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Research Article

Study on Certain Subclass of Analytic Functions Involving Mittag-Leffler Function

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We propose and explore a new subclass of regular functions described by a new derivative operator in this paper. Some coefficient estimations, growth and distortion aspects, extreme points, star-like radii, convexity, Fekete-Szego inequality, and partial sums are derived.

1. Introduction

Let $\mathscr{A}$ represent the regular function class u defined on the disk $U = \{w : |w| < 1\}$ normalized by u (i.e., $u(0) = 0$ and $u'(0) = 1$). The origin of the form is about the Taylor series expansion of such an equation

$$u(w) = w + \sum_{\eta=2}^{\infty} a_\eta w^\eta. \quad (1)$$

S indicates a subclass of $\mathscr{A}$ consists entirely of mappings that are the same as U.

For $u \in \mathscr{A}$ presented by (2) and $g(w)$ specified by

$$g(w) = w + \sum_{\eta=2}^{\infty} b_\eta w^\eta, \quad (2)$$

their convolution, represented by $(u * g)$, is specified as

$$(u * g)(w) = w + \sum_{n=2}^{\infty} a_n b_n w^n = (g * u)(w)(w \in U). \quad (3)$$

The $\mathscr{A}$ subclass consisting of the u-type function is specified by T as

$$u(w) = w - \sum_{\eta=2}^{\infty} a_\eta w^\eta. \quad (4)$$

Silverman [1] extensively examined this subclass.

The study of operators plays an important role in geometric function theory in complex analysis and its related fields. Many derivative and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to better understand the geometric properties of such operators. The Mittag-Leffler function [2, 3] is defined by the following power series, convergent in the whole complex plane:

$$E_{\upsilon}(w) := \sum_{\eta=0}^{\infty} \frac{w^{\eta}}{\Gamma(\upsilon\eta+1)}, \quad \upsilon > 0, w \in \mathbb{C}. \tag{5}$$

We recognize that it is an entire function of order $1/\upsilon$ providing a simple generalization of the exponential function $\exp(w)$ to which it reduces for $\upsilon = 1$. For detailed information on the Mittag-Leffler-type functions and their laplace transforms, the reader may consult, e.g., [4–6] and the recent treatise by Gorenflo et al. [7].

We also note that for the convergence of the power series in (5), the parameter υ may be complex provided that $\Re(\upsilon) > 0$. The most interesting properties of the Mittag-Leffler function are associated with its asymptotic expansions as $w \longrightarrow \infty$ in various sectors of the complex plane. A more general function $E_{\upsilon,\tau}$ generalizing $E_{\upsilon}(w)$ was introduced by Wiman [8] and defined by

$$E_{\upsilon,\tau}(w) = \sum_{\eta=0}^{\infty} \frac{w^{\eta}}{\Gamma(\upsilon\eta+\tau)} (w, \upsilon, \tau \in \mathbb{C}, \Re(\upsilon) > 0, \Re(\tau) > 0). \tag{6}$$

Observe that the function $E_{\upsilon,\tau}$ contains many well-known functions as its special case, for example, $E_{1,1}(w) = e^{w}, E_{1,2}(w) = (e^{w} - 1)/w, E_{2,1}(w^2) = \cosh w, E_{2,1}(-w^2) = \cos w, E_{2,2}(w^2) = \sinh w/w, E_{2,2}(-w^2) = \sinh w/w, E_4 = 1/2[\cos w^{1/4} + \cosh w^{1/4}]$, and $E_3 = 1/2[e^{w^{1/3}} + 2e^{-(1/3)} \cos((\sqrt{3}/2)w^{1/3})]$.

The Mittag-Leffler function arises naturally in the solution of fractional-order differential and integral equations and especially in the investigations of fractional generalization of kinetic equation, random walks, Levy flights, and super diffusive transport and in the study of complex systems. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found, e.g., in [9–16]. Observe that Mittag-Leffler function $E_{\upsilon,\tau}(w)$ does not belong to the family A. Thus, it is natural to consider the following normalization of Mittag-Leffler functions as below:

$$E_{\upsilon,\tau} = w\Gamma\tau E_{\upsilon,\tau}(w) = w + \sum_{\eta=2}^{\infty} \frac{\Gamma(\tau)}{\Gamma(\upsilon(\eta-1)+\tau)} w^{\eta}. \tag{7}$$

It holds for complex parameters υ, τ and $w \in \mathbb{C}$.
The function $Q_{\upsilon,\tau}(w)$ is specified by

$$Q_{\upsilon,\tau}(w) = w\Gamma(\tau)E_{\upsilon,\tau}(w). \tag{8}$$

Now, for $u \in \mathscr{A}$, the derivative operator that follows is defined by $\mathscr{D}_{\hbar}^{m}(\upsilon,\tau)u : \mathscr{A} \longrightarrow \mathscr{A}$ by

$$\begin{aligned}
\mathscr{D}_{\hbar}^{0}(\upsilon,\tau)u(w) &= u(w) * Q_{\upsilon,\tau}(w),\\
\mathscr{D}_{\hbar}^{1}(\upsilon,\tau)u(w) &= (1-\hbar)(u(w) * Q_{\upsilon,\tau}(w)) + \hbar w(u(w) * Q_{\upsilon,\tau}(w))',\\
&\vdots\\
\mathscr{D}_{\hbar}^{m}(\upsilon,\tau)u(w) &= \mathscr{D}_{\hbar}^{1}\left(\mathscr{D}_{\hbar}^{m-1}(\upsilon,\tau)u(w)\right).
\end{aligned} \tag{9}$$

