Hindawi
Journal of Function Spaces
Volume 2021, Article ID 6270711, 9 pages
https://doi.org/10.1155/2021/6270711

Research Article

A New Class of Analytic Normalized Functions Structured by a Fractional Differential Operator

Najla M. Alarifi[1] and Rabha W. Ibrahim[2]

[1] *Department of Mathematics, Imam Abdulrahman Bin Faisal University, Dammam 31113, Saudi Arabia*
[2] *IEEE: 94086547, Kuala Lumpur 59200, Malaysia*

Correspondence should be addressed to Rabha W. Ibrahim; rabhaibrahim@yahoo.com

Received 11 July 2021; Accepted 12 August 2021; Published 13 September 2021

Academic Editor: Sarfraz Nawaz Malik

Newly, the field of fractional differential operators has engaged with many other fields in science, technology, and engineering studies. The class of fractional differential and integral operators is considered for a real variable. In this work, we have investigated the most applicable fractional differential operator called the Prabhakar fractional differential operator into a complex domain. We express the operator in observation of a class of normalized analytic functions. We deal with its geometric performance in the open unit disk.

1. Introduction

The class of complex fractional operators (differential and integral) is investigated geometrically by Srivastava et al. [1] and generalized into two-dimensional fractional parameters by Ibrahim for a class of analytic functions in the open unit disk [2]. These operators are consumed to express different classes of analytic functions, fractional analytic functions [3] and differential equations of a complex variable, which are called fractional algebraic differential equations studding the Ulam stability [4, 5].

We carry on our investigation in the field of complex fractional differential operators. In this investigation, we formulate an arrangement of the fractional differential operator in the open unit disk refining the well-known Prabhakar fractional differential operator. We apply the recommended operator to describe new generalized classes of fractional analytic functions including the Briot-Bouquet types. Consequently, we study the classes in terms of the geometric function theory.

2. Methods

Our methods are divided into two subsections, as follows.

2.1. Geometric Methods. In this place, we clarify selected notions in the geometric function theory, which are situated in [6–8].

Definition 1. Let $\cup := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. Two analytic functions g_1, g_2 in $\cup$ are called subordinated denoting by $g_1 \prec g_2$ or $g_1(z) \prec g_2(z), z \in \mathbb{U}$, if there exits an analytic function ω, $|w| \le |z| < 1$ having the formula

$$g_1(z) = g_2(w(z)), \quad z \in \cup. \tag{1}$$

g_1 is majorized by g_2 denoting by $g_1 \ll g_2$ if and only if

$$g_1(z) = w(z) g_2(z), \quad z \in \cup; \tag{2}$$

equivalently, the coefficient inequality is held $|a_n| \le |b_n|$, respectively.

There is a deep construction between subordination and majorization [9] in $\cup$ for selected distinct classes comprising the convex class ($\mathscr{C}$):

$$1+\Re\left(\frac{zg''(z)}{g'(z)}\right)>0, \quad z\in \cup, \tag{3}$$

and starlike functions $(\mathcal{S}^*)$

$$\Re\left(\frac{zg'}{g(z)}\right)>0, \quad z\in \cup. \tag{4}$$

Definition 2. We present a class of analytic functions by

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n, \quad z\in \cup. \tag{5}$$

This class is denoted by Λ and known as the class of univalent functions which is normalized by $f(0)=f'(0)-1=0$.

Associated with the terms $\mathcal{S}^*$ and $\mathcal{C}$, we present the term $\mathcal{P}$ of all analytic functions p in $\cup$ with a positive real part in $\cup$ and $p(0)=1$.

Two analytic functions f, g are called convoluted, denoting by $f * g$ if and only if

$$(f*g)(z)=\left(\sum_{n=0}^{\infty}a_n z^n\right)*\left(\sum_{n=0}^{\infty}g_n z^n\right)=\sum_{n-0}^{\infty}a_n g_n z^n. \tag{6}$$

Definition 3. The generalized Mittag-Leffler function is defined by [10–12]

$$\Xi_{\nu,\mu}^{\wp}(z)=\sum_{n=0}^{\infty}\frac{(\wp)_n}{\Gamma(\nu n+\mu)}\frac{z^n}{n!}, \tag{7}$$

where $(\vartheta)_n$ represents the Pochhammer symbol and

$$\Xi_{\nu,\mu}^{1}(z):=\Xi_{\nu,\mu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\nu n+\mu)} \cdot\left((\wp)_0=1,(\wp)_n=\wp(\wp+1)\cdots(\wp+n-1)\right). \tag{8}$$

Note that $\Xi_{\nu,u}^{\wp}(z)$ is an ultimate traditional generalization of the function e^z, where $\Xi_{1,1}^{1}(z)=e^z$.

