m(m-1) + \left( \frac{\varepsilon-1}{4} \right) (m-1) \{3 \cos^2 \theta - 2\} \\ &\quad + m(m-1) |\mathbb{H}|^2 + (m |\mathbb{H}|^2 - S), \end{aligned} \quad (52)$$

$$\tilde{R} - m(m-1) = m(m-1) |\tilde{\mathbb{H}}|^2 + \left( m |\tilde{\mathbb{H}}|^2 - \tilde{S} \right). \quad (53)$$

Tracing (38), it can be established that

$$e^{2\rho} \tilde{R} = R - (m-2)(m-1) |\nabla_\rho|^2 - 2(m-1) \Delta_\rho, \quad (54)$$

which replacing together with (52) and (53) into (54) gives

$$\begin{aligned} e^{2\rho} &\left( m(m-1) + m(m-1) |\tilde{\mathbb{H}}|^2 + \left( m |\tilde{\mathbb{H}}|^2 - \tilde{S} \right) \right) \\ &= \left( \frac{\varepsilon+3}{4} \right) m(m-1) + \left( \frac{\varepsilon-1}{4} \right) (m-1) \{3 \cos^2 \theta - 2\} \\ &\quad + m(m-1) |\mathbb{H}|^2 + (m |\mathbb{H}|^2 - S) - (m-2)(m-1) |\nabla_\rho|^2 \\ &\quad - 2(m-1) \Delta_\rho. \end{aligned} \quad (55)$$

It implies the following:

$$\begin{aligned} e^{2\rho} \tilde{S} - S - (m-2)(m-1) |\nabla_\rho|^2 - 2(m-1) \Delta_\rho \\ &= m(m-1) \left\{ \left\{ e^{2\rho} - \left( \frac{\varepsilon+3}{4} \right) \right. \right. \\ &\quad \left. \left. - \left( \frac{\varepsilon-1}{4} \right) \left( \frac{3 \cos^2 \theta - 2}{m} \right) \right\} + \left( e^{2\rho} |\tilde{\mathbb{H}}|^2 - |\mathbb{H}|^2 \right) \right\} \\ &\quad + m \left( e^{2\rho} |\tilde{\mathbb{H}}|^2 - |\mathbb{H}|^2 \right). \end{aligned} \quad (56)$$

Now from (39) and (41), we derive

$$\begin{aligned} m(m-1) &\left\{ e^{2\rho} - \left( \frac{\varepsilon+3}{4} \right) - \left( \frac{\varepsilon-1}{4} \right) \left( \frac{3 \cos^2 \theta - 2}{n} \right) \right\} \\ &\quad + m(m-1) \sum_\alpha (\mathbb{H}^\alpha - \rho_\alpha)^2 = m(m-1) |\mathbb{H}|^2 \\ &\quad - (m-2)(m-1) |\nabla_\rho|^2 - 2(m-1) \Delta_\rho. \end{aligned} \quad (57)$$

Dividing by  $m(m-1)$  in the above equation, it implies that

$$\begin{aligned} e^{2\rho} &= \left\{ \left( \frac{\varepsilon+3}{4} \right) + \left( \frac{\varepsilon-1}{4} \right) \left( \frac{3 \cos^2 \theta - 2}{m} \right) + |\mathbb{H}|^2 \right\} \\ &\quad - \frac{2}{m} \Delta_\rho - \frac{m-2}{m} |\Delta_\rho|^2 - \left| (\nabla_\rho^\sim)^\perp - \mathbb{H} \right|^2. \end{aligned} \quad (58)$$