If u is specified by (1), then from the operator's definition $\mathscr{D}_{\hbar}^{m}u$, it is clear to see that

$$\mathscr{D}_{\hbar}^{m}(\upsilon,\tau)u(w) = w + \sum_{\eta=2}^{\infty} \phi_{\eta}^{m}(\hbar,\upsilon,\tau)a_{\eta}w^{\eta}, \tag{10}$$

where

$$\phi_{\eta}^{m}(\hbar,\upsilon,\tau) = \frac{\Gamma(\tau)}{\Gamma(\upsilon(\eta-1)+\tau)}[\hbar(\eta-1)+1]^{m}. \tag{11}$$

Keep in mind that

(1) the Al-Oboudi operator [17] is achieved when $\upsilon = 0$ and $\tau = 1$

(2) we get the Salagean operator [18] when $\upsilon = 0, \tau = 1$, and $\hbar = 1$

(3) when $m = 0$, we get $E_{\upsilon,\tau}(w)$, according to Srivastava et al. [19]

If $u \in T$ is represented by (4), then we have got it.

$$\mathscr{D}_{\hbar}^{m}(\upsilon,\tau)u(w) = w - \sum_{\eta=2}^{\infty} \phi_{\eta}^{m}(\hbar,\upsilon,\tau)a_{\eta}w^{\eta}. \tag{12}$$

Now, by utilizing the differential operator, $\mathscr{D}_{\hbar}^{m}(\upsilon,\tau)u$, a new subclass of functions belonging to the class $\mathscr{A}$ is specified.

Definition 1. For $0 \le \nu \le 1, \ell \ge 1, k \ge 0$, and $0 \le \wp < 1$, a mapping in a class is referred to as $u \in \mathscr{A}S_{\hbar,\upsilon,\tau}^{m}(\nu,\ell,k,\wp)$, if it satisfies the case

$$\Re\left\{\ell\frac{w\aleph'(w)}{\aleph(w)} - (\ell-1)\right\} > k\left|\ell\frac{w\aleph'(w)}{\aleph(w)} - \ell\right| + \wp, \tag{13}$$

where

$$\aleph(w) = (1-\nu)\mathscr{D}_{\hbar}^{m}u(w) + \nu w(\mathscr{D}_{\hbar}^{m}u(w))'. \tag{14}$$

We also define $TS_{\hbar,\upsilon,\tau}^{m}(\nu,\ell,k,\wp) = S_{\hbar,\upsilon,\tau}^{m}(\nu,\ell,k,\wp) \cap T$.

For special situations of characteristics, $S_{\hbar,\upsilon,\tau}^{m}(\nu,\ell,k,\wp)$ and $TS_{\hbar,\upsilon,\tau}^{m}(\nu,\ell,k,\wp)$, it can be reduced to new or known categories of functions studied in recent research [20–25].

The objective of this review is to look into a variety of properties for functions in the aforesaid class. For specific parameter instances.

2. Coefficient Estimates

To get our results, we will require the subsequent lemma.

Lemma 2 (see [26]). *Let $\wp$ be a real and z be a complex number. Then, $\Re(z) \geq \wp$.*

$$\Longleftrightarrow |z + (1-\wp)| - |z - (1+\wp)| \geq 0. \tag{15}$$

For beginnings, we have a coefficient that is relevant for functions in the classs $S^m_{\hbar,\upsilon,\tau}(\nu, \ell, k,\wp)$.

Theorem 3. *Let $u \in \mathscr{A}$ indicated by (1). If*

$$\sum_{\eta=2}^{\infty} [1-\wp+\ell(\eta-1)(1+k)]\chi_\eta(\hbar, \nu, m, \upsilon, \tau)|a_\eta| \leq 1-\wp, \tag{16}$$

where

$$\chi_\eta(\hbar, \nu, m, \upsilon, \tau) = [1 + \nu(m-1)]\phi^m_\eta(\hbar, \upsilon, \tau), \tag{17}$$

then $u \in S^m_{\hbar,\upsilon,\tau}(\nu, \ell, k,\wp)$.