Moreover, it can be formulated by the Fox-Write hypergeometric function, as follows:

$$\Xi_{\nu,\mu}^{\wp}(z)=\left(\frac{1}{\Gamma(\wp)}\right){}_1\Psi_1\left[\begin{matrix}(\wp,1)\\ \nu,\mu\end{matrix};z\right]. \tag{9}$$

2.2. Complex Prabhakar Operator (CPO). The Prabhakar integral operator is defined for analytic function

$$\psi(z)\in\mathcal{H}[0,1]=\left\{\psi(z)=\psi_1 z+\psi_2 z^2+\cdots,z\in\cup\right\} \tag{10}$$

by the formula [13, 14]

$$\begin{aligned}P_{\alpha,\beta}^{\gamma,\omega}\psi(z)&=\int_0^z(z-\zeta)^{\beta-1}\Xi_{\alpha,\beta}^{\gamma}\left[\omega(z-\zeta)^{\alpha}\right]\psi(\zeta)d\zeta\\&=\left(\psi\cdot\varrho_{\alpha,\beta}^{\gamma,\omega}\right)(z)\ (\alpha,\beta,\gamma,\omega\in\mathbb{C}, z\in\cup).\end{aligned} \tag{11}$$

Moreover [13, 14],

$$\begin{aligned}\varrho_{\alpha,\beta}^{\gamma,\omega}(z)&:=z^{\beta-1}\Xi_{\alpha,\beta}^{\gamma}(\omega z^{\alpha}),\\ \Xi_{\alpha,\beta}^{\gamma}(\chi)&=\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)\Gamma(\alpha n+\beta)}\frac{\chi^n}{n!}.\end{aligned} \tag{12}$$

For example, let $\psi(z)=z^{\varsigma-1}$, then (see [15], Corollary 2.3)

$$\begin{aligned}P_{\alpha,\beta}^{\gamma,\omega}z^{\varsigma-1}&=\int_0^z(z-\zeta)^{\beta-1}\Xi_{\alpha,\beta}^{\gamma}\left[\omega(z-\zeta)^{\alpha}\right]\left(\zeta^{\varsigma-1}\right)d\zeta\\&=\Gamma(\varsigma)z^{\beta+\varsigma-1}\Xi_{\alpha,\beta+\varsigma}^{\gamma}(\omega z^{\alpha}).\end{aligned} \tag{13}$$

The Prabhakar derivative can be computed by the formula [13]

$${}_kD_{\alpha,\beta}^{\gamma,\omega}f(\chi)=\frac{d^k}{d\chi^k}\left(P_{\alpha,k-\beta}^{-\gamma,\omega}f(\chi)\right). \tag{14}$$

Definition 4. Let $\psi\in\Lambda$. Then the complex Prabhakar differential operator (CPFDO) of (13) is formulated in terms of the Riemann-Liouville derivative, as follows:

$$\begin{aligned}{}_k^{\mathcal{R}}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}\psi(z)&=\frac{d^k}{dz^k}\int_0^z(z-\zeta)^{k-\beta-1}\Xi_{\alpha,k-\beta}^{-\gamma}\left[\omega(z-\zeta)^{\alpha}\right]\psi(\zeta)d\zeta\\&=\frac{d^k}{dz^k}\left(P_{\alpha,k-\beta}^{-\gamma,\omega}\psi(z)\right),\end{aligned} \tag{15}$$

and in terms of the Caputo derivative, as follows:

$$\begin{aligned}{}_k^{\mathcal{C}}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}\psi(z)&=\int_0^z(z-\zeta)^{k-\beta-1}\Xi_{\alpha,k-\beta}^{-\gamma}\left[\omega(z-\zeta)^{\alpha}\right]\left(\frac{d^k}{d\zeta^k}\psi(\zeta)\right)d\zeta\\&=P_{\alpha,k-\beta}^{-\gamma,\omega}\left(\frac{d^k}{dz^k}\psi(z)\right).\end{aligned} \tag{16}$$

Note that

$${}_k^{\mathcal{C}}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}\psi(z)={}_k^{\mathcal{R}}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}\psi(z)-\sum_{m=0}^{k-1}z^{m-\beta}\Xi_{\alpha,m-\beta}^{-\gamma}[\omega z^{\alpha}]\psi^{(m)}(0). \tag{17}$$

For example, let $\psi(z)=z^{\varepsilon}, \varepsilon \geq 1$, then in virtue of [15] (Corollary 2.3), we conclude that

$$\begin{aligned}
{}_{1}^{\mathscr{C}}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}(z^{\varepsilon}) &= \int_0^z (z-\zeta)^{1-\beta-1}\Xi_{\alpha,1-\beta}^{-\gamma}\left[\omega(z-\zeta)^{\alpha}\right]\left(\frac{d}{d\zeta}\psi(\zeta)\right)d\zeta: \\
&= \int_0^Z (z-\zeta)^{\mu-1}\Xi_{\alpha,\mu}^{-\gamma}\left[\omega(z-\zeta)^{\alpha}\right]\left(\frac{d}{d\zeta}(\zeta^{\varepsilon})\right)d\zeta \\
&= \varepsilon\int_0^z \zeta^{\varepsilon-1}(z-\zeta)^{\mu-1}\Xi_{\alpha,\mu}^{-\gamma}\left[\omega(z-\zeta)^{\alpha}\right]d\zeta \\
&= \Gamma(\varepsilon+1)z^{\mu+\varepsilon-1}\Xi_{\alpha,\mu+\varepsilon}^{-\gamma}[\omega z^{\alpha}], \quad \mu:=1-\beta.
\end{aligned} \tag{18}$$