Taking integration along  $dV$ , it is not complicated to get the following:

$$\begin{aligned} \Lambda_{1,\phi} \text{Vol}(\mathcal{N}^m) &\leq 2^{1-\frac{\phi}{2}}(k+1)^{1-\frac{\phi}{2}} m^{\frac{\phi}{2}} (\text{Vol}(\mathcal{N}^m))^{1-\frac{\phi}{2}} \left( \int_{\mathcal{N}^m} e^{2\rho} dV \right)^{\frac{\phi}{2}} \\ &\leq \frac{(2k+2)^{1-(\phi/2)} m^{\phi/2}}{(\text{Vol}(\mathcal{N}))^{\phi/2-1}} \left\{ \int_{\mathcal{N}^m} \left\{ \left( \frac{\varepsilon+3}{4} \right) \right. \right. \\ &\quad \left. \left. + \left( \frac{\varepsilon-1}{4} \right) \left( \frac{3 \cos^2 \theta - 2}{m} \right) + |\mathbb{H}|^2 \right\} dV \right\}^{\frac{\phi}{2}}. \end{aligned} \quad (59)$$

The above result is comparable to (8) as we desired to prove. In the case where  $\phi > 2$ , it is not possible to apply the Hölder inequality directly to govern  $\int_{\mathcal{N}^m} (e^{2\rho})^{\phi/2}$  by using  $\int_{\mathcal{N}^m} (e^{2\rho})$ . We did multiply both sides of (58) with the factor  $e^{(\phi-2)\rho}$  and then solve by using integration on  $\mathcal{N}^m$  (cf. [25]).

$$\begin{aligned} \int_{\mathcal{N}^m} e^{\phi\rho} dV &\leq \int_{\mathcal{N}^m} \left\{ \left( \frac{\varepsilon+3}{4} \right) + \left( \frac{\varepsilon-1}{4} \right) \left( \frac{3 \cos^2 \theta - 2}{m} \right) \right. \\ &\quad \left. + |\mathbb{H}|^2 \right\} e^{(\phi-2)\rho} dV - \left( \frac{m-2-2\phi+4}{m} \right) \int_{\mathcal{N}^m} e^{(\phi-2)\rho} |\Delta_\rho|^2 dV \\ &\leq \int_{\mathcal{N}^m} \left\{ \left( \frac{\varepsilon+3}{4} \right) + \left( \frac{\varepsilon-1}{4} \right) \left( \frac{3 \cos^2 \theta - 2}{m} \right) + |\mathbb{H}|^2 \right\} e^{(\phi-2)\rho} dV. \end{aligned} \quad (60)$$

Next, it follows from the assumption  $m \geq 2\phi - 2$ . We apply Young's inequality; then

$$\begin{aligned} & \int_{\mathcal{N}^m} \left\{ \left( \frac{\varepsilon + 3}{4} \right) + \left( \frac{\varepsilon - 1}{4} \right) \left( \frac{3 \cos^2 \theta - 2}{m} \right) + |\mathbb{H}|^2 \right\} e^{(\phi-2)\rho} dV \\ & \leq \frac{2}{\phi} \int_{\mathcal{N}^m} \left\{ \left| \left( \frac{\varepsilon + 3}{4} \right) + \left( \frac{\varepsilon - 1}{4} \right) \left( \frac{3 \cos^2 \theta - 2}{m} \right) + |\mathbb{H}|^2 \right| \right\}^{\frac{\phi}{2}} dV \\ & \quad + \frac{(\phi - 2)}{\phi} \int_{\mathcal{N}^m} e^{\frac{\phi}{2}\rho} dV. \end{aligned} \quad (61)$$

From (60) and (61) we deduce the following inequality:

$$\int_{\mathcal{N}^m} e^{\rho} dV \leq \int_{\mathcal{N}^m} \left( \left| \left( \frac{\varepsilon + 3}{4} \right) + \left( \frac{\varepsilon - 1}{4} \right) \left( \frac{3 \cos^2 \theta - 2}{m} \right) + |\mathbb{H}|^2 \right| \right)^{\frac{\phi}{2}} dV. \quad (62)$$

Now putting (62) into (43), we obtain (9). In the case of slant submanifolds, the equality case holds in (8); then considering the cases in (44) and (46), we get this:

$$\begin{aligned} |\omega^a|^2 &= |\omega^a|^\phi, \\ \Delta_\phi \omega^a &= -\Lambda_{1,\phi} |\omega^a|^{\phi-2} \omega^a, \end{aligned} \quad (63)$$

for each  $a = 1, \dots, 2k + 2$ . If  $1 < \phi < 2$ , then  $|\omega^a| = 0$  or  $1$ . So, there would be only one  $a$  for which  $|\omega^a| = 1$  and  $\Lambda_{1,\phi} = 0$ , which seems to be a contradiction as the eigenvalue is non-zero. For this reason, we consider  $\phi = 2$  and only restricted to Laplacian case. After this, we are able to apply Theorem 1.5 from [21].