Proof. In the definition by consequence of 1 and Lemma 2, it is enough to demonstrate that

$$\begin{aligned} &\left| \ell \frac{w\aleph'(w)}{\aleph(w)} - (\ell - 1) - k\left| \ell \frac{w\aleph'(w)}{\aleph(w)} - \ell \right| - (1+\wp) \right| \\ &\quad \leq \left| \ell \frac{w\aleph'(w)}{\aleph(w)} - (\ell - 1) - k\left| \ell \frac{w\aleph'(w)}{\aleph(w)} - \ell \right| + (1+\wp) \right|. \end{aligned} \tag{18}$$

For the R.H.S and L.H.S of (18), we may, respectively, write

$$\begin{aligned} R &= \left| \ell \frac{w\aleph'(w)}{\aleph(w)} - (\ell - 1) - k\left| \ell \frac{w\aleph'(w)}{\aleph(w)} - \ell \right| + (1-\wp) \right| \\ &= \frac{1}{|\aleph(w)|} \left| \ell w\aleph'(w) - (\ell-1)\aleph(w) - ke^{i\theta}\left| \ell w\aleph'(w) - \ell\aleph(w) \right| + (1-\wp)\aleph(w) \right| \\ &> \frac{|w|}{|\aleph(w)|}\left[2-\wp- \sum_{\eta=2}^{\infty} [2-\wp+\ell(\eta-1)(k+1)] \right] \chi_\eta(\hbar, \nu, m, \upsilon, \tau)|a_\eta|, \end{aligned} \tag{19}$$

and similarly,

$$\begin{aligned} L &= \left| \ell \frac{w\aleph'(w)}{\aleph(w)} - (\ell - 1) - k\left| \ell \frac{w\aleph'(w)}{\aleph(w)} - \ell \right| - (1+\wp) \right| \\ &= \frac{1}{|\aleph(w)|} \left| \ell w\aleph'(w) - (\ell-1)\aleph(w) - ke^{i\theta}\left| \ell w\aleph'(w) - \ell\aleph(w) \right| - (1+\wp)\aleph(w) \right| \\ &< \frac{|w|}{|\aleph(w)|}\left[\wp + \sum_{\eta=2}^{\infty} [\ell(\eta-1)(1+k)-\wp] \right] \chi_\eta(\hbar, \nu, m, \upsilon, \tau)|a_\eta|. \end{aligned} \tag{20}$$

Then,

$$R - L > \frac{|w|}{|\aleph(w)|}\left[2(1-\wp) - 2\sum_{\eta=2}^{\infty} [1-\wp+\ell(\eta-1)(1+k)]\chi_\eta(\hbar, \nu, m, \upsilon, \tau)|a_\eta| \right] \geq 0. \tag{21}$$

The condition (16) required is fulfilled. □

We have a necessary and adequate situation in the next theorem for a function $u \in T$ to be in the class $TS^m_{\hbar,\upsilon,\tau}(\nu, \ell, k,\wp)$.

Theorem 4. *Let $u \in T$ indicated by (3). Then, $u \in TS^m_{\hbar,\upsilon,\tau}(\nu, \ell, k,\wp)$.*

$$\Longleftrightarrow \sum_{\eta=2}^{\infty} [1-\wp+\ell(\eta-1)(1+k)]\chi_\eta(\hbar, \nu, m, \upsilon, \tau)a_\eta \leq 1-\wp, \tag{22}$$

where $\chi_\eta(\hbar, \nu, m, \upsilon, \tau)$ is defined by (17).

Proof. We can only prove the requirement in view 3 of the theorem. If $u \in TS^m_{\hbar,\upsilon,\tau}(\nu, \ell, k,\wp)$ and w is real, then

$$\begin{aligned} &\frac{1 - \sum_{\eta=2}^{\infty} [1 + \ell(\eta-1)]\chi_\eta(\hbar, \nu, m, \upsilon, \tau)a_\eta w^{\eta-1}}{1 - \sum_{\eta=2}^{\infty} \chi_\eta(\hbar, \nu, m, \upsilon, \tau)a_\eta w^{\eta-1}} - \wp \\ &\quad > k\left| \frac{\sum_{\eta=2}^{\infty} \ell(\eta-1)\chi_\eta(\hbar, \nu, m, \upsilon, \tau)a_\eta w^{\eta-1}}{1 - \sum_{\eta=2}^{\infty} \chi_\eta(\hbar, \nu, m, \upsilon, \tau)a_\eta w^{\eta-1}} \right|. \end{aligned} \tag{23}$$

We get the desired inequality from letting $w \longrightarrow 1^-$. □

Corollary 5. *If $u \in TS^m_{\hbar,\upsilon,\tau}(\nu, \ell, k,\wp)$, then*

$$a_\eta \leq \frac{1-\wp}{[1-\wp+\ell(\eta-1)(1+k)]\chi_\eta(\hbar, \nu, m, \upsilon, \tau)} (\eta \geq 2). \tag{24}$$

3. Growth and Distortion Theorem

Theorem 6. *Let $u \in TS^m_{\hbar,\upsilon,\tau}(\nu, \ell, k,\wp)$. Then, for $|w| = r < 1$,*

$$r - \frac{(1-\wp)}{B_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)} r^2 \leq |u(w)| \leq r + \frac{(1-\wp)}{B_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)} r^2, \tag{25}$$

$$1 - \frac{2(1-\wp)}{B_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)} r^2 \leq |u'(w)| \leq 1 + \frac{2(1-\wp)}{B_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)} r, \tag{26}$$

where

$$B_\eta(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp) = [1-\wp+\ell(\eta-1)(1+k)]\chi_\eta(\hbar, \nu, m, \upsilon, \tau)(\eta \geq 2). \tag{27}$$