In general, we have

$$\begin{aligned}
{}_{k}^{\mathscr{C}}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}(z^{\varepsilon}) &= \int_0^z (z-\zeta)^{k-\beta-1}\Xi_{\alpha,k-\beta}^{-\gamma}\left[\omega(z-\zeta)^{\alpha}\right]\left(\frac{d^k}{d\zeta^k}(\zeta^{\varepsilon})\right)d\zeta \\
&= \int_0^z (z-\zeta)^{k-\beta-1}\Xi_{\alpha,k-\beta}^{-\gamma}\left[\omega(z-\zeta)^{\alpha}\right]\left(\frac{d}{d\zeta}(\zeta^{\varepsilon})\right)d\zeta \\
&= (1-k+\varepsilon)_k\int_0^z \zeta^{\varepsilon-k}(z-\zeta)^{k-\beta-1}\Xi_{\alpha,k-\beta}^{-\gamma}\left[\omega(z-\zeta)^{\alpha}\right]d\zeta \\
&= (1-k+\varepsilon)_k\int_0^z \zeta^{(\varepsilon-k+1)-1} \\
&\quad\cdot (z-\zeta)^{k-\beta-1}\Xi_{\alpha,k-\beta}^{-\gamma}\left[\omega(z-\zeta)^{\alpha}\right]d\zeta: \\
&= (\nu)_k\int_0^z \zeta^{\nu-1}(z-\zeta)^{\mu-1}\Xi_{\alpha,\mu}^{-\gamma}\left[\omega(z-\zeta)^{\alpha}\right]d\zeta \\
&= (\nu)_k\Gamma(\nu)z^{\nu+\mu-1}\Xi_{\alpha,\mu+\nu}^{-\gamma}[\omega z^{\alpha}],
\end{aligned} \tag{19}$$

where $\mu := k-\beta$, $\nu := \varepsilon-k+1$, and $(\nu)_k = \Gamma(1+\varepsilon)/\Gamma(1+\varepsilon-k)$. Hence, we obtain

$$\begin{aligned}
{}_{k}^{\mathscr{C}}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}(z^{\varepsilon}) &= \Gamma(1+\varepsilon)z^{\nu+\mu-1}\Xi_{\alpha,\mu+\nu}^{-\gamma}[\omega z^{\alpha}] \\
&= \Gamma(k+\nu)z^{\nu+\mu-1}\Xi_{\alpha,\mu+\nu}^{-\gamma}[\omega z^{\alpha}].
\end{aligned} \tag{20}$$

We have the following property.

Proposition 5. *Let $\psi \in \Lambda$. Define a functional ${}_{k}^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega}: \cup \longrightarrow \cup$ by*

$${}_{k}^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega} := \left(\frac{z^{\beta}}{\Xi_{\alpha,2-\beta}^{-\gamma}[wz^{\alpha}]}\right)\left({}^{\mathscr{C}}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}\right). \tag{21}$$

Then ${}_{k}^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi = {}_{k}^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega} * \psi \in \Lambda(\alpha,\beta,\gamma,\omega \in \mathbb{C}, z \in \cup)$.

Proof. Let $\psi \in \Lambda$. Then a computation implies

$$\begin{aligned}
{}_{k}^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z) &= \left(\frac{z^{\beta}}{\Xi_{\alpha,2-\beta}^{-\gamma}[\omega z^{\alpha}]}\right)\left({}_{k}^{\mathscr{C}}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right) \\
&= \left(\frac{z^{\beta}}{\Xi_{\alpha,2-\beta}^{-\gamma}[\omega z^{\alpha}]}\right)\left({}_{k}^{\mathscr{C}}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}\left(z+\sum_{n=2}^{\infty}\psi_n z^n\right)\right) \\
&= \left(\frac{z^{\beta}}{\Xi_{\alpha,2-\beta}^{-\gamma}[\omega z^{\alpha}]}\right)\left({}_{k}^{\mathscr{C}}\mathbb{D}_{\alpha,\beta^z}^{\gamma,\omega} + \sum_{n=2}^{\infty}\psi_n {}_{k}^{\mathscr{C}}\mathbb{D}_{\alpha,\beta}^{\gamma,\omega}z^n\right) \\
&= \left(\frac{z^{\beta}}{\Xi_{\alpha,2-\beta}^{-\gamma}[\omega z^{\alpha}]}\right)\left(\Xi_{\alpha,2-\beta}^{-\gamma}[\omega z^{\alpha}]z^{1-\beta}\right. \\
&\quad\left. + \sum_{n=2}^{\infty}\psi_n\Gamma(n+1)z^{n-\beta}\Xi_{\alpha,n+1-\beta}^{-\gamma}[\omega z^{\alpha}]\right) \\
&= z+\sum_{n=2}^{\infty}\left(\psi_n\Gamma(n+1)\frac{\Xi_{\alpha,n+1-\beta}^{-\gamma}[\omega z^{\alpha}]}{\Xi_{\alpha,2-\beta}^{-\gamma}[\omega z^{\alpha}]}\right)z^n: \\
&= z+\sum_{n=2}^{\infty}\psi_n\delta_n z^n = \left(z+\sum_{n=2}^{\infty}\delta_n z^n\right) \\
&\quad * \left(z+\sum_{n=2}^{\infty}\psi_n z^n\right) = \left({}_{k}^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega} * \psi\right)(z),
\end{aligned} \tag{22}$$

where $\delta_n := \Gamma(n+1)\Xi_{\alpha,n+1-\beta}^{-\gamma}[\omega z^{\alpha}]/\Xi_{\alpha,2-\beta}^{-\gamma}[\omega z^{\alpha}]$. This indicates that ${}_{k}^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi \in \Lambda$. □

We call ${}_{k}^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega}$ the normalized complex Prabhakar operator (NCPO) in the open unit disk. Since ${}_{k}^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega} \in \Lambda$, then we can study it in view of the geometric function theory.