Let  $\phi > 2$  and equality remains valid in (9); then it shows that (49) and (50) become the equalities which indicates

$$|\omega^1|^\phi = \dots = |\omega^{2k+2}|^\phi, \quad (64)$$

and condition  $|\nabla \omega^a| = 0$  holds for existing  $a$ . It shows that  $\omega^a$  is a constant value and  $\Lambda_{1,\phi}$  is also equal to zero. This last result again represents a conflict that  $\Lambda_{1,\phi}$  is a nonnull eigenvalue. This completes the proof of the theorem.

**3.2. Proof of Theorem 5.** Suppose that  $1 < \phi \leq 2$ ; we have  $\phi / (2(\phi - 1)) \geq 1$ . Then, by the Hölder inequality, we have

$$\begin{aligned} & \int_{\mathcal{N}^m} \left\{ \left( \frac{\varepsilon + 3}{4} \right) + \left( \frac{\varepsilon - 1}{4} \right) \left( \frac{3 \cos^2 \theta - 2}{m} \right) + |\mathbb{H}|^2 \right\} dV \\ & \leq \left( \text{Vol}(\mathcal{N}) \right)^{1 - \frac{2(\phi-1)}{\phi}} \times \left\{ \int_{\mathcal{N}^m} \left\{ \left( \frac{\varepsilon + 3}{4} \right) \right. \right. \\ & \quad \left. \left. + \left( \frac{\varepsilon - 1}{4} \right) \left( \frac{3 \cos^2 \theta - 2}{m} \right) + |\mathbb{H}|^2 \right\}^{\frac{\phi}{2(\phi-1)}} dV \right\}^{\frac{2(\phi-1)}{\phi}}. \end{aligned} \quad (65)$$

Thus, combining Equation (8) with (65), we get the desired result (12). This completes the proof of the theorem.

As a result of the observations in Remark 11, the next result will be specified as a special variant of Theorem 2. To be precise, we determine the following result by replacing  $\varepsilon = 1$  in (8) and (9), respectively.

**Corollary 14.** Assume  $\mathcal{N}^m$  is an  $m(\geq 2)$ -dimensional closed orientated slant submanifold in  $(2k + 1)$ -sphere  $\mathbb{S}^{2k+1}(1)$ . Then,  $\Lambda_{1,\phi}$  satisfies the following inequality for the  $\phi$ -Laplacian:

$$\begin{aligned} \Lambda_{1,\phi} & \leq \frac{2^{1-\phi/2}(k+1)^{(1-\phi/2)} m^{\phi/2}}{(\text{Vol}(\mathcal{N}))^{\phi/2}} \left\{ \int_{\mathcal{N}^m} (1 + |\mathbb{H}|^2) dV \right\}^{\frac{\phi}{2}} \text{ for } 1 < \phi \leq 2, \\ \Lambda_{1,\phi} & \leq \frac{2^{1-\phi/2}(k+1)^{(1-\phi/2)} m^{\phi/2}}{\text{Vol}(\mathcal{N})} \int_{\mathcal{N}^m} (1 + |\mathbb{H}|^2)^{\frac{\phi}{2}} dV \text{ for } 2 < \phi \leq \frac{m}{2} + 1. \end{aligned} \quad (66)$$

There is an additional corollary derived from Corollary 14 as follows.