Equations (25) and (26) are sharp for the u given function

$$u(w) = w - \frac{(1-\wp)}{B_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)} w^2. \quad (28)$$

Proof. Since $u \in TS^m_{\hbar,\upsilon,\tau}(\nu, \ell, k,\wp)$ and it follows from 4 of the theorem,

$$\sum_{\eta=2}^{\infty} B_\eta(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp) a_\eta \le (1-\wp), \quad (29)$$

where $B_\eta(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)$ is given by (27), we have

$$\begin{aligned} B_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp) \sum_{\eta=2}^{\infty} a_\eta &= \sum_{\eta=2}^{\infty} B_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp) a_\eta \\ &\le \sum_{\eta=2}^{\infty} B_\eta(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp) a_\eta \\ &\le 1-\wp, \end{aligned} \quad (30)$$

and therefore,

$$\sum_{\eta=2}^{\infty} a_\eta \le \frac{(1-\wp)}{B_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)}. \quad (31)$$

Since u is given by (3), we get

$$\begin{aligned} |u(w)| &\le |w| + |w|^2 \sum_{\eta=2}^{\infty} a_\eta |w|^{\eta-2} \le r + r^2 \sum_{\eta=2}^{\infty} a_\eta \\ &\le r + \frac{(1-\wp)}{B_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)} r^2, \\ |u(w)| &\ge |w| - |w|^2 \sum_{\eta=2}^{\infty} a_\eta |w|^{\eta-2} \ge r - r^2 \sum_{\eta=2}^{\infty} a_\eta \\ &\ge r - \frac{(1-\wp)}{B_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)} r^2. \end{aligned} \quad (32)$$

In light of Theorem 4, we have

$$\begin{aligned} \frac{B_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)}{2} \sum_{\eta=2}^{\infty} \eta a_\eta &= \sum_{\eta=2}^{\infty} \frac{B_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)}{2} \eta a_\eta \\ &\le \sum_{\eta=2}^{\infty} B_\eta(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp) a_\eta \le (1-\wp), \end{aligned} \quad (33)$$

which yields

$$\sum_{\eta=2}^{\infty} \eta a_\eta \le \frac{2(1-\wp)}{B_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)}. \quad (34)$$

Thus,

$$\begin{aligned} |u'(w)| &\le 1 + \sum_{\eta=2}^{\infty} \eta a_\eta |w|^{\eta-1} \le 1 + r \sum_{\eta=2}^{\infty} \eta a_\eta \\ &\le 1 + \frac{2(1-\wp)}{B_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)} r, \\ |u'(w)| &\ge 1 - \sum_{\eta=2}^{\infty} \eta a_\eta |w|^{\eta-1} \ge 1 - r \sum_{\eta=2}^{\infty} \eta a_\eta \\ &\ge 1 - \frac{2(1-\wp)}{B_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)} r. \end{aligned} \quad (35)$$

Hence, the proof is complete. □

Consider that $|u(w)| = |w - ((1-\wp)/(B_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)) B_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp))w^2|$ is sharp is to (25).

And $|u'(w)| = |1 - ((2(1-\wp))2(1-\wp)/(B_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp))B_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp))w|$ is sharp is to (26).

4. Extreme Points

Now, for the function class, we look at the extreme points $TS^m_{\hbar,\upsilon,\tau}(\nu, \ell, k,\wp)$.

Theorem 7. *Let the functions* $u_1(w) = w$ *and*

$$u_\eta(w) = w - \frac{(1-\wp)}{B_\eta(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)} w^\eta (\eta \ge 2). \quad (36)$$

Then, $u \in TS^m_{\hbar,\upsilon,\tau}(\nu, \ell, k,\wp)$.

$$\Longleftrightarrow u(w) = \sum_{\eta=2}^{\infty} \hbar_\eta u_\eta(w) (w \in U), \quad (37)$$

where $\hbar_\eta \ge 0 (\eta \ge 1)$ *and* $\sum_{\eta=1}^{\infty} \hbar_\eta = 1$.

Proof. Assume that it is possible to write u as in (37). Then,

$$\begin{aligned} u(w) &= \hbar_1 w + \sum_{\eta=2}^{\infty} \hbar_\eta \left[w - \frac{(1-\wp)}{B_\eta(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)} w^\eta \right] \\ &= w - \sum_{\eta=2}^{\infty} \hbar_\eta \frac{(1-\wp)}{B_\eta(\hbar, \nu, m, \ell, k,\wp)} w^\eta, \end{aligned} \quad (38)$$

since

$$\begin{aligned} &\sum_{\eta=2}^{\infty} B_\eta(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp) \hbar_\eta \frac{(1-\wp)}{B_\eta(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)} \\ &= (1-\wp) \sum_{\eta=2}^{\infty} \hbar_\eta = (1-\wp)(1 - \hbar_1) \le (1-\wp). \end{aligned} \quad (39)$$

By virtue 4 of the theorem, it follows that $u \in TS^m_{\hbar,\upsilon,\tau}(\nu, \ell, k,\wp)$.