Our aim is to formulate it in terms of some well-known classes of analytic functions. It is clear that δ_n is a complex connection (coefficient) of the operator and it is a constant when $\alpha = 0$.

Remark 6. The integral operator corresponding to the fractional differential operator ${}_{k}^{C}\Delta_{\alpha\beta}^{\gamma\omega}$ is expanded by the series

$${}_{k}^{C}\Upsilon_{\alpha,\beta}^{\gamma,\omega}\psi(z) = z+\sum_{n=2}^{\infty}\left(\psi_n\frac{\Xi_{\alpha,2-\beta}^{-\gamma}[\omega z^{\alpha}]}{\Gamma(n+1)\Xi_{\alpha,n+1-\beta}^{-\gamma}[\omega z^{\alpha}]}\right)z^n. \tag{23}$$

It is clear that

$$\left({}_{k}^{C}\Upsilon_{\alpha,\beta}^{\gamma,\omega} * {}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\right)\psi(z) = \left({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega} * {}_{k}^{C}\Upsilon_{\alpha,\beta}^{\gamma,\omega}\right)\psi(z) = \psi(z). \tag{24}$$

The linear convex combination of the operators ${}_{k}^{C}\Upsilon_{\alpha,\beta}^{\gamma,\omega}$ and ${}_{k}^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega}$ can be recognized by the formula

$${}_{k}^{\complement}\sum_{\alpha,\beta}^{\gamma,\omega}\psi(z) = \complement\,{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z) + (1-\complement)\,{}_{k}^{C}\Upsilon_{\alpha,\beta}^{\gamma,\omega}\psi(z), \tag{25}$$

where $\complement \in [0,1]$. Clearly, ${}_{k}^{\complement}\sum_{\alpha,\beta}^{\gamma,\omega}\psi(z) \in \Lambda$, *where* $\psi \in \Lambda$.

2.3. Subclasses of NCPO. In terms of the NCPO, we formulate the next classes.

Definition 7. A function $\psi \in \Lambda$ is considered to be in the class ${}_{k}^{c}S_{\alpha,\beta}^{*\gamma,\omega}(\sigma)$ if and only if

$$ {}_{k}^{c}S_{\alpha,\beta}^{*\gamma,\omega}(\sigma) = \left\{ \psi \in \Lambda : \frac{z\left({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)} \prec \sigma(z),\ \sigma(0) = 1 \right\}. \tag{26} $$

We shall deal with the conditions of a function ψ to be in ${}_{k}^{c}S_{\alpha,\beta}^{*\gamma,\omega}(\sigma)$ whenever $\sigma \in C$ is convex as well as nonconvex.

Definition 8. A function $\psi \in \Lambda$ is considered to be in the class ${}_{k}^{c}J_{\alpha,\beta}^{\gamma,\omega}(\mathfrak{A}, \mathfrak{B}, \flat)$ if and only if

$$ {}_{k}^{c}J_{\alpha,\beta}^{\gamma,\omega}(\mathfrak{A}, \mathfrak{B}, \flat) = \left\{ \psi \in \Lambda : 1 + \frac{1}{\flat}\left(\frac{2{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z) - {}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(-z)} \right) \prec \frac{1+\mathfrak{A}z}{1+\mathfrak{B}z} \right\}. \tag{27} $$

We request the next result, which can be located in [6].

Lemma 9. *Define the class of analytic functions as follows: for $\varrho \in \mathbb{C}$ and a positive integer n*

$$ \mathbb{H}[\psi, n] = \left\{ \psi : \psi(z) = \varrho + \varrho_n z^n + \varrho_{n+1} z^{n+1} + \cdots \right\}. \tag{28} $$

(i) *Let $\ell \in \mathbb{R}$. Then $\Re(\psi(z) + \ell z\psi'(z)) > 0 \longrightarrow \Re(\psi(z)) > 0$. In addition, if $\ell > 0$ and $\psi \in \mathbb{H}[1, n]$, then there are constants $\wp > 0$ and $\kappa > 0$ such that $\kappa = \kappa(\ell, \wp, n)$ and*

$$ \psi(z) + \ell z\psi'(z) \prec \left(\frac{1+z}{1-z}\right)^{\kappa} \longrightarrow \psi(z) \prec \left(\frac{1+z}{1-z}\right)^{\wp} \tag{29} $$

(ii) *Let $c \in [0, 1)$ and $\psi \in \mathbb{H}[1, n]$. Then there exists a fixed real number $\ell > 0$ so that*

$$ \Re\left(\psi^2(z) + 2\psi(z)\cdot z\psi'(z)\right) > c \longrightarrow \Re(\psi(z)) > \ell \tag{30} $$

(iii) *Let $\psi \in \mathbb{H}[\psi, n]$ with $\Re(\psi) > 0$. Then*

$$ \Re\left(\psi(z) + z\psi'(z) + z^2\psi''(z)\right) > 0 \tag{31} $$

or for $\aleph : \cup \longrightarrow R$ such that

$$ \Re\left(\psi(z) + \aleph(z)\frac{z\psi'(z)}{\psi(z)}\right) > 0. \tag{32} $$

Then $\Re(\psi(z)) > 0$.

3. Results

Our results are as follows.