**Corollary 15.** Assuming that  $\mathcal{N}^m$  is an  $m(\geq 2)$ -dimensional closed orientated slant submanifold in  $(2k + 1)$ -sphere  $\mathbb{S}^{2k+1}(1)$ , then  $\Lambda_{1,\phi}$  satisfies the following inequality for the  $\phi$ -Laplacian:

$$\Lambda_{1,\phi} \leq \frac{(2k+2)^{(1-\phi/2)} m^{\phi/2}}{(\text{Vol}(\mathcal{N}))^{(\phi-1)}} \left\{ \int_{\mathcal{N}^m} (1 + |\mathbb{H}|^2)^{\frac{\phi}{2(\phi-1)}} dV \right\}^{(\phi-1)}, \quad (67)$$

for  $1 < \phi \leq 2$ .

*Remark 16.* It is noticed that Corollaries 14 and 15 are exactly same as Theorem 1.5 in [20].

**3.3. Application to Invariant Submanifolds of Sasakian Space Forms.** Using Remark 7 and Theorem 2, we have the following results.

**Corollary 17.** Let  $\mathcal{N}^m$  be an  $m(\geq 2)$ -dimensional closed orientated invariant submanifold in a Sasakian space form  $\tilde{\mathbb{M}}^{2k+1}(\varepsilon)$ . Then,  $\Lambda_{1,\phi}$  satisfies the following inequality for the  $\phi$ -Laplacian:

$$\begin{aligned} \Lambda_{1,\phi} & \leq \left( \frac{2^{(1-\phi/2)}(k+1)^{(1-\phi/2)} m^{\phi/2}}{(\text{Vol}(\mathcal{N}))^{\phi/2}} \right) \\ & \quad \times \left\{ \int_{\mathcal{N}^m} \left\{ \left( \frac{\varepsilon + 3}{4} \right) + \left( \frac{\varepsilon - 1}{4m} \right) + |\mathbb{H}|^2 \right\} dV \right\}^{\phi/2} \text{ for } 1 < \phi \leq 2, \\ \Lambda_{1,\phi} & \leq \left( \frac{2^{(1-\phi/2)}(k+1)^{(\phi/2-1)} m^{\phi/2}}{\text{Vol}(\mathcal{N})} \right) \\ & \quad \times \int_{\mathcal{N}^m} \left\{ \left( \frac{\varepsilon + 3}{4} \right) + \left( \frac{\varepsilon - 1}{4m} \right) + |\mathbb{H}|^2 \right\}^{\phi/2} dV \text{ for } 2 < \phi \leq \frac{m}{2} + 1. \end{aligned} \quad (68)$$



From Corollary 4 for  $\phi = 2$  and Remark 7, we have the following.

**Corollary 18.** *Assuming that  $\mathcal{N}^m$  is an  $m$ -dimensional closed orientated invariant submanifold in a Sasakian space form  $\tilde{\mathbb{M}}^{2k+1}(\varepsilon)$ , then  $\Lambda_1^\Delta$  satisfies the following inequality for the Laplacian:*

$$\Lambda_1^\Delta \leq \frac{m}{\text{Vol}(\mathcal{N})} \int_{\mathcal{N}} \left\{ |\mathbb{H}|^2 + \left(\frac{\varepsilon + 3}{4}\right) + \left(\frac{\varepsilon - 1}{4m}\right) \right\} dV. \quad (69)$$

Similarly, from Theorem 5, we obtain the following corollary.

**Corollary 19.** *Assuming that  $\mathcal{N}^m$  is an  $m(\geq 2)$ -dimensional closed orientated invariant submanifold in a Sasakian space form  $\tilde{\mathbb{M}}^{2k+1}(\varepsilon)$ , then  $\Lambda_{1,\phi}$  satisfies the following inequality for the  $\phi$ -Laplacian:*

$$\begin{aligned} \Lambda_{1,\phi} \leq & \left( \frac{2^{1-\phi/2}(k+1)^{(1-\phi/2)}m^{\phi/2}}{(\text{Vol}(\mathcal{N}))^{(\phi-1)}} \right) \\ & \times \left\{ \int_{\mathcal{N}^m} \left( \left(\frac{\varepsilon + 3}{4}\right) + \left(\frac{\varepsilon - 1}{4m}\right) + |\mathbb{H}|^2 \right)^{\frac{\phi}{2(\phi-1)}} dV \right\}^{(\phi-1)}, \end{aligned} \quad (70)$$

for  $1 < \phi \leq 2$ .