Conversely, suppose $u \in TS^{m}_{\hbar,\upsilon,\tau}(\nu, \ell, k,\wp)$ and consider

$$\hbar_\eta = \frac{B_\eta(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)}{(1-\wp)} a_\eta, \quad \eta \geq 2, \qquad \hbar_1 = 1 - \sum_{\eta=2}^{\infty} \hbar_\eta. \tag{40}$$

Then,$u(w) = \sum_{\eta=1}^{\infty} \hbar_\eta u_n(w)$, hence the theorem. □

5. Radii of Starlikeness, Convexity, and Close-to-Convexity

Theorem 8. *Let $u \in TS^{m}_{\hbar,\upsilon,\tau}(\nu, \ell, k,\wp)$. Then, u is star-shaped of order $\rho(0 \leq \rho < 1)$ in $|w| < r_1(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)$, where*

$$r_1(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp) = \inf_{\eta \geq 2} \left[\frac{(1-\rho)B_\eta(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)}{(\eta-\rho)(1-\wp)}\right]^{1/(\eta-1)}. \tag{41}$$

Proof. To be able to prove the theorem, we have to demonstrate that

$$\left|\frac{wu'(w)}{u(w)} - 1\right| \leq 1 - \rho, \tag{42}$$

$0 \leq \rho < 1$ for $w \in U$ with $|w| < r_1(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)$. We have

$$\left|\frac{wu'(w)}{u(w)} - 1\right| = \left|\frac{-\sum_{\eta=2}^{\infty} (\eta-1)a_\eta w^{\eta-1}}{1 - \sum_{\eta=2}^{\infty} a_\eta w^{\eta-1}}\right| \leq \frac{\sum_{\eta=2}^{\infty} (\eta-1)a_\eta |w|^{\eta-1}}{1 - \sum_{\eta=2}^{\infty} a_\eta |w|^{\eta-1}}. \tag{43}$$

Thus,

$$\left|\frac{wu'(w)}{u(w)} - 1\right| \leq 1 - \rho \quad \text{if} \sum_{\eta=2}^{\infty} \frac{(\eta-\rho)}{(1-\rho)} a_\eta |w|^{\eta-1} \leq 1. \tag{44}$$

In virtue of (22), we have

$$\frac{\sum_{\eta=2}^{\infty} B_\eta(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)}{1-\wp} a_\eta \leq 1. \tag{45}$$

The inequality of (43) would then be valid if

$$\frac{(\eta-\rho)}{(1-\rho)} |w|^{\eta-1} \leq \frac{B_\eta(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)}{1-\wp} (\eta \geq 2), \tag{46}$$

or if

$$|w| \leq \left[\frac{(1-\rho)B_\eta(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)}{(\eta-\rho)(1-\wp)}\right]^{1/(\eta-1)} (\eta \geq 2). \tag{47}$$

Hence, the proof is complete. □

The evidence 9 and 10 of the subsequent theorems is comparable to 8 of the theorem, so the evidence is excluded.

Theorem 9. *Let $u \in TS^{m}_{\hbar,\upsilon,\tau}(\nu, \ell, k,\wp)$. Then, u is convex of order $\rho(0 \leq \rho < 1)$ in $|w| < r_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)$, where*

$$r_2(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp) = \inf_{\eta \geq 2} \left[\frac{(1-\rho)B_\eta(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)}{\eta(\eta-\rho)(1-\wp)}\right]^{1/(\eta-1)}. \tag{48}$$

Theorem 10. *Let the function u given by (3) be in the class $TS^{m}_{\hbar,\upsilon,\tau}(\nu, \ell, k,\wp)$. Then, u in close-to-convex of order $\rho(0 \leq \rho < 1)$ in $|w| < r_3(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)$, where*

$$r_3(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp) = \inf_{\eta \geq 2} \left[\frac{(1-\rho)B_\eta(\hbar, \nu, m, \upsilon, \tau, \ell, k,\wp)}{\eta(1-\wp)}\right]^{1/(\eta-1)}. \tag{49}$$

6. Fekete-Szego Inequality

In this section, for the mapping in the class, we get the Fekete-Szego inequality $S^{m}_{\hbar,\upsilon,\tau}(\nu, \ell, k,\wp)$. To illustrate our fundamental result, we will identify the appropriate lemma.

Lemma 11 (see [27]). *If $p(w) = 1 + c_1 w + c_2 w + c_3 w^2 + \cdots$ is an analytic mapping with positive real part in U, then*

$$\left|c_2 - \jmath c_1^2\right| = \begin{cases} -4\jmath + 2, & \jmath \leq 0, \\ 2, & 0 \leq \jmath \leq 1, \\ 4\jmath - 2, & \jmath \geq 1. \end{cases} \tag{50}$$

When $\jmath < 0$ or $\jmath > 1$, the inequality holds iff $p(w) = (1 + w)/(1 - w)$ or one of its rotations. If $0 < \jmath < 1$, then the equality holds iff

$$p(w) = \frac{1 + w^2}{1 - w^2} \tag{51}$$

or one of its rotations. If $\jmath = 0$, the equality holds iff

$$p(w) = \left(\frac{1+\delta}{2}\right)\frac{1+w}{1-w} + \left(\frac{1-\delta}{2}\right)\frac{1-w}{1+w} (0 \leq \delta \leq 1) \tag{52}$$

or one of its rotations.