Theorem 10. *Let $\psi \in \Lambda$. If one of the next inequalities is considered,*

(i) *${}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)$ is of bounded turning function*

(ii) *$({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z))' \prec (1 + z/1 - z)^{\kappa}$, $\kappa > 0$, $z \in \cup$*

(iii) *$\Re(({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)')({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)/z)) > c/2$, $c \in [0, 1)$, $z \in \cup$*

(iv) *$\Re((z{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z))'' - ({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z))' + 2({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)/z)) > 0$*

(v) *$\Re((z({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z))'/{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)) + 2({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)/z)) > 1$*

then ${}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)/z \in \mathscr{P}(\lambda)$ for some $\lambda \in [0, 1)$.

Proof. Define a function ρ as follows:

$$ \rho = \frac{{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}, \quad z \in \cup. \tag{33} $$

Then a computation implies that

$$ z\rho'(z) + \rho(z) = \left({}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'. \tag{34} $$

In virtue of the first inequality, we have that ${}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)$ is of bounded turning function, which leads to $\Re(z\rho'(z) + \rho(z)) > 0$. Therefore, Lemma 9(i) indicates that $\Re(\rho(z)) > 0$ which gives the first part of the theorem. Consequently, the second part is confirmed. In virtue of Lemma 9(i), we have a fixed real number $\ell > 0$ such that $\kappa = \kappa(\ell)$ and

$$ \frac{{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{\ell}. \tag{35} $$

This implies that

$$ \Re\left(\frac{{}_{k}^{C}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\right) > \lambda, \quad \lambda \in [0, 1). \tag{36} $$

Suppose that

$$\begin{aligned}\Re\left(\rho^2(z)+2\rho(z)\cdot z\rho'(z)\right)\\ =2\Re\left(\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\left(\left({}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'-\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{2z}\right)\right)>\varsigma.\end{aligned}\tag{37}$$

According to Lemma 9(ii), there exists a fixed real number $\ell>0$ satisfying $\Re(\rho(z))>\ell$ and

$$\rho(z)=\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\in\mathscr{P}(\lambda),\quad \lambda\in[0,1).\tag{38}$$

It follows from (37) that $\Re({}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z))')>0$; consequently, by Noshiro-Warschawski and Kaplan theorems, ${}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)$ is univalent and of bounded turning function in $\cup$. Taking the derivative (33), then we get

$$\begin{aligned}\Re\left(\rho(z)+z\rho'(z)+z^2\rho''(z)\right)\\ =\Re\left(z\left({}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)''-\left({}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'\right.\\ \left.+2\left(\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\right)\right)>0.\end{aligned}\tag{39}$$

Hence, Lemma 9(ii) implies $\Re({}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)/z)>0$.
The logarithmic differentiation of (33) yields

$$\begin{aligned}\Re\left(\rho(z)+\frac{z\rho'(z)}{\rho(z)}+z^2\rho''(z)\right)\\ =\Re\left(\frac{z\left({}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}+2\left(\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\right)-1\right)>0.\end{aligned}\tag{40}$$

Hence, Lemma 9(iii) implies, where $\aleph(z)=1$,

$$\Re\left(\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\right)>0.\tag{41}$$

□

The next results show the upper bound of the operator ${}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)$ utilizing the exponential integral in the open unit disk provided that the function $\psi\in{}_k^CS_{\alpha,\beta}^{*\gamma,\omega}(\sigma)$.

Theorem 11. *Suppose that $\psi\in{}_k^CS_{\alpha,\beta}^{*\gamma,\omega}(\sigma)$, where $\sigma(z)$ is convex in $\cup$. Then*

$${}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\prec z\,\exp\left(\int_0^z\frac{\sigma(\Psi(\omega))-1}{\omega}d\omega\right),\tag{42}$$

where $\Psi(z)$ is analytic in $\cup$, with $\Psi(0)=0$ and $|\Psi(z)|<1$. Also, for $|z|=\xi$, ${}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)$ satisfies the inequality

$$\begin{aligned}\exp\left(\int_0^1\frac{\sigma(\Psi(\xi))-1}{\xi}\right)d\xi\\ \le\left|\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\right|\le\exp\left(\int_0^1\frac{\sigma(\Psi(\xi))-1}{\xi}\right)d\xi.\end{aligned}\tag{43}$$

Proof. By the hypothesis, we receive the following conclusion:

$$\left(\frac{z\left({}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}\right)\prec\sigma(z),\quad z\in\cup.\tag{44}$$

This gives the occurrence of a Schwarz function with $\Psi(0)=0$ and $|\Psi(z)|<1$ such that

$$\left(\frac{z\left({}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}\right)=\sigma(\Psi(z)),\quad z\in\cup.\tag{45}$$

That is,

$$\left(\frac{\left({}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}\right)-\frac{1}{z}=\frac{\sigma(\Psi(z))-1}{z}.\tag{46}$$

Integrating the above equality, we get

$$\log\left({}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)-\log(z)=\int_0^z\left(\frac{\sigma(\Psi(\omega))-1}{\omega}\right)d\omega.\tag{47}$$

Consequently, we get

$$\log\left(\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\right)=\int_0^z\frac{\sigma(\Psi(\omega))-1}{\omega}d\omega.\tag{48}$$

By the definition of subordination, we arrive at the following inequality

$${}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\prec z\,\exp\left(\int_0^z\frac{\sigma(\Psi(\omega))-1}{\omega}d\omega\right).\tag{49}$$