**3.4. Application to Anti-Invariant Submanifolds of Sasakian Space Forms.** Using Remark 8 and Theorem 2, we have the following results:

**Corollary 20.** *Let  $\mathcal{N}^m$  be an  $m(\geq 2)$ -dimensional closed orientated anti-invariant submanifold in a Sasakian space form  $\tilde{\mathbb{M}}^{2k+1}(\varepsilon)$ . Then,  $\Lambda_{1,\phi}$  satisfies the following inequality for the  $\phi$ -Laplacian:*

$$\begin{aligned} \Lambda_{1,\phi} \leq & \left( \frac{2^{(1-\phi/2)}(k+1)^{(1-\phi/2)}m^{\phi/2}}{(\text{Vol}(\mathcal{N}))^{\phi/2}} \right) \\ & \cdot \left\{ \int_{\mathcal{N}^m} \left\{ \left(\frac{\varepsilon + 3}{4}\right) - \left(\frac{\varepsilon - 1}{2m}\right) + |\mathbb{H}|^2 \right\} dV \right\}^{p/2} \text{ for } 1 < \phi \leq 2, \\ \Lambda_{1,\phi} \leq & \left( \frac{2^{(1-\phi/2)}(k+1)^{(\phi/2-1)}m^{p/2}}{\text{Vol}(\mathcal{N})} \right) \int_{\mathcal{N}^m} \\ & \cdot \left\{ \left| \left(\frac{\varepsilon + 3}{4}\right) - \left(\frac{\varepsilon - 1}{2m}\right) + |\mathbb{H}|^2 \right| \right\}^{\phi/2} dV \text{ for } 2 < \phi \leq \frac{m}{2} + 1. \end{aligned} \quad (71)$$

From Corollary 4 for  $\phi = 2$  and Remark 8, we have the following.

**Corollary 21.** *Assuming that  $\mathcal{N}^m$  is an  $m$ -dimensional closed orientated anti-invariant submanifold in a Sasakian space form  $\tilde{\mathbb{M}}^{2k+1}(\varepsilon)$ , then  $\Lambda_1^\Delta$  satisfies the following inequality for*

*the Laplacian:*

$$\Lambda_1^\Delta \leq \frac{m}{\text{Vol}(\mathcal{N})} \int_{\mathcal{N}} \left\{ |\mathbb{H}|^2 + \left(\frac{\varepsilon + 3}{4}\right) - \left(\frac{\varepsilon - 1}{2m}\right) \right\} dV. \quad (72)$$

Similarly, from Theorem 5, we obtain the following corollary.

**Corollary 22.** *Assuming that  $\mathcal{N}^m$  is an  $m(\geq 2)$ -dimensional closed orientated anti-invariant submanifold in a Sasakian space form  $\tilde{\mathbb{M}}^{2k+1}(\varepsilon)$ , then  $\Lambda_{1,\phi}$  satisfies the following inequality for the  $\phi$ -Laplacian:*

$$\begin{aligned} \Lambda_{1,\phi} \leq & \left( \frac{2^{1-\phi/2}(k+1)^{(1-\phi/2)}m^{\phi/2}}{(\text{Vol}(\mathcal{N}))^{(\phi-1)}} \right) \\ & \cdot \left\{ \int_{\mathcal{N}^m} \left( \left(\frac{\varepsilon + 3}{4}\right) - \left(\frac{\varepsilon - 1}{2m}\right) + |\mathbb{H}|^2 \right)^{\frac{\phi}{2(\phi-1)}} dV \right\}^{(\phi-1)}, \end{aligned} \quad (73)$$

for  $1 < \phi \leq 2$ .

### Data Availability

There is no data used for this manuscript.

### Conflicts of Interest

The authors declare no competing of interest.

### Authors' Contributions

All authors have equal contribution and have finalized the manuscript.

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