If $\jmath = 1$, the equality holds iff $p(w)$ is the reciprocal of one of the mapping such that the equality holds when it comes to $\jmath = 0$.

Theorem 12. *Let $\ell \geq 1, 0 \leq k \leq \wp < 1$. If $u \in S^m_{\hbar,\upsilon,\tau}(\nu, \ell, k, \wp)$ is given by (1), then*

$$\left|a_3 - \mu a_2^2\right| = \begin{cases} \dfrac{(1-\wp)}{\ell^2(1-k)^2\chi_3(\hbar,\nu,m,\upsilon,\tau)}\left[\ell(1-k) + 2(1-\wp) - 4\mu(1-\wp)\dfrac{\chi_3(\hbar,\nu,m,\upsilon,\tau)}{\chi_2^2(\hbar,\nu,m,\upsilon,\tau)}\right], & \mu \leq \sigma_1, \\ \dfrac{(1-\wp)}{\ell(1-k)\chi_3(\hbar,\nu,m,\upsilon,\tau)}, & \sigma_1 \leq \mu \leq \sigma_2, \\ \dfrac{-(1-\wp)}{\ell^2(1-k)^2\chi_3(\hbar,\nu,m,\upsilon,\tau)}\left[\ell(1-k) + 2(1-\wp) - 4\mu(1-\wp)\dfrac{\chi_3(\hbar,\nu,m,\upsilon,\tau)}{\chi_2^2(\hbar,\nu,m,\upsilon,\tau)}\right], & \mu \geq \sigma_2, \end{cases} \quad (53)$$

where

$$\sigma_1 = \frac{\chi_2^2(\hbar,\nu,m,\upsilon,\tau)}{2\chi_3(\hbar,\nu,m,\upsilon,\tau)}, \quad \sigma_2 = \frac{\chi_2^2(\hbar,\nu,m,\upsilon,\tau)[1-\wp+\ell(1-k)]}{2\chi_3(\hbar,\nu,m,\upsilon,\tau)(1-\wp)}. \quad (54)$$

The outcome is sharp.

Proof. Since, for complex numbers, $\Re(z) \leq |z|$, $u \in S^m_{\hbar,\upsilon,\tau}(\nu, \ell, k, \wp)$ implies that

$$\Re\left[\ell\frac{w\aleph'(w)}{\aleph(w)} - (\ell-1)\right] > k\Re\left[\ell\frac{w\aleph'(w)}{\aleph(w)} - \ell\right] + \wp \quad (55)$$

or that

$$\Re\left[\frac{w\aleph'(w)}{\aleph(w)}\right] > \frac{\wp-1+\ell(1-k)}{\ell(1-k)}. \quad (56)$$

Hence,

$$\aleph \in S^*\left(\frac{\wp-1+\ell(1-k)}{\ell(1-k)}\right). \quad (57)$$

Let

$$p(w) = \frac{w\aleph'(w)/\aleph(w) - ((\wp-1+\ell(1-k)/\ell(1-k))}{(1-\wp)/(\ell(1-k))} = 1 + c_1 w + c_2 w^2 + \cdots. \quad (58)$$

We then have, by way of (10) and (14),

$$a_2 = \frac{(1-\wp)}{\ell(1-k)\chi_2(\hbar,\nu,m,\upsilon,\tau)}c_1, \quad a_3 = \frac{(1-\wp)}{2\ell(1-k)\chi_2(\hbar,\nu,m,\upsilon,\tau)}\left[c_2 + \frac{1-\wp}{\ell(1-k)}c_1^2\right]. \quad (59)$$

Therefore, we obtain

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{(1-\wp)}{2\ell(1-k)\chi_3(\hbar,\nu,m,\upsilon,\tau)}\left[c_2 + \frac{1-\wp}{\ell(1-k)}c_1^2\right] - \mu\frac{(1-\wp)^2}{\ell^2(1-k)^2\chi_2^2(\hbar,\nu,m,\upsilon,\tau)}c_1^2 \\ &= \frac{(1-\wp)}{2\ell(1-k)\chi_3(\hbar,\nu,m,\upsilon,\tau)} \cdot \left[c_2 - \frac{1-\wp}{\ell(1-k)}c_1^2\left(2\mu\frac{\chi_3(\hbar,\nu,m,\upsilon,\tau)}{A_1^2(\hbar,\nu,m,\upsilon,\tau)} - 1\right)\right]. \end{aligned} \quad (60)$$

We write

$$a_3 - \mu a_2^2 = \frac{(1-\wp)}{2\ell(1-k)\chi_3(\hbar,\nu,m,\upsilon,\tau)}\left(c_2 - \rho c_1^2\right), \quad (61)$$

where

$$\rho = \frac{(1-\wp)}{\ell(1-k)}\left[2\mu\frac{\chi_3(\hbar,\nu,m,\upsilon,\tau)}{\chi_2^2(\hbar,\nu,m,\upsilon,\tau)} - 1\right]. \quad (62)$$

The implementation of the lemma above follows our conclusion. Denote

$$\xi = \frac{\wp-1+\ell(1-k)}{\ell(1-k)}. \quad (63)$$

If $\mu < \sigma_1$ or $\mu > \sigma_2$, it is true that equality exists.