Note that the function $\sigma(z)$ plots the disk $0<|z|<\xi<1$ onto a reign, which is convex and symmetric with respect to the real axis. That is,

$$\sigma(-\xi|z|)\le\Re(\sigma(\Psi(\xi z)))\le\sigma(\xi|z|),\quad \xi\in(0,1),\tag{50}$$

then we have the inequalities

$$\sigma(-\xi)\le\sigma(-\xi|z|),\sigma(\xi|z|)\le\sigma(\xi)\,;\tag{51}$$

consequently, we get

$$\int_0^1 \frac{\sigma(\Psi(-\xi|z|))-1}{\xi}d\xi \le \Re\left(\int_0^1 \frac{\sigma(\Psi(\xi))-1}{\xi}d\xi\right) \le \int_0^1 \frac{\sigma(\Psi(\xi|z|))-1}{\xi}d\xi. \tag{52}$$

In view of Equation (48), we obtain the general log-inequality

$$\int_0^1 \frac{\sigma(\Psi(-\xi|z|))-1}{\xi}d\xi \le \log\left|\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\right| \le \int_0^1 \frac{\sigma(\Psi(\xi|z|))-1}{\xi}d\xi\,; \tag{53}$$

that is,

$$\exp\left(\int_0^1 \frac{\sigma(\Psi(-\xi|z|))-1}{\xi}d\xi\right) \le \left|\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\right| \le \exp\left(\int_0^1 \frac{\sigma(\Psi(\xi|z|))-1}{\chi}d\xi\right). \tag{54}$$

Hence, we have

$$\exp\left(\int_0^1 \frac{\sigma(\Psi(-\xi))-1}{\xi}d\xi\right) \le \left|\frac{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}{z}\right| \le \exp\left(\int_0^1 \frac{\sigma(\Psi(\xi))-1}{\xi}d\xi\right). \tag{55}$$

□

Proceeding, we illustrate the sufficient condition of ψ to be in the class ${}_k^C S_{\alpha,\beta}^{*\gamma,\omega}\psi(\sigma)$, where σ is convex univalent satisfying $\sigma(0)=1$.

Theorem 12. *If $\psi \in \Lambda$ satisfies the inequality*

$$\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}\left(2+\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)''}{\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'} - \left(\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}\right)\right) \prec \sigma(z), \tag{56}$$

then $\psi \in {}_k^C S_{\alpha,\beta}^{*\gamma,\omega}(\sigma)$.

Proof. The proof directly comes from [6] (Theorem 3.1a). Taking

$$p(z) = \frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}, \tag{57}$$

and $P(z)=1$ in the inequality

$$p(z) + P(z)\cdot\left(zp'(z)\right) \prec \sigma(z), \tag{58}$$

then we obtain

$$\begin{aligned} p(z) + P(z)\cdot\left(zp'(z)\right) &= \frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)} \times \left(2+\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)''}{\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'} - \left(\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}\right)\right) \prec \sigma(z). \end{aligned} \tag{59}$$

This implies that

$$p(z) = \frac{z\left({}_k^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^{\mathscr{C}}\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)} \prec \sigma(z), \quad \sigma \in \mathscr{C}, \tag{60}$$

that is $\psi \in {}_k^C S_{\alpha,\beta}^{*\gamma,\omega}(\sigma)$. □

Corollary 13. *Let the assumption of Theorem 12 hold. Then*

$$\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)} \times \left(1+\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)''}{\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'} - \left(\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}\right)\right) \ll \sigma'(z). \tag{61}$$

Proof. Let

$$p(z) = \frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}. \tag{62}$$

In view of Theorem 12, we have

$$\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)} \prec \sigma(z), \tag{63}$$

where $\sigma \in C$. Then by [9] (Theorem 3), we get $p'(z) \ll \sigma'(z)$ for some $z \in \cup$, where

$$p'z = \left(\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}\right)\left(1+\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)''}{\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'} - \left(\frac{z\left({}^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)\right)'}{{}_k^C\Delta_{\alpha,\beta}^{\gamma,\omega}\psi(z)}\right)\right). \tag{64}$$

□

It is well known that the function $\sigma(z)=e^{\epsilon z}$, $1<|\epsilon|\leq\pi/2$ is not convex in $\cup$, where the domain $\sigma(\cup)$ is lima-bean (see [6] (P123)). One can obtain the same result of Theorem 12 as follows.

Theorem 14. *If $\psi\in\Lambda$ satisfies the inequality*

$$1+\frac{z\left({}^{C}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)\right)''}{{}^{C}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)'}\prec e^{\epsilon z},\quad 1<|\epsilon|\leq\frac{\pi}{2},\tag{65}$$

then $\psi\in{}^{C}_{k}S^{*\gamma,\omega}_{\alpha,\beta}(e^{\epsilon z})$.

Proof. Let

$$p(z):=\frac{z\left({}^{C}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)\right)'}{{}^{C}_{k}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)}.\tag{66}$$

Then a computation implies

$$\begin{aligned}
&p(z)+\frac{zp'(z)}{p(z)}\\
&=\left(\frac{z\left({}^{C}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)\right)'}{{}^{C}_{k}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)}\right)+\frac{\left(z\left({}^{C}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)\right)'/{}^{C}_{k}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)\right)\left(1+z\left({}^{C}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)\right)''/\left({}^{C}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)\right)'-\left(z\left({}^{C}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)\right)'/{}^{C}_{k}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)\right)\right)}{z\left({}^{C}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)\right)'/{}^{C}_{k}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)}\\
&=\left(1+\frac{z\left({}^{C}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)\right)''}{{}^{C}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)'}\right)\prec e^{\epsilon z}.
\end{aligned}\tag{67}$$