$$\Longleftrightarrow \aleph(w) = \frac{w}{\left(1 - e^{i\theta}w\right)^{2(1-\xi)}} (\theta \in \mathbb{R}). \quad (64)$$

When $\sigma_1 < \mu < \sigma_2$, it is true that equality exists, iff

$$\aleph(w) = \frac{w}{\left(1 - e^{i\theta}w^2\right)^{(1-\xi)}} (\theta \in \mathbb{R}). \quad (65)$$

If $\mu = \sigma_1$, then it is true that equality exists, iff

$$\aleph(w) = \left[\frac{w}{\left(1 - e^{i\theta}w\right)^{2(1-\xi)}}\right]^{(1+\delta)/2} \left[\frac{w}{\left(1 + e^{i\theta}w\right)^{2(1-\xi)}}\right]^{(1-\delta)/2} = \frac{w}{\left[\left(1 - e^{i\theta}w\right)^{1+\delta}\left(1 + e^{i\theta}w\right)^{1-\delta}\right]^{1-\xi}}, \quad 0 \le \delta \le 1, \theta \in \mathbb{R}. \tag{66}$$

Finally, if it is true that equality exists $\Longleftrightarrow p(w)\,\mu = \sigma 2$, it is the inverse of one of the equality functions and holds true in the case of $\mu = \sigma 2$ □

7. Partial Sums

Consider the recent works on partial analytic function sums by Silverman [28] and Silvia [29]. Partial function in this class is considered in this section to be $TS^{m}_{\hbar,\upsilon,\tau}(\nu, \ell, k, \wp)$ giving sharp lower boundaries to the reap part ratios of $u(w)$ to $uq(w)$ and $u'(w)$ to $u'q(w)$.

Theorem 13. *Let $u \in TS^{m}_{\hbar,\upsilon,\tau}(\nu, \ell, k, \wp)$ and indicate $u_1(w)$ and $u_q(w)$ as partial sums*

$$u_1(w) = w, \quad u_q(w) = w + \sum_{\eta=2}^{q} a_\eta w^\eta \ (q \in \mathbb{N} \setminus \{1\}). \tag{67}$$

Suppose that

$$\sum_{\eta=2}^{\infty} d_\eta |a_\eta| \le 1, \tag{68}$$

where

$$d_\eta = \frac{[1-\wp+\ell(\eta-1)(1+k)]A_\eta(\hbar, \nu, m, \upsilon, \tau)}{1-\wp}. \tag{69}$$

Then, $u \in TS^{m}_{\hbar,\upsilon,\tau}(\nu, \ell, k, \wp)$.
Furthermore,

$$\Re\left[\frac{u(w)}{u_q(w)}\right] > 1 - \frac{1}{d_{q+1}} (w \in U, q \in \mathbb{N}), \tag{70}$$

$$\Re\left[\frac{u_q(w)}{u(w)}\right] > \frac{d_{q+1}}{1 + d_{q+1}}. \tag{71}$$

Proof. It is not crucial to verify that the d_η coefficients supplied by (69) are correct.

$$d_{\eta+1} > d_\eta > 1. \tag{72}$$

So we have

$$\sum_{\eta=2}^{q} |a_\eta| + d_{q+1} \sum_{\eta=q+1}^{\infty} |a_\eta| \le \sum_{\eta=2}^{\infty} d_\eta |a_\eta| \le 1. \tag{73}$$

The hypothesis used (69), by setting

$$g_1(w) = d_{q+1}\left[\frac{u(w)}{u_q(w)} - \left(1 - \frac{1}{d_{q+1}}\right)\right] = 1 + \frac{d_{q+1}\sum_{\eta=q+1}^{\infty} a_\eta w^{\eta-1}}{1 + \sum_{\eta=2}^{q} a_\eta w^{\eta-1}}. \tag{74}$$

If we use and apply (73), we find that

$$\left|\frac{g_2(w) - 1}{g_2(w) + 1}\right| \le \frac{d_{q+1}\sum_{\eta=q+1}^{\infty} |a_\eta|}{2 - 2\sum_{\eta=2}^{q} |a_\eta| - d_{q+1}\sum_{\eta=q+1}^{\infty} |a_\eta|} \le 1. \tag{75}$$

That immediately leads in a conclusion (70) of Theorem 13. To find out that

$$u(w) = w + \frac{w^{q+1}}{d_{q+1}} \tag{76}$$

gives sharp result, we observe that for $w = re^{i\pi/q}$,

$$\frac{u(w)}{u_q(w)} = 1 + \frac{w^q}{d_{q+1}} \longrightarrow 1 - \frac{1}{d_{q+1}} \quad \text{as } w \longrightarrow 1^-. \tag{77}$$

Similarly, if we take

$$g_2(w) = \left(1 + d_{q+1}\right)\left(\frac{u_q(w)}{u(w)} - \frac{d_{q+1}}{1 + d_{q+1}}\right) = 1 - \frac{\left(1 + d_{\eta+1}\right)\sum_{\eta=q+1}^{\infty} a_\eta w^{\eta-1}}{1 + \sum_{\eta=2}^{\infty} a_\eta w^{\eta-1}}, \tag{78}$$

we can deduce, and make use of (73), that

$$\left|\frac{g_2(w) - 1}{g_2(w) + 1}\right| \le \frac{\left(1 + d_{q+1}\right)\sum_{\eta=q+1}^{\infty} |a_\eta|}{2 - 2\sum_{\eta=2}^{q} |a_\eta| - \left(1 - d_{q+1}\right)\sum_{\eta=q+1}^{\infty} |a_\eta|}. \tag{79}$$

This leads directly to the statement (71) of Theorem 13.
For each $q \in \mathbb{N}$ with the external mapping $u(w)$, the bound in (71) is sharp indicated by (76).
Thus, the evidence of the Theorem 13 is complete. □

Theorem 14. *Let $u \in TS^{m}_{\hbar,\upsilon,\tau}(\nu, \ell, k, \wp)$ and fulfill (16). Then,*

$$\Re\left[\frac{u'(w)}{u'_q(w)}\right] \ge 1 - \frac{q+1}{d_{q+1}}. \tag{80}$$

Proof. By setting

$$g(w)=d_{q+1}\left[\frac{u'(w)}{u'_q(w)}\right]-\left(1-\frac{q+1}{d_{q+1}}\right)$$
$$=\frac{1+\left(d_{q+1}/(q+1)q+1\right)\sum_{\eta=q+1}^{\infty} na_\eta w^{\eta-1}+\sum_{\eta=2}^{\infty}\eta a_\eta w^{\eta-1}}{1+\sum_{\eta=2}^{\infty}\eta a_\eta w^{\eta-1}}$$
$$=1+\frac{\left(d_{q+1}/(q+1)q+1\right)\sum_{\eta=q+1}^{\infty}\eta a_\eta w^{\eta-1}}{1+\sum_{\eta=2}^{\infty}\eta a_\eta w^{\eta-1}}. \tag{81}$$

Now,

$$\left|\frac{g(w)-1}{g(w)+1}\right|\le\frac{\left(d_{q+1}/(q+1)q+1\right)\sum_{\eta=q+1}^{\infty}\eta|a_\eta|}{2-2\sum_{\eta=2}^{q}\eta|a_\eta|-\left(d_{q+1}/(q+1)q+1\right)\sum_{\eta=q+1}^{\infty}\eta|a_\eta|}. \tag{82}$$

Now,

$$\left|\frac{g(w)-1}{g(w)+1}\right|\le 1 \quad \text{if} \sum_{\eta=2}^{q}\eta|a_\eta|+\frac{d_{q+1}}{q+1}\sum_{\eta=q+1}^{\infty}\eta|a_\eta|\le 1, \tag{83}$$

since the L.H.S. of (83) is bounded above by $\sum_{\eta=2}^{q} d_\eta|a_\eta|$ if

$$\sum_{\eta=2}^{q}\left(d_\eta-\eta\right)|a_\eta|+\sum_{\eta=q+1}^{\infty} d_\eta-\frac{d_{q+1}}{q+1}\eta|a_\eta|\ge 0, \tag{84}$$

and the proof is complete. □

The consequence of the extreme function is sharp $u(w) = w + w^{q+1}/d_{q+1}$.

Theorem 15. *Let $u\in TS_{\hbar,\upsilon,\tau}^{m}(\nu,\ell,k,\wp)$ and fulfill (16). Then,*

$$\Re\left[\frac{u'_q(w)}{u'(w)}\right]\ge\frac{d_{q+1}}{q+1+d_{q+1}}. \tag{85}$$

Proof. By setting

$$g(w)=\left[q+1+d_{q+1}\right]\left[\frac{u'_q(w)}{u'(w)}-\frac{d_{q+1}}{q+1+d_{q+1}}\right]$$
$$=1-\frac{\left(1+\left(d_{q+1}/(q+1)q+1\right)\right)\sum_{\eta=q+1}^{\infty}\eta a_\eta w^{\eta-1}}{1+\sum_{\eta=2}^{q}\eta a_\eta w^{\eta-1}}. \tag{86}$$

Using (84) and making use of it, we deduce that

$$\left|\frac{g(w)-1}{g(w)+1}\right|\le\frac{\left(1+\left(d_{q+1}/(q+1)q+1\right)\right)\sum_{\eta=q+1}^{\infty}\eta|a_\eta|}{2-2\sum_{\eta=2}^{q}\eta|a_\eta|-\left(1+\left(d_{q+1}/(q+1)q+1\right)\right)\sum_{\eta=q+1}^{\infty}\eta|a_\eta|}\le 1 \tag{87}$$

that immediately leads us to the statement 15 of the theorem. □

8. Conclusions

This research has introduced study a new differential operator related to analytic function and studied some basic properties of geometric function theory. Accordingly, some results to coefficient estimates, grouth and distortion theorem, Fekete-Szego inequalityy, and partial sums have also been considered, inviting future research for this field of study.

Data Availability

No data were used to find this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

This work was equally contributed by all writers. The final version of the work has been read and approved by all of the authors.

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