This implies that [6] (P123)

$$p(z)=\frac{z\left({}^{C}_{k}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)\right)'}{{}^{C}_{k}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)}\prec e^{\epsilon z};\tag{68}$$

that is, $\psi\in{}^{C}_{k}S^{*\gamma,\omega}_{\alpha,\beta}(e^{\epsilon z})$. □

Theorem 15. *If $\psi\in{}^{c}_{k}J^{\gamma,\omega}_{\alpha,\beta}(\mathfrak{A},\mathfrak{B},\flat)$ then the function*

$$\mathbb{B}(z)=\frac{1}{2}[\psi(z)-\psi(-z)],\quad z\in\cup,\tag{69}$$

satisfies

$$\begin{aligned}
&1+\frac{1}{\flat}\left(\frac{{}^{C}_{k+1}\Delta^{\gamma,\omega}_{\alpha,\beta}\mathbb{B}(z)}{{}^{C}_{k}\Delta^{\gamma,\omega}_{\alpha,\beta}\mathbb{B}(z)}-1\right)\prec\frac{1+\mathfrak{A}z}{1+\mathfrak{B}z},\\
&\mathfrak{R}\left(\frac{z\mathbb{B}(z)'}{\mathbb{B}(z)}\right)\geq\frac{1-\eth^2}{1+\eth^2},\quad |z|=\eth<1.
\end{aligned}\tag{70}$$

Proof. Let $\psi\in{}^{c}_{k}J^{\gamma,\omega}_{\alpha,\beta}(\mathfrak{A},\mathfrak{B},\flat)$. Then there occurs a function $J(z)$ such that

$$\begin{aligned}
\flat(J(z)-1)&=\left(\frac{2{}^{C}_{k+1}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)}{{}^{C}_{k}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)-{}^{C}_{k}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(-z)}\right),\\
\flat(J(-z)-1)&=\left(\frac{2{}^{C}_{k+1}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(-z)}{{}^{C}_{k}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(-z)-{}^{C}_{k}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)}\right).
\end{aligned}\tag{71}$$

This confirms that

$$1+\frac{1}{\flat}\left(\frac{{}^{C}_{k+1}\Delta^{\gamma,\omega}_{\alpha,\beta}\mathbb{B}(z)}{{}^{C}_{k}\Delta^{\gamma,\omega}_{\alpha,\beta}\mathbb{B}(z)}-1\right)=\frac{J(z)+J(-z)}{2}.\tag{72}$$

However, J satisfies

$$J(z)\prec\frac{1+\mathfrak{A}z}{1+\mathfrak{B}z},\tag{73}$$

which is univalent, then we get

$$1+\frac{1}{\flat}\left(\frac{{}^{C}_{k+1}\Delta^{\gamma,\omega}_{\alpha,\beta}\mathbb{B}(z)}{{}^{C}_{k}\Delta^{\gamma,\omega}_{\alpha,\beta}\mathbb{B}(z)}-1\right)\prec\frac{1+\mathfrak{A}z}{1+\mathfrak{B}z}.\tag{74}$$

Also, $\mathbb{B}(z)$ is starlike in $\cup$ which implies that

$$\hbar(z):=\frac{z\mathbb{B}(z)'}{\mathbb{B}(z)}\prec\frac{1-z^2}{1+z^2}.\tag{75}$$

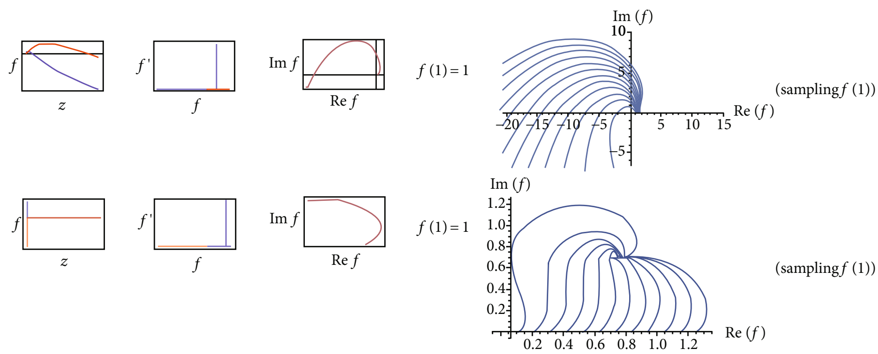

Figure 1: Plot of the solution for $zf(z/f(z))$ and $f(z)+zf(z/f(z))$, respectively.

Hence, a Schwarz function $\Bbbk \in \cup$, $|k(z)| \le |z| < 1$, $\Bbbk(0)=0$ gets

$$\hbar(z) \prec \frac{1-\Bbbk(z)^2}{1+\Bbbk(z)^2}, \tag{76}$$

which leads to

$$\Bbbk^2(\zeta) = \frac{1-\hbar(\zeta)}{1+\hbar(\zeta)}, \quad \zeta \in \cup, \zeta, |\zeta| = r < 1. \tag{77}$$

A calculation yields

$$\left|\frac{1-\hbar(\zeta)}{1+\hbar(\zeta)}\right| = |\Bbbk(\zeta)|^2 \le |\zeta|^2. \tag{78}$$

Therefore, we get the following inequality: or

$$\left|\hbar(\zeta) - \frac{1+|\zeta|^4}{1-|\zeta|^4}\right|^2 \le \frac{4|\zeta|^4}{\left(1-|\zeta|^4\right)^2} \tag{79}$$

$$\left|\hbar(\zeta) - \frac{1+|\zeta|^4}{1-|\zeta|^4}\right| \le \frac{2|\zeta|^2}{\left(1-|\zeta|^4\right)}. \tag{80}$$

Thus, we have

$$\Re(\hbar(z)) \ge \frac{1-\eth^2}{1+\eth^2}, \quad |\zeta| = \eth < 1. \tag{81}$$

This completes the assertion of the theorem. □

Example 16.

(i) Let

$$\frac{zf'(z)}{f(z)} := \frac{z\left({}^{C}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)\right)'}{{}^{C}_{k}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)},$$
$${}^{C}_{k}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z) = \frac{z}{(1-z)^2}, \quad \psi \in \Lambda. \tag{82}$$

Then the solution of $zf'(z)/f(z) = ((1+z)/(1-z))$ is formulated, as follows:

$$\left({}^{C}\Delta^{\gamma,\omega}_{\alpha,\beta}\psi(z)\right) = \frac{z}{(1-z)^2}, \quad \psi \in \Lambda. \tag{83}$$

Moreover, the solution of the equation

$$f(z) + \frac{zf'(z)}{f(z)} = \left(\frac{1+z}{1-z}\right) \tag{84}$$

is approximated to $f(z) = z/(1-z)$.

(ii) The solution of $zf'(z)/f(z) = ((1+z)/(1-z))^{0.25}$ is given in terms of the hypergeometric function, as follows (see Figure 1):

$$f(z) = c\exp\left(1.8(z+1)\left(\frac{z+1}{1-z}\right)^{0.25} \cdot \frac{F_1(1.25; 0.25, 1; 2.25; 0.5z+0.5, z+1)}{z(2.25F_1(1.25; 0.25, 1; 2.25; 0.5z+0.5, z+1) + (z+1)F_1(2.25; 0.25, 2; 3.25; 0.5z+0.5, z+1) + (0.125z+0.125)F_1(2.25; 1.25, 1; 3.25; 0.5z+0.5z+1))}\right). \tag{85}$$

4. Conclusion

The Prabhakar fractional differential operator in the complex plane is formulated for a class of normalized function in the open unit disk. We formulated the modified operator in two classes of analytic functions to investigate its geometric behavior. Differential inequalities are formulated to include them. Examples showed the behavior of solutions and the formula. The suggested operators can be utilized to formulate some classes of analytic functions or to generalize other types of differential operators such a conformable, quantum, or fractal operators.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally and significantly to writing this article. All authors read and agreed to the published version of the manuscript.

References

[1] H. M. Srivastava and S. Owa, Eds., *Univalent Functions, Fractional Calculus, and Their Applications*, Chichester: Ellis Horwood, New York; Toronto: Halsted Press, 1989.

[2] R. W. Ibrahim, "On generalized Srivastava-Owa fractional operators in the unit disk," *Advances in Difference Equations*, vol. 2011, no. 1, Article ID 55, 2011.

[3] H. M. Srivastava, M. Darus, and R. W. Ibrahim, "Classes of analytic functions with fractional powers defined by means of a certain linear operator," *Integral Transforms and Special Functions*, vol. 22, no. 1, pp. 17–28, 2011.

[4] R. W. Ibrahim, "Ulam stability for fractional differential equation in complex domain," *Abstract and Applied Analysis*, vol. 2012, Article ID 649517, 8 pages, 2012.

[5] R. W. Ibrahim, "Generalized Ulam-Hyers stability for fractional differential equations," *International Journal of mathematics*, vol. 23, no. 5, article 1250056, 2012.

[6] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, CRC Press, 2000.

[7] A. W. Goodman, "An invitation on the study of univalent and multivalent functions," *International Journal of Mathematics and Mathematical Sciences*, vol. 2, no. 2, pp. 163–186, 1979.

[8] T. H. MacGregor, "Majorization by univalent functions," *Duke Mathematical Journal*, vol. 34, no. 1, 1967.

[9] D. M. Campbell, "Majorization-subordination theorems for locally univalent functions, II," *Canadian Journal of Mathematics*, vol. 25, no. 2, pp. 420–425, 1973.

[10] H. M. Srivastava, "Some families of Mittag-Leffler type functions and associated operators of fractional calculus (survey)," *Turkic World Mathematical Society (TWMS) Journal of Pure and Applied Mathematics*, vol. 7, pp. 123–145, 2016.

[11] A. K. Shukla and J. C. Prajapati, "On a generalization of Mittag-Leffler function and its properties," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 2, pp. 797–811, 2007.

[12] H. J. Haubold, A. M. Mathai, and R. K. Saxena, "Mittag-Leffler functions and their applications," *Journal of Applied Mathematics*, vol. 2011, Article ID 298628, 51 pages, 2011.

[13] T. R. Prabhakar, "A singular integral equation with a generalized Mittag-Leffler function in the kernel," *Yokohama Mathematical Journal*, vol. 19, pp. 7–15, 1971.

[14] R. Garra, R. Gorenflo, F. Polito, and Z. Tomovski, "Hilfer-Prabhakar derivatives and some applications," *Applied Mathematics and Computation*, vol. 242, pp. 576–589, 2014.

[15] A. A. Kilbas, M. Saigo, and R. K. Saxena, "Generalized Mittag-Leffler function and generalized fractional calculus operators," *Integral Transforms and Special Functions*, vol. 15, no. 1, pp. 31–49, 2004.